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Some of the following definitions are more general than necessary for immediate purposes. They are so stated partly for their own interest and partly for later reference.

Let R^m be any set of points in one-to-one correspondence with a point set in a euclidean m-space, E^m. By identifying the points of R^m with their respective images, we define continuity, open sets, and so on, on R^m. Thus a set on R^m is *open* if and only if the corresponding set is open relative to E^m. Let rectangular cartesian coordinates be introduced in E^m, and assign to each point of R^m the coordinates of its image. We thus obtain a coordinate system (u) with R^m for its *domain*. The ***inner domain*** of (u) will mean the set of inner points of R^m. It may be vacuous.

If the domains of two coordinate systems $(u)\equiv(u_1, \cdots, u_m)$ and $(v)\equiv(v_1, \cdots, v_n)$ have common points, then the *transformation between* (u) *and* (v) will mean the correspondence

$$v_i = v_i(u) \qquad (i = 1, \cdots, n) \tag{2.1}$$

giving the (v)-coordinates in terms of the (u)-coordinates for every point common to the two domains.

Let D be any open subset of the domain of (u). Then a function $f(u)$ will be called *of class* C^r *on* D, where r is a non-negative integer, if $f(u)$ and all its derivatives of orders not greater than r are defined and continuous on D. It will be called of class C^r on[2] $\bar{D}$ if, in addition, $f(u)$ and all its derivatives of orders not greater than r have continuous limiting values[3] on $\bar{D}-D$. If $f(u)$ is of class C^r on D (or $\bar{D}$) for every positive integer r, we shall refer to it as being *of class* C^∞ *on* D (*or* $\bar{D}$). A function *of class* C^ω *on* D means one which is analytic on D. If, in addition, it is of class C^∞ on $\bar{D}$, it is said to be *of class* C^ω *on* $\bar{D}$.

We shall say that the transformation (2.1) between two coordinate systems $(u)\equiv(u_1, \cdots, u_m)$ and $(v)\equiv(v_1, \cdots, v_m)$ is *of class* C^r, $r\in(0, 1, 2, \cdots, \infty, \omega)$, if (1) the inner domains of (u) and (v) have a nonvacuous intersection, D, (2) the correspondence defined by (2.1), interpreted as a mapping of part of (u)-space into (v)-space, is a homeomorphism, (3) each of the functions $v_i(u)$ is of class C^r on D, and (4) in case $r>0$, the jacobian of the transformation is nowhere zero on D (or, in case the domains of (u) and (v) are closed, on $\bar{D}$).

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[2] The symbol for a point set, modified by a bar, denotes the closure of the set.

[3] Whitehead [13] restricts "differentiable functions" on a closed set, as $\bar{D}$, to those which can be differentiably extended to an open neighborhood of the closed set. He makes a similar restriction with respect to the "non-degeneracy" (in terms of the rank of a jacobian matrix) of a mapping of a closed set. The merits of these restrictions are discussed later.

Consider the totality of coordinate systems $(x_1, \cdots, x_m)$ whose domains are m-cells on the topological m-manifold M. Suppose that, for some value $r \in (0, 1, \cdots, \infty, \omega)$, there exists a subset, S, of these coordinate systems such that (1) every point of M is on the domain of at least one of the systems S and (2) every transformation between any two of the systems S is of class C^r. Under these conditions, we shall say that M is *of class C^r in terms*[4] *of* S. If $r > 0$, then M is said to be *differentiable in terms of* S, and if $r = \omega$, it is referred to as *analytic in terms of* S.

3. **Manifolds in euclidean spaces.** By an *m-manifold of class C^k* $(k = 1, 2, \cdots, \infty, \omega)$ in E^n is meant a set of points, M, in E^n, any point P of which has an m-cell neighborhood on M which can be defined by giving $(n-m)$ of the y's as class C^k functions of the remaining y's. These remaining y's can then be interpreted as local coordinates on the neighborhood of P in question. In terms of such local coordinates, M also satisfies the intrinsic definition (§2) of an m-manifold of class C^k.

It was natural to inquire whether the intrinsically defined manifolds of class C^k $(k > 0)$ are more general in their topological structure than manifolds of class C^k in euclidean spaces. Whitney [40] answered this question in the negative, by showing that every manifold of class C^k, intrinsically defined, is homeomorphic to a manifold of class C^k in a euclidean space. His results include a number of valuable properties, which we proceed to set forth, in so far as they are relevant to the present discussion.

Let M be an m-manifold of class C^k in terms of a set S of local coordinate systems. Suppose, for some $n > m$, that it is possible to define n functions of position $(y_1, \cdots, y_n)$ over the entire manifold M in such a way that, for some positive integer, κ, (1) the y's are continuous functions of the local coordinates S, (2) all their derivatives of orders $(1, \cdots, \kappa)$ are defined and continuous in terms of each of the coordinate systems S throughout its domain, (3) the jacobian matrix $\|\partial y_i/\partial x_j\|$ $(i = 1, \cdots, n;\ j = 1, \cdots, m)$ is of rank m on the domain of each system $(x_1, \cdots, x_m)$ of S, and (4) the same set $(y_1, \cdots, y_n)$ of values is not assumed at two different points of M.

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[4] It is important to note that for a manifold to be of class C^r does not necessarily imply a structural restriction, since the class is defined as a property of a set of local coordinate systems. According to one of the important theorems quoted below (§3(B)), there is no topological distinction between manifolds of class C^1 and those of any class C^r $(r > 1)$. The question whether manifolds of class C^0 (that is, topological manifolds in general) are topologically equivalent to those of class C^1 is considered in §14.

(A) *If* $(y_1, \cdots, y_n)$ *are now interpreted as rectangular cartesian coordinates in a euclidean space* E^n, *then the functions just described afford a nonsingular parametric representation* M^* *of* M *in* E^n, *with the coordinate systems* S *serving as local parameters. Such a representation is referred to as a class* C^κ *map or imbedding of* M. *In the above description, we necessarily have* $\kappa \leqq k$, *if* M *is of class* C^k *(but not of class* C^{k+1}*) in terms of* S.

(B) *In the preceding statement, it is easy to show, with the aid of the implicit function theorem, that the parameters can be locally eliminated, in which case* M^* *is locally represented by giving* $(n-m)$ *of the* y*'s as functions of the remaining* m, *and these functions are of class* C^κ. *They might possibly also be of class* C^λ *for some value* $\lambda > \kappa$. *Whitney's work* [40] *included a proof that if* M *is of class* C^k, $k \in (1, \cdots, \infty)$, *then it has a class* C^k *imbedding,* M^*, *in* E^{2m+1}, *and it can further be required that* M^* *be of arbitrarily high class (analytic for example) in terms of the euclidean metric of* E^{2m+1}.

Among the implications of the last statement, it may be noted that differentiable manifolds are no more general in their topological structure than analytic manifolds. In other words, for manifolds of class C^r $(r > 0)$, topological properties depend in no way on the class number r. If M is analytic, it is still not known whether it can be analytically mapped into a manifold in an E^n; though it can, as just stated, be mapped in class C^∞ fashion into an analytic manifold in E^n.

It is possible to establish [27], very briefly and simply, a weaker imbedding theorem, to the effect merely that if M is of class C^k, for any $k \in (1, 2, \cdots, \infty)$, then M has a class C^k imbedding in a euclidean space of sufficiently high dimensionality.

4. The general triangulation problem. The complexes with which this discussion is concerned are cellular complexes, defined in the spirit of euclidean geometry rather than in a more abstract manner. These complexes represent the most general triangulated spaces, in the terminology of the present treatment. The following definitions are adapted from Alexandroff and Hopf [16].

A euclidean n-space E^n is separated by an $(n-1)$-dimensional plane into two parts, the closure of either of which is called a *half-space.* A *convex cell* is a bounded subset of E^n representable as the common part of a finite number of half-spaces.[6] For any such cell, s, there exists a number $r \in (0, 1, \cdots, n)$ such that s lies in some r-plane, E^r, but not in any $(r-1)$-plane. This number is called the *dimensionality* of s, which is referred to as a *convex* r*-cell* s_r. Such a cell s_r is the closure of an open region in E^r, called the *interior of the cell*. The boundary of s_r (s_r minus its interior) falls uniquely into a finite set of convex $(r-1)$-cells, called the *bounding* $(r-1)$*-cells of* s_r. Their bounding $(r-2)$-cells are called the *bounding* $(r-2)$*-cells of* s_r, and so on. The bounding 0-cells are also called *vertices*. A convex r-cell with just $(r+1)$ vertices is an *r-simplex*.

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[6] Cells are thus defined as closed sets, contrary to the usage in §§1–3 above.

A finite or denumerable set of convex cells in E^n forms a *rectilinear complex* if (1) each bounding cell of a member of the set belongs to the set, (2) the intersection of any two cells of the set is a common bounding cell,[4] and (3) no point is vertex of infinitely many cells of the set. A rectilinear complex K is described as *simplicial* if all its cells are simplexes. It is referred to as *locally finite* if each of its points has a neighborhood which intersects at most a finite number of cells of the complex. The set of all points of E^n each on one of the cells of a locally finite rectilinear complex is called a *euclidean polyhedron*.

Any topological image of a locally finite rectilinear complex K is called a *complex*, and the images of the cells of K are called the *cells* of the complex. Thus an *n-cell* is defined as a topological image of the closure of a finite convex region of E^n.

Any topological space will be referred to as *locally polyhedral* if it can be covered with a denumerable set of neighborhoods each homeomorphic with a neighborhood on some euclidean polyhedron.

THE GENERAL TRIANGULATION PROBLEM. *Is it possible to subdivide every locally polyhedral space into the cells of a complex?*

Otherwise expressed, the problem is to discover whether a topological space with the local structure of a complex necessarily has such a structure in the large. Failing this, it is of interest to discover necessary and sufficient conditions (either or both) that a locally polyhedral space be triangulable. The triangulation problem can be solved, with the aid of well known theorems, for locally polyhedral spaces of dimensionality 2 (or less), but it has not yet been solved, in the absence of differentiability conditions, for the general locally polyhedral space (or even for the general manifold) of any higher dimension.

The triangulation problem for manifolds is of basic importance in the more general triangulation problem, by virtue of the following easily proved result.

(A) *Any locally polyhedral space* Π *can be expressed as a sum*

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[4] For the sake of this definition, the vacuous set is regarded as a bounding cell (of dimensionality -1) of every cell, and an r-cell is counted among its own bounding cells.

$$\Pi = \sum M_i^m \qquad (m = 0, 1, \cdots ; i = 1, 2, \cdots)$$

where the M_i^m are distinct topological m-manifolds such that, given any pair (M_i^m, M_j^n), either the intersection $M_i^m \cdot \overline{M}_j^n$ is vacuous or else $n > m$ and M_i^m is on $\overline{M}_j^n$.

In other words, a locally polyhedral space is built up of manifolds with incidence relations analogous to those of the cells of a complex. This does not mean that the triangulability of topological manifolds would imply the triangulability of locally polyhedral spaces in general; for it is not certain that a triangulation of the boundary of a manifold could be extended over the manifold, even if both should be proved triangulable.

5. **Significance of the problem.** The manifolds whose topology was studied by Poincaré were essentially the same as our differentiable manifolds, although the present definitions had not been made at the time. Poincaré introduced the cellular complexes of combinatorial topology as a device for dealing numerically with certain problems which presented difficulty for his manifolds. It was, however, not obvious that the results of the combinatorial theory were directly applicable to these manifolds, and the entire development of combinatorial topology was carried out with no assurance of such applicability until the triangulation problem was solved for differentiable manifolds.

The triangulation problem is thus basic in the relationship between point theoretic and combinatorial topology. Its solution, however, even if carried through in the most general case, would not suffice to lay a solid foundation for a theory of topological manifolds from the combinatorial viewpoint. In comparing the point theoretic and combinatorial methods, it is seen that analogous roles are played by homeomorphisms in the former theory and by combinatorial equivalence in the latter. Two complexes are combinatorially equivalent if they have subdivisions which are isomorphic with respect to incidence relations. A general triangulation theorem for manifolds would thus need to be supplemented by a theorem regarding the combinatorial equivalence of two triangulations of the same manifold; or, what amounts to the same thing, of a pair of homeomorphic manifolds.

As brought out in later sections, adequate triangulation and combinatorial equivalence theorems have now been proved for differentiable manifolds, so that the results of combinatorial topology are now directly applicable to the spaces for the sake of which the theory was initiated.

In the case of topological manifolds, and other locally polyhedral spaces, much work now done with the aid of approximations by complexes could probably be more readily accomplished with the aid of appropriate triangulation and equivalence theorems. However, the difficulties presented by these more general cases are of an entirely different order from those encountered under hypotheses of differentiability.

Aside from the primarily topological aspects of these problems, there exist many possibilities, some of which have already been realized, for the employment of triangulation theorems in analysis. Analysis and differential geometry are frequently concerned with spaces (algebraic or analytic varieties for example) defined by equations, or else with differentiable (analytic) manifolds of the sort defined in §§2 and 3 above, and also in Veblen and Whitehead's book [37] on the foundations of differential geometry. The homology theory of combinatorial topology, for example, is of frequent applicability and can be carried over to the spaces in question with the aid of triangulations. Such considerations are touched upon in a later section of this paper.

A large class of applications of cellular subdivisions and related methods can be grouped into (1) the extension of local results to results in the large (cf. §1 above) and (2) the extension of theorems proved for topological spaces subject to various restrictions, to apparently less restricted, but topologically equivalent, spaces. The significance of the foregoing remark can, fortunately, be illustrated by results in the literature, for the triangulation theorems thus far obtained have received applications of both of the types just mentioned.

We proceed, in §§6–11, to give an account of research thus far published with a direct bearing on the triangulation problem. This account lays a foundation for a more detailed discussion of the role of the problem in analysis and of various ramifications of the problem.

6. **Triangulation theorems in the literature.** Consider a locally polyhedral space, Π, expressed as a sum of topological manifolds as in §4(A). By the *coordinate systems on* Π, we mean the set of all coordinate systems $(x)=(x_1, \cdots, x_n)$ $(n=1, 2, \cdots)$ whose domains are (closed) n-cells on Π, each inner domain being on one of the manifolds M_i^n. We will say that Π is *piecewise of class C^r in terms of S* $(r=0, 1, \cdots, \infty, \omega)$, where S is a subset of these coordinate systems, if two conditions are fulfilled. The first is that M_i^n $(n=1, 2, \cdots;$ $i=1, 2, \cdots)$ be of class C^r in terms of the subset of the systems S consisting of all those whose inner domains are on M_i^n. The second

condition relates to coordinate systems on incident manifolds. Suppose $M_i^m \subset \bar{M}_j^n$ (cf. §4(A)), and let (x) and (y) be systems of the set S with inner domains on M_i^m and M_j^n respectively. We then require (1) that if the domains of (x) and (y) intersect, their intersection shall be the closure of an open set, D, relative to M_i^m, (2) that the y's be class C^r functions of the x's on $\bar{D}$, and (3) in case $r > 0$, that the jacobian matrix $\|\partial y_i/\partial x_j\|$ be of rank m on $\bar{D}$. We use the term *piecewise differentiable* as equivalent to piecewise of class C^1.

The most general triangulation theorem thus far established, so far as the writer is aware, is the following:

TRIANGULATION THEOREM. *Every piecewise differentiable locally polyhedral space is triangulable.*

This is the culmination of a series of results which appeared from time to time in the literature. A brief chronological outline of published material bearing directly on the problem is as follows:

(1) A proof [1] by van der Waerden of the triangulability of algebraic varieties.

(2) An abstract [2] by the writer, presenting triangulation theorems for (a) a region of euclidean n-space bounded by a finite number of differentiable manifolds, (b) a differentiable manifold in a euclidean space, and (c) a region of euclidean 3-space bounded by a finite number of piecewise differentiable 2-manifolds. This abstract represented work done on a doctoral thesis at Harvard, under Professor Marston Morse, and the writer's later research on the subject can be regarded as an outgrowth of that work.

(3) A proof [3], outlined by Lefschetz in his colloquium lectures on topology, of the triangulability of analytical varieties.

(4) An article [4] by Koopman and Brown, in which (a) it was stated that, while van der Waerden's proof is valid for algebraic varieties, it cannot be extended to analytic varieties in general, (b) it was asserted that Lefschetz' discussion [3] was incomplete, and (c) an independent proof was given of the triangulability of analytic varieties.

(5) A paper [5] giving in detail the proof of the first triangulation theorem mentioned under (2) above.

(6) A detailed proof [6] of the triangulability of analytic loci by Lefschetz and Whitehead along the lines of the outlined proof in Lefschetz' book [3].

(7) A proof [7], by the writer, of the triangulability of a regular r-locus in a euclidean space. Such a locus has the same sort of definition as the piecewise differentiable locally polyhedral space. The pa-

per employed a superfluous imbedding assumption, later shown to involve no essential loss of generality. Incidentally, it may be noted that the theorem of this paper includes all previously established triangulability results.

(8) An attack [8] by Nöbeling on the general triangulation problem for manifolds and on the problem of showing, if true, that any two triangulations of homeomorphic manifolds have equivalent subdivisions; that is, subdivisions isomorphic with respect to incidence relations of cells. The attack appeared successful, and its success would have afforded a foundation (cf. §5 above) for a complete development of a combinatorial theory of topological manifolds. However, essential errors were noted, after the article was published, and were pointed out, for example, by Seifert [9] in a brief review of the paper. These two basic problems still remain unsolved in the general case.

(9) A proof [10] by the writer of the triangulability of the differentiable manifold. The purpose of this proof was to apply the triangulation methods and results of an earlier paper [7] explicitly to the manifolds of class one (or class C^1) as defined, for example, by Veblen and Whitehead [37], by Hodge [27], and in §4 above.

(10) Papers [11, 12] by Brouwer and Freudenthal, respectively, on the triangulation of differentiable manifolds. These papers were written without knowledge of other publications on the subject.

(11) A treatment [13] by Whitehead, based partly on the writer's work, of triangulation theorems and related questions for differentiable loci.

(12) The writer's establishment [14] of the triangulability of piecewise differentiable locally polyhedral spaces.

7. Certain local properties of imbedded manifolds. It is natural, in the case of euclidean polyhedra (see §4 above), to employ rectilinear methods of subdivision. These methods have the advantage, basic in the development of combinatorial topology, that any two rectilinear triangulations of the same euclidean polyhedron have isomorphic rectilinear subdivisions (cf. §5), a property not known to hold for more general triangulations.

The most straightforward methods [7, 10, 13] of triangulation for differentiable loci are extensions and adaptations of rectilinear simplicial subdivisions. These methods depend for their success upon the manner in which a differentiable m-manifold can be locally approximated by inscribed r-simplexes ($r=1, 2, \cdots, m$).

Let M denote a closed differentiable m-manifold in E^n. In view of Whitney's results, M has all the generality, when proving its triangulability, of an intrinsically defined[7] differentiable manifold. Let $(y)=(y_1, \cdots, y_n)$ be a rectangular cartesian coordinate system in E^n. The notation Y^i $(i=1, 2, \cdots, N=C_{n,m})$ will be employed for the various combinations $Y^i=(y_{i_1}, \cdots, y_{i_m})$ of the y's in sets of m. The coordinate m-plane of the set Y^i will be referred to as the *Y^i-plane.* The projection of a point (or a set of points) on the Y^i-plane is obtained by equating to zero all the y's not in the set Y^i and is referred to as the *Y^i-projection.* If V is an m-dimensional volume on the tangent m-plane to M at P, and V^i is the Y^i-projection of V, then the ratios $\gamma^i=V^i/V$ $(i=1, \cdots, N)$ are, numerically, generalized direction cosines of M at P. The sum of their squares is unity [7], so that the largest of them at any point P on M is at least $1/N^{1/2}$. Let D^i denote the subset of M on which $\gamma^i > 1/2N^{1/2}$. Then $(D^1, \cdots, D^N)$ are an overlapping set of open regions covering M. On D^i, euclidean concepts will be employed (in the small), carried over from the Y^i-plane by the Y^i-projection (or, rather, its inverse). To avoid confusion, euclidean terminology, when thus employed, will be modified by the prefix Y^i. Thus, if Δ^i denotes any m-cell on D^i whose Y^i-projection is a convex cell, then the *Y^i-segment* joining two points (P, Q) on Δ^i means the arc on Δ^i whose Y^i-projection is the line-segment joining the projections of P and Q. A Y^i-r-simplex similarly means an r-dimensional cell on Δ^i, whose Y^i-projection is a simplex. Euclidean terms without prefixes have, of course, the usual meanings with reference to the euclidean metric of E^n.

It is not difficult to show that, for any $\epsilon > 0$, however small, there exists a $\delta > 0$ so small that if P_0 and P_1 are two distinct points on D^i at distance less than δ apart, then P_0 and P_1 determine a Y^i-1-simplex, σ_1, whose tangent lines make angles less than ϵ with P_0P_1.

A set $(P_0, P_1, \cdots, P_r)$ of points on D^i is called *Y^i-independent* if these points determine a nondegenerate Y^i-r-simplex, σ_r. They then also determine a nondegenerate rectilinear r-simplex, s_r, and are thus independent in the usual sense. The most obvious higher-dimensional analogue of the foregoing paragraph would appear to be a statement that if s_r is of sufficiently small diameter, then each tangent r-plane to σ_r makes an angle[8] less than ϵ with the plane of s_r. This is, however, false. For example, three points (however close together) on a great circle of a sphere determine a plane perpendicular to the sphere. However, the desired conclusion holds provided s_r is bounded away from degeneracy; for example, following Whitehead, in terms of the *relative thickness*, $\tau(s_r)=\rho/l$, where ρ is the distance from the centroid of s_r to its boundary, and l is the length of the longest edge of s_r.

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[7] The triangulation methods hereafter described can be carried out without any imbedding [10], using only the local euclidean geometries of a set of coordinate systems in terms of which M is differentiable. However, the absolute euclidean geometry of E^n is convenient and permits the incidental establishment of a number of important results.

[8] The cosine of such an angle is the inverse ratio of an m-dimensional volume on one plane to its projection on the other.

(A) *For any two positive constants ϵ and τ, however small, there exists a $\delta>0$ so small that if the simplex determined by $(P_0, P_1, \cdots, P_r)$ is of thickness not less than τ and diameter not greater than δ, then $(P_0, P_1, \cdots, P_r)$ determine a nondegenerate Y^i-r-simplex whose tangent r-plane at any point makes an angle less than ϵ with the plane of $s_r=P_0P_1\cdots P_r$.*

The first proof [7, 10] that M is triangulable consisted in spreading a triangulation over M by a recurrent process, giving preference to Y^i-simplexes on D^i $(i=1, \cdots, N)$. The Y^i-simplexes for different values of i were separated from one another by "cells of mixed straightness," which we proceed to describe.

Let Δ^{ij} be an m-cell on the intersection of D^i and D^j, for some two different values of i and j. Let $(P_0, P_1, \cdots, P_r)$ be a set of points satisfying the conditions of (A) above. Then $(P_0, P_1, \cdots, P_r)$ determine both a Y^i-r-simplex, σ_r, and a Y^j-r-simplex, σ_r'. Let σ_{r-1} and σ_{r-1}' be the bounding $(r-1)$-cells of σ_r and σ_r', respectively, opposite P_0.

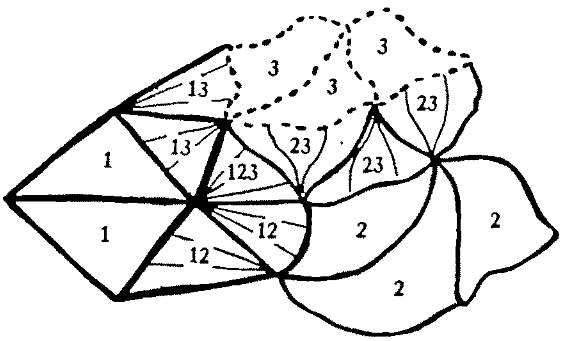
(B) *Under the conditions of* (A) *above, as the diameter of s_r approaches zero, its relative thickness remaining bounded away from zero, σ_{r-1}' and σ_{r-1} become increasingly good approximations to one another in such a way that, for δ small enough, the Y^j-segments joining P_0 to σ_{r-1} are all distinct and therefore determine a nondegenerate r-cell, σ_r^*. Since σ_{r-1} is a Y^i-simplex, it is natural to refer to σ_r^* as a Y^{ij}-r-cell.*

More generally, let $D^{j_1\cdots j_q}$ be the set of common inner points, assumed to be nonvacuous, of $D^{j_1}, \cdots, D^{j_q}$.

Let $\sigma_{r-1}{}^{i_1\cdots i_p}$ be a $Y^{i_1\cdots i_p}$-$(r$-$1)$-cell on $D^{j_1\cdots j_q}$, assuming the term defined, for some r of the set $(1, \cdots, m-1)$, where $p<r$ and $(i_1, \cdots, i_p)$ is a subset of $(j_1, \cdots, j_q)$. Let P be a point on $\Delta^{j_1\cdots j_q}$, and let $j\neq(i_1, \cdots, i_{p-1})$ be one of the numbers $(j_1, \cdots, j_q)$. If the Y^j-segments from P to points of $\sigma_{r-1}{}^{i_1\cdots i_p}$ are all distinct, and thus constitute a nonsingular r-cell σ_r, then σ_r is called a *$Y^{i_1\cdots i_p j}$-r-cell* or a *$Y^{i_1\cdots i_p}$-r-cell*, according as $j\neq i_p$ or $j=i_p$. Commencing with an arbitrary point on D^i as a *Y^i-0-cell*, we thus have a recurrent definition of $Y^{i_1\cdots i_p}$-r-cells.

(C) *If, for a given $\tau<0$, $(P_0, P_1, \cdots, P_r)$ determine an r-simplex of thickness not less than τ and of sufficiently small diameter on $D^{j_1\cdots j_q}$,* *then* $(P_0, P_1, \cdots, P_r)$ *determine nondegenerate* $Y^{j_1}\cdots{}^{j_q}$*-r-cells of all conceivable types, consistent with the definitions. In other words, the vertices* $(P_0, P_1, \cdots, P_r)$ *can be introduced, one at a time, in any order, in building up a simplicial cell as follows. With the first* κ *vertices introduced* $(\kappa \leqq r+1)$ k_1*-straightness can be employed for any* $k_1 \in (j_1, \cdots, j_q)$*. If* $\kappa \leqq r$*, then with the next* λ *points introduced,* $\lambda \leqq r+1-\kappa$*,* k_2*-straightness can be employed, where* (k_1, k_2) *are any two different numbers of the set* $(j_1, \cdots, j_q)$*, and so on. This leads to the concept of "*$(j_1, \cdots, j_q)$*-independence" for a set such as* $(P_0, P_1, \cdots, P_r)$*.*

8. **A recurrent triangulation procedure** $(m=2)$. In a lecture at the International Congress of Mathematicians at Zurich in 1932, Alexander [15] referred to the establishment of triangulation theorems for differentiable manifolds as "merely a matter of honest toil." It is indeed true that more patience than ingenuity was required to carry through the detailed modifications of rectilinear methods, which depend in a straightforward way on the facts that the metrics of two overlapping coordinate systems are continuously related and that the parallelism of one is a first approximation to that of the other. This "honest toil," a sort of mathematical pick-and-shovel work, was, however, a necessary task, if the combinatorial theory was to be directly applied to differentiable manifolds. Certain by-products of the labor, and certain of the methods employed, appear worthy of note and are presented below, free from the more tedious details.



Y^i-simplexes and transitional cells in two dimensions

A brief outline of the most direct triangulation procedure will first be presented in the readily visualized case of a closed differentiable 2-manifold M in E^3. The accompanying schematic diagram should be of assistance in following the steps of the process. Let $(y)\equiv(y_1, y_2, y_3)$ be rectangular cartesian coordinates. Using the notation of §7, let $Y^1\equiv(y_2, y_3)$, $Y^2\equiv(y_3, y_1)$ and $Y^3\equiv(y_1, y_2)$. Then D^i $(i=1, 2, 3)$

is the region on M where the direction cosine γ^i exceeds $12^{-1/2}$. Let D^{ij} denote the intersection of D^i and D^j $(i\neq j)$, and let D^{123} denote the common part of D^1, D^2, D^3. A convenient set of nonvacuous distinct regions covering M is defined as follows:

$$
\begin{aligned}
&D_0^i = D^i - (D^{ij} + D^{ik}) &&((ijk) \text{ a permutation of } (123)),\\
(8.1)\qquad &D_0^{ij} = D^{ij} - D^{123} &&(i \neq j),\\
&D^{123}.
\end{aligned}
$$

The closed regions D_0^i $(i=1, 2, 3)$ are bounded apart on M by the open region $(D^{12}+D^{23}+D^{31})$. Also, the subregions D_0^{12}, D_0^{23}, and D_0^{31} of D^{12}, D^{23}, and D^{31}, respectively, are bounded apart on $(D^{12}+D^{23}+D^{31})$ by D^{123}.

Let the Y^1-plane be triangulated into equilateral 2-simplexes $(c)'$, and let $(\sigma)'$ be the set of all Y^1-2-simplexes, plus bounding cells, each of which has one of the 2-simplexes $(c)'$ for its Y^1-projection. The following subcomplexes of $(\sigma)'$ will be used, it being understood in each case that all bounding cells of the specified 2-cells are included in the subcomplex.

$(\sigma)^1$: those 2-cells of $(\sigma)'$ each having at least one vertex on the closure of $D^1 - D^{12}$.

$(\tau)^2$: those 2-cells of $(\sigma)'$ entirely on D^{12} and having no vertices in common with cells of $(\sigma)^1$.

$(\beta)^{12}$: those 2-cells not belonging to $(\sigma)^1$ or $(\tau)^2$ but having all their vertices on $(\sigma)^1+(\tau)^2$. Each 2-cell of this set has one or two of its vertices on $(\sigma)^1$ and the remaining vertex or vertices on $(\tau)^2$.

(A) *The complex* $(\sigma)_0' = (\sigma)^1 + (\beta)^{12} + (\tau)^2$ *may not exhaust* $(\sigma)'$. *There may, for example, be 2-simplexes in* $(\sigma)'$ *lying entirely on* D^{12} *and each having a vertex not belonging to either* $(\sigma)^1$ *or* $(\tau)^2$. *However, for any preassigned neighborhood* N^1 *of the boundary* $\bar{D}^1 - D^1$ *of* D^1, *the complex* $(c)'$, *which determines* $(\sigma)'$ *and hence* $(\sigma)_0'$, *can be made so fine that* $(\sigma)_0'$ *will contain* $D^1 - N^1$.

(B) *The fineness of* $(c)'$ *will be restricted not only by the foregoing type of condition but also by the requirement that the vertices of each cell of* $(\beta)^{12} + (\tau)^2$ *be* Y^{12}*-independent* (§7(C)).

As a consequence of (B), it is possible to modify $(\sigma)_0'$ by substituting for $(\tau)^2$ and $(\beta)^{12}$, respectively, the following complexes:

$(\sigma)_0^2$: the set of all Y^2-simplexes each determined by the vertices of one of the Y^1-simplexes $(\tau)^2$.

$(\sigma)^{12}$: the set of all Y^{12}-2-cells, plus bounding cells, each determined by the vertices of one of the cells $(\beta)^{12}$ in such a way that Y^2-straightness prevails for bounding 1-cells with both vertices on

$(\sigma)_0^2$ and Y^1-straightness prevails otherwise. (See the cells labeled Y^{12} in the above diagram.)

(C) *The complex* $(\sigma)^{12}_{*}=(\sigma)^1+(\sigma)^{12}+(\sigma)_0^2$ *has the same vertices as* $(\sigma)_0'$ *and covers approximately the same region. In particular, the condition of statement* (A) *above can be assumed with* $(\sigma)^{12}_{*}$ *in place of* $(\sigma)_0'$, *since this condition can be assured by a further restriction, if necessary, on the fineness of* $(c)'$.

The introduction of the cells $(\sigma)_0^2$ facilitates a subdivision and extension of the triangulation $(\sigma)^{12}_{*}$ to cover D^1+D^2, save for an arbitrarily restricted neighborhood, N^{12}, of the boundary of D^1+D^2. Let the Y^2-plane be triangulated into simplexes $(c)''$, and let $(\sigma)''$ denote the set of all Y^2-simplexes on D^2 each corresponding under the Y^2-projection to one of the simplexes of $(c)''$. Let $(\sigma)_0''$ denote the subcomplex of $(\sigma)''$ consisting only of those 2-cells thereof, plus bounding cells, which are free from points on $(\sigma)^1+(\sigma)^{12}$. Then $(\sigma)_0''$ and $(\sigma)^1+(\sigma)^{12}+(\sigma)_0^2$ overlap only on subcomplexes made up of Y^2-simplexes. Hence these complexes can be subdivided, using only Y^2-simplexes, so as to have in common exactly a subcomplex of both. There results a complex $(\sigma)^{12}_{**}=(\sigma)^1+(\sigma)^{12}+(\sigma)^2$ which, by restrictions on the fineness of $(c)'$ and $(c)''$, can be required to cover all of M save for the preassigned neighborhood N^{12}.

(D) *The above work can be carried out* [7, 10] *in such a way as to ensure that the vertices of each cell of* $(\sigma)^{12}_{**}$ *on* D^{i3} $(i=1,\ 2)$ *or* D^{123} *satisfy the conditions of* §7(C) *for* (2, 3)*-independence or* (1, 2, 3)*-independence. The details are here omitted.*

It remains only to extend the triangulation $(\sigma)^{12}_{**}$ over the remainder of M, which is a subset of D^3 and covers all save a neighborhood N_0^3 of $\bar{D}_0^3$, where N_0^3 can be arbitrarily restricted in advance. The following subcomplexes of $(\sigma)^{12}_{**}$ are useful, it being understood that bounding cells are included in each case:

$(\sigma)_0^{12}$: the 2-cells of $(\sigma)^{12}_{**}$ each having at least one vertex on $D_0^1+D_0^{12}+D_0^2$.

$(\tau)^3$: the subcomplex of $(\sigma)^{12}_{**}$ consisting of all 2-cells thereof having no vertices in common with the subcomplex $(\sigma)_0^{12}$.

(β): those 2-cells of $(\sigma)^{12}_{**}$ which belong to neither $(\sigma)_0^{12}$ nor $(\tau)^3$.

As a consequence of (D), it is possible to modify $(\sigma)^{12}_{**}$ by substituting for $(\tau)^3$ and (β), respectively, the following complexes:

$(\sigma)_0^3$: the set of all Y^3-simplexes each determined by the vertices of one of the simplexes of $(\tau)^3$.

$(\beta)^{13}+(\beta)^{23}+(\beta)^{123}$: the $(\beta)^{i3}$ $(i=1,\ 2)$ are Y^{i3}-cells determined by vertices of Y^i-simplexes of (β) and the $(\beta)^{123}$ are Y^{123}-cells determined by vertices of Y^{12}-cells of (β), it being understood that

Y^3-straightness is to be used only relative to vertices on $(\tau)^3$.

The above description is intended to be suggestive rather than detailed.

(E) *Let* $(\sigma)_* = (\sigma)_0^{12} + (\beta)_0^3 + (\beta)^{13} + (\beta)^{23} + (\beta)^{123}$. *Then the boundary of* $(\sigma)_*$ *(and, with suitable restrictions, several "layers" of cells near the boundary) is made up exclusively of* Y^3*-simplexes. Since the remainder of* M *is on* D^3*, it is easy to extend* $(\sigma)_*$ *into a complete triangulation of* M *with the use of only* Y^3*-simplexes.*

9. **Generalization to higher dimensions.** (A) *The method outlined in* §8 *generalizes directly to a recurrent process for spreading a triangulation over an entire differentiable m-manifold* M *in* E^n. *After the jth step of the recurrency, the region* $(D^1+D^2+ \cdots D^j)$*, save for an arbitrary preassigned neighborhood of its boundary, is covered by the triangulation, and all the cells are* $Y^{i_1 \cdots i_p}$*-cells, for various subsets* $(i_1, \cdots, i_p)$ *of* $(1, \cdots, j)$. *When the triangulation is extended over* D^{j+1}*, preference is given to* Y^{j+1}*-straightness, and the transition thereto is made with the aid of* $Y^{i_1 \cdots i_p, j+1}$*-cells.*

The whole process depends, as suggested by the results in §7, on the existence of arbitrarily fine local subdivisions of M which match up in a certain way and whose cells are bounded away from degeneracy in a manner not dependent on the fineness of the subdivision.

It should be noted that $Y^{i_1 \cdots i_p}$-cells are differentiable, insofar as interior points are concerned. However, they may have conical points (or loci of conical points) on their boundaries; as, for example, in the case of a 2-cell made up of straight segments from a point to a nonplanar differentiable arc in E^3. The presence of conical points is unimportant from the viewpoint of proving triangulability, but it is undesirable for a number of other reasons.

10. **C^1-complexes.** The piecemeal triangulation process of §§8 and 9 was presented because of its directness. Whitehead's work, however, has the advantage of leading to a sort of "preferred class" of triangulations, referred to as *C^1-triangulations* or *C^1-complexes*, which resemble rectilinear complexes in possessing the basically important property of combinatorial equivalences,[9] in the following sense.

(A) *Any two C^1-triangulations of the same manifold have rectilinear models which possess isomorphic rectilinear subdivisions.*

This property puts the topology of differentiable manifolds on a basis equivalent to that of euclidean polyhedra and provides a foundation for theorems involving both differential geometry and combinatorial topology.

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[9] This property is also shared by the writer's triangulations, by virtue of statement (B) below. However, it was not previously proved for them.

A map $f(s)$ in E^n of a k-simplex s_k in E^k is described as *of class* C^1 if it is continuously differentiable and can be extended in class C^1 fashion throughout some open neighborhood of s_k in E^k. The map is referred to as *nondegenerate* if its jacobian is of rank k, and if the map can be extended over an open neighborhood of s_k so as to preserve this property.

Let K be a rectilinear simplicial complex in a euclidean space. Then $f(K) \subset E^n$ is called a *C^1-map* or a *map of class* C^1 if it is of class C^1 throughout each simplex of K. It is called *nondegenerate* if it is nondegenerate on each simplex. Such a map is referred to also as a *C^1-complex*. By virtue of this definition, the cells of a (nondegenerate) C^1-complex have no conical points on their boundaries.

The presence of conical points, noted at the end of §9, prevents the triangulations there described from being C^1-complexes. However, as Whitehead pointed out, a C^1-complex can be readily derived, as follows, from such a triangulation (σ) of M in E^n. Suppose, without further argument [19], that the vertices of each m-cell of (σ) determine a nondegenerate m-simplex in E^n and that the totality of such simplexes is a nonsingular inscribed polyhedron, Π, approximating to M in E^n. (Such approximations are further discussed below.) Each face of Π is assumed to make small angles with M at its vertices. If M is of class C^k (but not of class C^{k+1}) in E^n, the normal $(n-m)$-planes to M in E^n form a system of class C^{k-1}. Whitney [40], however, with the aid of his analytic approximations, showed how to construct a class C^k family of approximately normal $(n-m)$-planes.

(B) *By means of a class C^k family of $(n-m)$-planes approximately normal to M, an approximating polyhedron, Π, can be projected back into a triangulation (σ) of M, where (σ) is not merely a C^1-complex, but a C^k-complex.*[10]

Whitehead [13] proved directly that an arbitrary differentiable manifold possesses a C^1-triangulation. His arguments, while in many respects refinements of the piecemeal triangulation of §§8 and 9, are sufficiently different to constitute an independent proof.

11. **The associated approximation theory.** Let $f(K)$ be a nondegenerate C^1-map of a rectilinear simplicial complex, K, in a euclidean space. Suppose (cf. §10) that $f(K)=M$, where M is a differentiable m-manifold in E^n. In other words, suppose $f(K)$ to be a C^1-triangula-

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[10] The definition of a C^k-complex is an obvious modification of that of a C^1-complex.

tion of M. Let K' be a subdivision of K, and let $f'(K')$ be a C^1-map of K' in E^n. This map is described as an (ϵ, ρ)*-approximation to* $f(K)$, if, throughout each simplex s of K', we have

$$\|f' - f\| \leq \epsilon \quad \text{and} \quad \|df' - df\| \leq \rho\|df\|,$$

where (1) $\|y\|$ stands for $(\sum_{i=1}^{n} y_i^2)^{1/2}$ and (2) df and df' are tangent vectors to $f(K')$ and $f'(K')$ respectively, corresponding to a vector dx in s. If f' and f agree, point for point, then $f'(K') = f(K')$ is a $(0, 0)$-approximation to $f(K)$ and represents a subdivision of the C^1-triangulation $f(K)$ of M.

The concept of an (ϵ, ρ)-approximation can be applied in connection with a theory of polyhedral approximations to M. Let (σ) denote the cells of the C^1-complex $f(K)$ covering M. An approximating euclidean polyhedron, Π, to M can be obtained by replacing each of the cells (σ) by the simplex in E^n determined by its vertices. In the absence of further restrictions, there is no assurance that Π will be nonsingular or that its faces will be nondegenerate.

The polyhedron Π, determined as just described by $f(K) = M$, can be represented by a map $L_f(K) = \Pi$, defined by the requirements that it be linear on each simplex of K and coincide with f at each of the vertices of $(\sigma) = f(K)$. The notation $L_f(K') = \Pi'$ then represents a similarly defined piecewise linear map with reference to the subdivision K' of K. Thus Π' is, in a certain sense, a finer inscribed polyhedron than Π.

The subdivision K' of K is called a (δ, σ)*-subdivision* of K if each simplex of K' is of diameter at most δ and of relative thickness at least σ.

(A) *Given the map* $f(K)$, *two arbitrary positive constants* (ϵ, ρ), *and any positive thickness* σ, *there exists a* $\delta > 0$ *so small that* $L_f(K') = \Pi'$ *will be an* (ϵ, ρ)*-approximation to* $f(K) = M$ *if* K' *is a* (δ, σ)*-subdivision of* K. *Furthermore, if* (ϵ, ρ) *are small enough, then the* (ϵ, ρ)*-approximation* Π' *to* M *will be nonsingular.*

The above result and the next following are due to Whitehead. Proofs [13] are here omitted. Taken together, these results lead to the form given below for the polyhedral approximation theorem.

(B) *For an arbitrarily small* $\delta > 0$ *and some fixed* $\sigma > 0$, σ *being independent of* δ, *there exist* (δ, σ)*-subdivisions of* K.

Freudenthal's triangulation methods [12] employ an upper bound on the flatness of a simplex as a means of bounding it away from degeneracy, the *flatness* being defined as the ratio $l^r/v(s_r)$ where $v(s_r)$ is the r-dimensional volume of s_r and l is its longest edge. His theory included the use of arbitrarily fine simplicial subdivisions of limited flatness. Thus the rudiments, but not the development, of a polyhedral approximation theory were present. He employed a piecemeal method, analogous to that reported in §§8 and 9 of this paper.

Now let δ take on successively the values $\delta', \delta'', \delta''', \cdots$ in a sequence of positive numbers converging to zero, and let $(K', K'', K''', \cdots)$ denote a sequence of (δ, σ)-subdivisions of K, σ remaining fixed while δ approaches zero (cf. (B) above). The successive polyhedra $\Pi' = L_f(K')$, $\Pi'' = L_f(K'')$, $\cdots$ then constitute a sequence of inscribed approximating polyhedra to M. In consequence of (A), the values $(\delta', \delta'', \cdots)$ can be chosen so that $(\Pi', \Pi'', \cdots)$ are, respectively, (ϵ', ρ')-approximations, (ϵ'', ρ'')-approximations, and so on, to M, where the ϵ's and ρ's are preassigned sequences of positive numbers converging to zero.

POLYHEDRAL APPROXIMATION THEOREM. *The polyhedra* $(\Pi', \Pi'', \cdots)$ *constitute a sequence of inscribed polyhedral approximations to* M. *Under the homeomorphisms induced by the mapping* $f(K) = M$ *and the mappings* $L_f(K') = \Pi'$, $L_f(K'') = \Pi''$, $\cdots$, *these polyhedra converge to* M *as point sets. Their m-dimensional volumes also converge to the usual integral for the m-dimensional measure of* M.

This theorem, stated for convenience in Whitehead's terminology, was established by the writer [19], using angles instead of relative thickness to keep the simplexes bounded away from degeneracy. It affords a direct generalization of the usual dèfinition and formulation of arc length along a curve. As compared with examples revealing that any nonplanar differentiable surface in 3-space can be represented as the limit of a sequence of homeomorphic inscribed approximating polyhedra whose areas increase without limit instead of converging to the integral for surface area, the success of the present method depends on keeping the simplexes of the approximating polyhedra bounded away from degeneracy during the approximating process.

12. **Some new results.** The theorems stated in this section, while not previously published, are fairly direct and potentially useful consequences of the results outlined above. In the present discussion, details are avoided, but indications are given of method for constructing detailed proofs.

Let M be an m-manifold of class $C^k (k>0)$ in E^n, and let $(y) = (y_1, \cdots, y_n)$ be rectangular cartesian coordinates in E^n. Let S be a set of local coordinate systems on M such that (1) M is of class C^k in terms of S and (2) the functions y_i on M are everywhere of class C^k

in terms of the systems S and have a jacobian matrix of maximal rank m. In Whitney's paper [41] on imbedding manifolds in families of analytic manifolds, methods are employed whereby the systems S can be augmented and extended as follows. Let $(u) \equiv (u_1, \cdots, u_m)$, with domain Δ, be one of the systems S. In the terminology of Theorem I of Whitney's paper [41], Δ is an m-manifold (in particular, an m-cell) of class C^1 in "regular position" in E^n. It can, therefore, be imbedded in an $(n-m)$-parameter family, $\Delta(c_1, \cdots, c_{n-m})$, of m-cells, in such a way that (1) $\Delta(c_1, \cdots, c_{n-m})$ is analytic if $(c_1, \cdots, c_{n-m}) \neq 0$, (2) $\Delta(0, \cdots, 0) = \Delta$, (3) $\Delta(c_1, \cdots, c_{n-m})$ is homeomorphic to Δ, (4) tangential directions at corresponding points are approximately equal (see the original source for details), and (5) the cells $\Delta(c_1, \cdots, c_{n-m})$ fill out, in a single-valued way, an open region Δ^* of E^n containing Δ. Then $(u)^* \equiv (u_1, \cdots, u_m, c_1, \cdots, c_{n-m})$ can be regarded as a coordinate system with Δ^* for domain.

(A) *Let each coordinate system* (u) *of the set* S *be augmented and extended, by the process just described, into a system* $(u)^*$. *There then results a set* S^* *of local coordinate systems in* E^n *with the following properties*: (1) *Each of the systems* S^* *is related to* (y) *by a transformation of class* C^1 *with nonvanishing jacobian*, (2) *the domains of the systems* S^* *cover a neighborhood* N^* *of* M *in* E^n, *and* (3) *the part of* M *on the domain of each local coordinate system appears as part of a coordinate m-plane, relative to that system.*

Now let the whole space E^n be regarded as an n-manifold of class C^1 in terms of the systems S^* plus the system (y), and let the piecemeal triangulation procedure described in §§8 and 9 be applied to E^n in the following manner. The triangulation shall first be spread over the domains of systems in the set S^*, and then, as the last step of the recurrency, extended over the remainder of E^n. Consider any stage of the recurrency where the triangulation is being spread over the domain Δ^* of a system $(u)^* = (u_1, \cdots, u_m, c_1, \cdots, c_{n-m})$. The recurrency involves a rectilinear simplicial subdivision of Δ^* (save for a neighborhood of its boundary) in terms of $(u)^*$-straightness. In this subdivision, we can require that the coordinate m-space of $(u_1, \cdots, u_m)$, which coincides with M in the domain Δ^*, be covered by a subcomplex. This property can be preserved throughout the entire recurrent process, until all the domains of the systems S^* have been used. It will then be true that a neighborhood of M is covered by a complex in which M appears as a subcomplex. The final step of the recurrency, in which (1) a transition is made to straightness in terms of the metric of E^n and (2) the subdivision is spread over the rest of E^n, does not affect any of the cells incident with M. Hence we have the following new theorem.

THEOREM 12.1. *If M is an m-manifold of class C^k in E^n, then the entire space E^n can be triangulated into a C^k-complex*[11] *in such a way that* (1) *M coincides with a subcomplex of the triangulation and* (2) *all cells outside an arbitrary preassigned neighborhood of M are rectilinear simplexes in the euclidean metric of E^n.*

COROLLARY. *It can be required that every cell of the triangulation of E^n which has no vertices on M shall be a rectilinear simplex.*

For, once the triangulation is made, it can be arbitrarily finely subdivided, while the cells remain bounded away from degeneracy. The vertices of each cell can thus be made to determine a rectilinear simplex, such that the totality (s) of these simplexes is a rectilinear simplicial subdivision of E^n. The subset $(s)^0$ of (s) consisting of all simplexes thereof each having all its vertices on M covers a polyhedral approximation, Π, to M of the sort involved in the polyhedral approximation theorem in §11. Now, using the process of the recurrency outlined in §§8 and 9, let the cells $(s)^0$, and all cells incident with them, be modified so as to give precedence only on the cells $(s)^0$ to straightness in terms of the systems S^*. The corollary then follows directly.

THEOREM 12.2. *Let M^r be an r-manifold of class C^k on an n-manifold M^n of class C^k $(k>0)$. Then the entire manifold M^n can be covered with a C^1-complex, a subcomplex of which covers M^r.*

This theorem can be proved after the same fashion as Theorem 12.1. Local coordinate systems on M^n, whose domains cover a neighborhood of M^r, then play the role of the systems (y) in E^n.

In the calculus of variations [30, 33] in the large, for example, the configuration of a differentiable r-manifold on a differentiable n-manifold is occasionally considered. It appears likely that Theorem 12.2 might prove useful in this connection; also in various other situations where submanifolds, chains or cycles are employed, and where it might prove convenient to regard them as subcomplexes of some covering triangulation.

Consider a closed differentiable $(n-1)$-manifold, M^{n-1}, in E^n, and let Π^{n-1} be an inscribed approximating euclidean polyhedron, in the sense of the polyhedral approximation theorem in §11. With the aid of a class C^1 family of approximately normal lines to M^{n-1}, it is possible to deform E^n isotopically so that (1) all points remain fixed outside an arbitrary preassigned distance from M^{n-1} and (2) Π^{n-1} is carried into M^{n-1}. Hence we have the following theorem.

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[11] In the foregoing argument, some cells might have conical points. This can be avoided by an adaptation of Whitehead's methods.

THEOREM 12.3. *The regions into which E^n is separated by M^{n-1} are topologically equivalent to the regions into which E^n is separated by a polyhedral approximation to M^{n-1}.*

COROLLARY. *A differentiable manifold in E^3 homeomorphic to a 2-sphere separates E^3 into two parts, the bounded one of which is a 3-cell.*

The corollary follows with the aid of Alexander's proof[13] of such a separation property for a euclidean polyhedron in E^3.

13. **Applications.** The divergence theorem and the generalized Theorem of Stokes offer the readiest illustrations of the use of cellular subdivisions in extending local results to results in the large. The simplest case, namely Green's Theorem in two dimensions, is outlined in §1 above, and reference is there made to Kellogg's treatment [28] in E^3. The writer [18] proved the generalized theorem of Stokes for a differentiable orientable r-manifold M^r with piecewise differentiable boundary on a differentiable n-manifold, M^n, shortly after establishing the necessary triangulation results. Previous proofs had applied only to a manifold M in the space of a single coordinate system, where M was either analytic or was subject to conditions not known to be fulfilled save by analytic manifolds. Let $Y_{i_1 \cdots i_{r-1}}$ be an alternating tensor such that the partial derivatives $\partial Y_{i_1 \cdots i_{r-1}}/\partial y_i$ are defined and continuous in a neighborhood of $\overline{M}^r$, where the y's are a typical member of a set of local coordinates in terms of which M^n is differentiable. Then the *Stokes tensor* of $Y_{i_1 \cdots i_{r-1}}$ is defined as

$$D_{i_1 \cdots i_r} = \frac{1}{r!} \delta^{\alpha_1 \cdots \alpha_r}_{i_1 \cdots i_r} \frac{\partial Y_{\alpha_2 \cdots \alpha_r}}{\partial y_{\alpha_1}}, \tag{13.1}$$

the summation convention holding for the α's, and the δ's being generalized Kronecker δ's. If

$$y_i = f_i(u_1, \cdots, u_r) \qquad (i = 1, \cdots, n) \tag{13.2}$$

and

$$y_i = g_i(v_1, \cdots, v_{r-1}) \qquad (i = 1, \cdots, n) \tag{13.3}$$

are equations, respectively, of a typical r-cell of M and of a typical

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[13] J. W. Alexander, Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 6–8.

$(r-1)$-cell of B, then Stokes' theorem can be formulated as the identity

$$
\begin{aligned}
(13.4)\qquad &\int_M \epsilon D_{\alpha_1\cdots\alpha_r}\frac{\partial(y_{\alpha_1}\cdots y_{\alpha_r})}{\partial(u_1\cdots u_r)}\,du_1\cdots du_r \\
&\qquad = \pm\int_B \epsilon' Y_{\alpha_1\cdots\alpha_{r-1}}\frac{\partial(y_{\alpha_1}\cdots y_{\alpha_{r-1}})}{\partial(v_1\cdots v_{r-1})}\,dv_1\cdots dv_{r-1}
\end{aligned}
$$

where (1) the integrals are to be evaluated over the separate cells of the triangulation and the results summed, and (2) the ϵ's are, on each cell, $+1$ or -1 according as the orientation of the cells by the parameters (u) or (v) agrees or disagrees with arbitrarily preassigned orientations of M_r and B_{r-1}. The $\pm$ in identity (13.4) depends on the relative orientations of M_r and B_{r-1}. The integrals are absolute integral invariants under transformations of parameters and under transformations between coordinate systems. If summations are made, on both sides of (13.4), over all combinations of the α's instead of being made as the α's run independently from 1 to n, then the factor r drops out.

The establishment of the theorem reduces, much as in the work of §1, to the problem of proving the identity for a typical cell of a triangulation and then summing over all the cells. For such a typical cell, the theorem of Stokes is obtained [18] with the aid of a class C^1 mapping, from the generalized divergence theorem as applied to an m-cell in euclidean m-space.

Hodge's work [27] includes (chap. 2) related material on the generalized theorem of Stokes. In general, Hodge's research on harmonic integrals was developed with the aid of local and global considerations pertaining to manifolds of class C^r. In this work, an interrelation is to be observed between topological properties and properties of differentiable geometry. The general fields of applicability described in Hodge's book appear to be essentially coextensive with those of the triangulation theorems here described. In both cases, the applications thus far made appear to be isolated and of a preliminary nature, relative to the possible applications, and one of the principle reasons for mentioning them is to call attention to the desirability of further investigations.

There is a considerable and growing body of literature to which reference should here be made. It would be impracticable to attempt an exhaustive bibliography, especially since the boundaries of this literature cannot easily be specified. However, reference should be made to contributions of Cartan [23], Chern [24–26], and de Rham

[32]. Chern's papers, like Hodge's book, should be read not only for content, but for additional bibliographic references.

The homology groups of a manifold M have proved important in a number of applications, including Hodge's theory of multiple integrals evaluated over chains of various dimensions and differentiability classes. Any covering complex on M can serve to determine its homology groups. Algebraic geometry offers another field of application for homology theory and other aspects of combinatorial topology. This fact motivated van der Waerden's triangulation [1] of algebraic varieties. Other applications in algebraic geometry are due to Lefschetz [3, 29]. In particular, one may note the adaptation to algebraic varieties of the multiplicity concept involved in the index of an intersection of two complexes. An agreement was revealed, with the aid of homology classes, between the topologic and algebraic concepts of multiplicity, in the sense that the algebraic multiplicity of an intersection of two algebraic manifolds having a finite number of points of intersection is equal to the topological index.

Further discussion of applications would be beyond the scope of the present paper. These fragmentary comments and the accompanying bibliography can do little more than serve as an initial guide to possible further study and research in the field. The same may be said of the material in the next section of this paper.

14. The general triangulation problem for manifolds and various ramifications thereof. In Alexander's lecture [15], already cited, the opinion was expressed that the triangulation and combinatorial equivalence problems reduce, for the general topological manifold, to showing the existence of an analytic (or a piecewise linear) homeomorphism approximating, with suitable restrictions, to an arbitrary topological mapping of an n-simplex in an E^n. This opinion is closely related to the following converse of the triangulation problem for differentiable manifolds, and also to Theorem 14.1 below. The attempt of Nöbeling [8] was based on analogous "smoothing" methods. Even in three dimensions, very difficult questions of a point theoretic nature are encountered.

THE REGULARITY PROBLEM. *Let M be an arbitrary topological m-manifold, and consider the set, S_0, of all coordinate systems whose domains are m-cells on M. The regularity problem is the problem of discovering whether S_0 contains any subset, S_1, in terms of which M is of class C^1.*

Since differentiable manifolds are triangulable, an affirmative solution of the regularity problem would imply a solution of the triangulation problem for the general topological manifold. While the regularity problem thus presents a method of attacking the general triangulation problem for manifolds, it is possible that the regularity problem is too strong. In other words, there might possibly exist triangulable manifolds which cannot be made differentiable. This consideration suggested an investigation of the regularity problem not only for topological manifolds in general but for the much simpler case of triangulable manifolds. The following results were obtained.

THEOREM 14.1. *For manifolds of dimensionality less than* 5, *the regularity problem is entirely equivalent to the triangulation problem. That is, on any triangulable m-manifold M* ($m<5$), *there exists a set S of coordinate systems in terms of which M is differentiable.*

THEOREM 14.2. *If M is a triangulable m-manifold* ($m<5$), *then an analytic Riemannian geometry can be defined on M.*

This follows from Theorem 14.1 with the aid of Whitney's results stated in §3 above.

The articles [20–22] in which the above theorems were proved include a number of results bearing on the regularity problem for the general value of m. The mode of attack involves approximations. As in the case of the theorem affirming the topological equivalence of differentiable and analytic manifolds, the most promising approach appears to be with the aid of an imbedding in a euclidean space followed by the construction of a homeomorphic approximating manifold. In the present case, the given triangulable manifold is represented as a euclidean polyhedron, Π^m, in E^n. The writer's attempt, successful for $m<5$, to construct a differentiable m-manifold, M, approximating to Π^m involved a study of spaces of $(n-m)$-planes transversal to Π^m. Transversality is a generalization of orthogonality. A connection is thus established with certain aspects of the study of sphere-spaces and fibre-spaces as developed by Whitney [42, 43], Stiefel [35] and others. In this connection, Chapter VIII of Lefschetz' *Algebraic topology* [29] is of interest and contains a number of references to related literature.

Theorem 14.2, with the triangulation theorem for the analytic case, establishes the topological equivalence of the classes of triangulable manifolds and analytic Riemannian manifolds. This suggests the possibility of interpreting, with respect to triangulable manifolds, theorems of topological content proved by methods of differential geometry. The Gauss-Bonnet theorem for Riemannian

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polyhedra, as treated by Allendoerfer and Weil [17], and also by Chern [25], appears relevant in this connection.

In closing, it may again be emphasized that the whole body of applications, and, in particular, the use of triangulation and approximation theorems to carry known results over to more general spaces, is largely a matter for future research. A mere beginning has been made.

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TAYLOR'S SERIES AND APPROXIMATION TO ANALYTIC FUNCTIONS

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Taylor's series, as the simplest expansion in terms of rational functions of an arbitrary analytic function of a complex variable, has shown itself extremely useful as a guide to other such expansions. Taylor's series itself suggests other expansions and their properties, for instance series of polynomials defined by interpolation or best approximation, and is frequently a special case or a limiting case of these expansions.

The object of this address is to indicate that Taylor's series is also useful both as a guide and as a tool in the study of still other general expansions of analytic functions, namely where each approximating function is assumed merely analytic and bounded in a given region, and where the function approximated is assumed merely analytic in a closed subregion. We shall establish certain results in detail, and later indicate some new unpublished results and some open problems.

For convenient reference we mention some properties of the Taylor development

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \tag{1}$$

of a function $f(z)$ analytic throughout the circle $|z| < R$ (>1) but analytic throughout no larger concentric circle. If we set

$$S_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \tag{2}$$

then we have

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1/R, \tag{3}$$

from which we find

$$\limsup_{n\to\infty} [\max |f(z) - S_n(z)|, \text{ for } |z| = r]^{1/n} = r/R, \qquad r < R, \tag{4}$$

$$\limsup_{n\to\infty} [\max |S_n(z)|, \text{ for } |z| = r]^{1/n} = r/R, \qquad r > R. \tag{5}$$

Let us now consider the following problem, with relation to this same function $f(z)$. Let $R_0 > R$ be fixed, and let $M > 0$ be chosen; we shall later allow M to become infinite. Denote by $f_M(z)$ the (or a)

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function analytic and of modulus not greater than M in the region $|z| < R_0$, such that

$$m_M = [\max |f(z) - f_M(z)|, \text{ for } |z| = 1] \tag{6}$$

is least; thus $f_M(z)$ is a function of modulus not greater than M in the region $|z| < R_0$, of *best approximation* to $f(z)$ on the circle $|z| = 1$. It is an easily proved consequence of the theory of normal families of functions that at least one extremal function $f_M(z)$ exists. Moreover we have $m_M > 0$, for otherwise $f_M(z)$ and $f(z)$ would coincide on the circle $|z| = 1$ and throughout the region $|z| < R_0$, whereas $f_M(z)$ is analytic throughout that region and $f(z)$ has a singularity on the circle $|z| = R < R_0$. Obviously m_M decreases or remains unchanged as M increases. *We wish to study the asymptotic relationship between M and m_M*, and shall proceed by employing the Taylor partial sums $S_n(z)$ as a comparison sequence for $f_M(z)$.

If $R_1 < R$ is arbitrary, we have from (5)

$$|S_n(z)| \leq M_1 R_0^n / R_1^n, \qquad \text{for } |z| = R_0, \tag{7}$$

where M_1 is independent of n and z; we likewise have from (4)

$$|f(z) - S_n(z)| \leq M_2 / R_1^n, \qquad \text{for } |z| = 1, \tag{8}$$

where M_2 is independent of n and z. The function $S_n(z)$ is admitted to competition in the determination of the extremal function $f_M(z)$ if we choose (as we now do) n corresponding to each M so that we have

$$M_1 R_0^n / R_1^n \leq M < M_1 R_0^{n+1} / R_1^{n+1}; \tag{9}$$

these inequalities enable us to compare the set $f_M(z)$, depending on the continuous parameter M, with the sequence $S_n(z)$. We have by (8) and the first of inequalities (9)

$$\begin{aligned} m_M &= [\max |f(z) - f_M(z)|, \text{ for } |z| = 1] \\ &\leq M_2 / R_1^n \leq M_2 M / M_1 R_0^n. \end{aligned} \tag{10}$$

By the second inequality of (9) we have

$$(\log M - \log M_1)/(\log R_0 - \log R_1) < n + 1,$$

so (10) can be written in the form

$$m_M \leq M_3 M \exp[-(\log M - \log M_1) \log R_0 / (\log R_0 - \log R_1)],$$

where M_3 is independent of M and n, whence by allowing M to become infinite and then by allowing R_1 to approach R:

$$\limsup_{M\to\infty} m_M^{1/\log M} \leqq \exp\,[(-\log R)/(\log R_0 - \log R)]. \tag{11}$$

Inequality (11) is one part of a main asymptotic result, for we shall now establish the following theorem.

THEOREM 1. *Under the conditions imposed on $f(z)$, we have*

$$\limsup_{M\to\infty} m_M^{1/\log M} = \exp\,[(-\log R)/(\log R_0 - \log R)]. \tag{12}$$

We establish the equality sign in (11) by use of Hadamard's three-circle theorem, that for an analytic function $\Phi(z)$, the function $\log[\max|\Phi(z)|$, for $|z| = r]$ is a convex function of $\log r$. It turns out that the strong inequality sign in (11) would imply the uniform convergence of a sequence of the functions $f_M(z)$ throughout a region $|z| < r$ $(> R)$, which is impossible by the definition of R. Choose for definiteness the values $M = e^n$, $n = 1, 2, \cdots$, and denote the corresponding functions $f_M(z)$ by $F_n(z)$ respectively. Thus we have on the circle $|z| = R_0$ (using the Fatou boundary values on $|z| = R_0$, which necessarily exist)

$$|F_n(z)| \leqq e^n, \quad |F_{n+1}(z)| \leqq e^{n+1}, \quad |F_n(z) - F_{n+1}(z)| \leqq 2e^{n+1}. \tag{13}$$

If we assume the first member of (11) less than the second member, we can write for M sufficiently large and for suitably chosen R_1 $(R < R_1 < R_0)$

$$m_M^{1/\log M} \leqq \exp\,[(-\log R_1)/(\log R_0 - \log R_1)],$$

whence from (6) for $M = e^n$ and $M = e^{n+1}$ we have on the circle $|z| = 1$

$$|F_n(z) - F_{n+1}(z)| \leqq 2 \exp\,[-n \log R_1)/(\log R_0 - \log R_1)]. \tag{14}$$

The last inequality of (13), together with (14), yields by the three-circle theorem $(1 \leqq r \leqq R_0)$

$$\limsup_{n\to\infty}\,[\max|F_n(z) - F_{n+1}(z)|^{1/n}, \text{ for } |z| = r]$$

$$\leqq \exp\left[\frac{-\log R_1}{\log R_0 - \log R_1}\cdot\frac{\log R_0 - \log r}{\log R_0} + \frac{\log r}{\log R_0}\right];$$

this last member is less than unity for every $r < R_1$, hence is less than unity for some value of r, $R < r < R_1$; the sequence $F_n(z)$ converges to $f(z)$ uniformly for $|z| < r$ and $f(z)$ is analytic for $|z| < r$, which contradicts the definition of R. Theorem 1 is established.

Suppose now we envisage a set of functions $g_M(z)$, analytic and respectively of modulus not greater than M in the region $|z| < R_0$. We assume the functions $g_M(z)$ defined for every M. It is natural to study the asymptotic behavior of $g_M(z)$, considered to approach $f(z)$ on the circle $|z| = 1$. We have by the definition of m_M

$$\mu_M = [\max |f(z) - g_M(z)|, \text{ for } |z| = 1] \geqq m_M,$$

whence by (12):

THEOREM 2. *If $f(z)$ and the functions $g_M(z)$ satisfy the conditions given, we have*

$$\limsup_{M \to \infty} \mu_M^{1/\log M} \geqq \exp [(-\log R)/(\log R_0 - \log R)]. \tag{15}$$

Inequality (15) is our fundamental relation, which holds for the arbitrary functions $g_M(z)$, assumed merely analytic and individually bounded in the region $|z| < R_0$. However, there are numerous sets of functions, such as the partial sums of Taylor's series, and various sequences of extremal polynomials and more general rational functions for which the equality sign in (15) can be established.

In both Theorem 1 and Theorem 2 it is in fact not essential to allow M to become continuously infinite; the conclusion is valid if we consider a sequence of values of M such that the set of quotients of each value to its predecessor is bounded from unity and from infinity.

Theorems 1 and 2 suggest of themselves a broader problem, namely that of a more general geometric situation. Let the closed region S be contained in the region R; let the function $f(z)$ be analytic on S but not throughout R; we study approximation to $f(z)$ on S by functions $f_M(z)$ respectively analytic and with upper bound not greater than M in R. This broader problem might first be attacked by direct application of Theorems 1 and 2 using a conformal map. This method is sufficient in the case that each of the regions S and R is bounded by a circle—in every such case the boundaries can be transformed into concentric circles, namely the situation of Theorems 1 and 2, by means of a linear transformation of the complex variable; the second member of (12) and (15) is to be defined in terms of the largest region in which $f(z)$ is analytic, the region bounded by a level curve of the harmonic function $u(x, y)$ which takes on constant values on the boundary of S and (different) constant values on the boundary of R. Also in case S is bounded by a level curve of the Green's function for R, a suitable conformal transformation is adequate; the same comment on evaluating the second member of (12) and (15) applies. In more general geometric situations this simple method of conformal transformation is not adequate; more powerful tools must be employed. However, the new situations just treated suggest that the secret of treating the most general situation must include the use of a harmonic function and its level curves, the harmonic function being constant on the boundaries of S and R.

We return to the situation of Theorem 1. Instead of using in the proof of Theorem 1 the partial sums of the Taylor development of $f(z)$, we may use the sequence of rational functions $r_n(z)$ of respective degrees n with poles uniformly distributed on the circle $|z|=R_1>R$ (the poles of $r_n(z)$ may be taken as the nth roots of R_1^n), with $r_n(z)$ determined by interpolation to $f(z)$ in the $(n+1)$th roots of unity. The convergence properties of the sequence $r_n(z)$ are of extremal type, in the sense that equation (12) is fulfilled, even as for the partial sums of Taylor's series. From these functions $r_n(z)$ we obtain not merely a new proof of Theorems 1 and 2, but a proof that in character extends to regions S and R of very general nature, say each bounded by a finite number of disjoint Jordan curves. In this general situation the analogue of rational functions with points of interpolation and poles uniformly distributed on circles is the set of rational functions with points of interpolation and poles uniformly distributed with respect to a parameter taken as the conjugate of the harmonic function $u(x, y)$, where $u(x, y)$ is harmonic in the region $R-S$, continuous in the corresponding closed region, and takes constant values on the boundary of R and different constant values on the boundary of S; the points of interpolation can be uniformly distributed on the boundary of S; by use of a suitable conformal map, the boundary of R can be chosen as one or more disjoint analytic Jordan curves; the functions $u(x, y)$ can be harmonically extended across these curves, and the poles can be prescribed on level curves of $u(x, y)$ exterior to R. These approximating functions were defined and studied in the present writer's book[1] in the Colloquium Series, and these functions are the only functions known which can be used for the present purpose. The three-circle theorem likewise is no longer sufficiently powerful for use in the new geometric situation, but a generalization again based on harmonic functions and due to Nevanlinna is adequate. We omit here the details, but merely refer to the generalizations of Theorems 1 and 2 set forth in papers by Walsh[2] and Nilson and Walsh.[3]

Even in the situation of Theorem 1 itself there are numerous unsolved problems.[4] Is the extremal function $f_M(z)$ for prescribed M unique? How can it be characterized? Does $|f_M(z)|$ take on (or approach) the value M in an infinity of points of $|z|=R$? Does $|f(z)-f_M(z)|$ take on the value m_M in an infinity of points of $|z|=1$? Do $f_M(z)$ and m_M depend continuously on M? What is the relation of the entire problem of $f_M(z)$ to the companion problem of a prescribed $\epsilon=[\max|f(z)-\phi_\epsilon(z)|, \text{ for } |z|=1]>0$, with $\phi_\epsilon(z)$ analytic and of smallest least upper bound M_ϵ in $|z|<R$? If the function $f_M(z)$ corresponds to the maximum error m_M, and if we choose $\epsilon=m_M$, can we conclude $M_\epsilon=M$, $\phi_\epsilon(z)\equiv f_M(z)$? If $f(z)$ is required to satisfy certain continuity conditions on $|z|=R$, does $f_M(z)$ satisfy corresponding conditions on $|z|=R_0$? What other measures can be used for approximation of $f_M(z)$ to $f(z)$ on $|z|=1$, and what norm can be used for $f_M(z)$ other than the least upper bound on $|z|=R_0$?

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Merely for the sake of simplicity, we place ourselves anew in the situation of Theorem 1, where equation (12) holds with m_M defined by (6), with $|f_M(z)|\leqq M$ in $|z|<R_0$, although $f_M(z)$ need no longer be the extremal function. The three-circle theorem as used in the proof of Theorem 1 yields further results on the degree of convergence of the set of functions $f_M(z)$:

$$\begin{aligned}&\limsup_{M\to\infty}\,[\max\,|f(z)-f_M(z)|, \text{ for } |z|=r]^{1/\log M}\\ &\qquad=\exp\,[(\log r-\log R)/(\log R_0-\log R)], \quad 1\leqq r<R,\end{aligned} \tag{16}$$

$$\begin{aligned}&\limsup_{M\to\infty}\,[\max\,|f_M(z)|, \text{ for } |z|=r]^{1/\log M}\\ &\qquad=\exp\,[(\log r-\log R)/(\log R_0-\log R)], \quad R\leqq r\leqq R_0.\end{aligned} \tag{17}$$

In particular the functions $f_M(z)$ converge to $f(z)$ throughout the circle $|z|=R$, and converge in no region exterior to that circle but contained in the circle $|z|=R_0$. The asymptotic behavior of the set $f_M(z)$ on the circle $|z|=r$, $1<r<R$, is the same as for the extremal functions analytic and bounded in $|z|<R_0$ defined by best approximation to $f(z)$ on any circle $|z|=r_0\leqq r$. Equations (16) and (17) are the precise analogues of (4) and (5) for Taylor's series, and indeed (4) and (5) are included in (16) and (17). Equations (4) and (5) are intimately connected with the properties of partial sums of Taylor's series: overconvergence, degree of convergence, and zeros, as studied by Porter, Jentzsch, Ostrowski, Szegö, Pólya, Carlson, Bourion, and others. A general theory of these topics has been recently developed by the

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[4] Some partial results here have quite recently been obtained by Professor M. H. Heins.

present writer, and is as yet unpublished. This theory is based on the continued use of harmonic majorants, and applies in the situation of Theorem 1 and its generalization to further geometric situations. For instance, in the situation of Theorem 1, special subsequences of functions $f_M(z)$ may converge uniformly in a region containing an arc of the circle $|z|=R$; such overconvergence is not a local property on an arc of that circle but when it occurs must occur across every arc of that circle on which $f(z)$ is analytic. Whenever overconvergence takes place for a particular sequence of the functions $f_M(z)$, the first members of both (16) for $r\neq 1$ and (17) for $r\neq R$ for that sequence are less than the respective second members. When overconvergence occurs, $f(z)$ can have no isolated singularity on the circle $|z|=R$. Special subsequences of the set $f_M(z)$ which converge with sufficient rapidity must converge throughout the largest region in $|z|<R_0$ which contains $|z|<1$ and within which $f(z)$ is analytic. Every point of the circle $|z|=R$ is a limit point of the zeros of the entire set $f_M(z)$, and these properties hold whether $f_M(z)$ is extremal or not, provided equation (12) is satisfied.

We turn to another kind of problem, still for the present in the situation of Theorem 1. Can Theorem 1 be sharpened if the functions $f(z)$ and $f_M(z)$, the latter not necessarily extremal functions, satisfy certain continuity conditions (for instance Lipschitz conditions on the function or on some derivative) on $|z|=R$ and $|z|=R_0$ respectively? That is to say, Theorem 1 is primarily a relation involving on the one hand *regions* of analyticity of $f(z)$ and of $f_M(z)$ and on the other hand degree of convergence of $f_M(z)$ to $f(z)$. What sharper results exist when we consider in addition the behavior of $f(z)$ and $f_M(z)$ on the boundaries of their assigned regions of analyticity? A somewhat analogous question arises in the study of approximation of arbitrary functions by polynomials. If the functions approximated are merely assumed analytic in certain regions, a satisfactory theory can be developed relating these regions of analyticity to geometric degree of convergence of approximating polynomials. If more delicate measures of convergence are used, to construct a comprehensive theory it is essential to bring into account the behavior of the functions approximated on the boundaries of their regions of analyticity; this theory is based on studies in the real domain by S. Bernstein, Jackson, Montel, and de la Vallée-Poussin, was developed mainly by Sewell, Curtiss, and Walsh, and was recently set forth in a book by Sewell.[8] The theory of approximation by polynomials, although adequate for our present problem in case the region R is bounded by a level curve of Green's function for the exterior of S, is not alone sufficient for the broader application we have been considering. It would seem to be necessary as a preliminary to study the convergence properties on the boundaries of regions of the sequences of rational functions used in extending the geometric situation of Theorem 1; no such study has ever been made. Such a study, analogous to the one made for polynomials, is highly desirable, and would have also other applications than to the principal theory considered here. Taylor's series again serves as a guide and as a model; partial but apparently typical results can be obtained readily, which seem to indicate the possibility of carrying through these programs to completion.

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HARVARD UNIVERSITY

THE APRIL MEETING IN NEW YORK

The four hundred fifteenth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 26–27, 1946. The attendance included the following two hundred eighty-eight members of the Society:

C. R. Adams, C. F. Adler, R. P. Agnew, R. L. Anderson, R. G. Archibald, R. F. Arens, H. A. Arnold, L. A. Aroian, K. J. Arrow, R. N. Ascher, Silvio Aurora, M. C. Ayer, E. G. Baker, Valentine Bargmann, P. T. Bateman, Richard Bellman, A. A. Bennett, Stefan Bergman, D. L. Bernstein, Felix Bernstein, Lipman Bers, Garrett Birkhoff, M. L. Boas, R. P. Boas, Salomon Bochner, G. L. Bolton, Samuel Borofsky, S. G. Bourne, A. H. Bowker, C. B. Boyer, A. T. Brauer, Richard Brauer, A. B. Brown, G. W. Brown, E. F. Buck, R. C. Buck, R. S. Burington, Herbert Busemann, Hobart Bushey, J. H. Bushey, S. S. Cairns, H. H. Campaigne, W. M. Chen, Alonzo Church, Paul Civin, R. F. Clippinger, I. S. Cohen, R. M. Cohn, Nancy Cole, H. R. Cooley, T. F. Cope, Richard Courant, Franco Croci, H. B. Curry, J. H. Curtiss, M. D. Darkow, Norman Davids, L. S. Dederick, C. R. DePrima, C. L. Dolph, H. L. Dorwart, C. H. Dowker, Arnold Dresden, Jacques Dutka, Bernard Epstein, G. C. Evans, W. H. Fagerstrom, Ky Fan, Herbert Federer, J. M. Feld, Will Feller, A. D. Fialkow, N. J. Fine, W. B. Fite, R. M. Foster, F. H. Fowler, G. A. Foyle, A. A. Fraenkel, Gerald Freilich, Bernard Friedman, K. O. Friedrichs, Orrin Frink, R. E. Fullerton, A. S. Galbraith, G. N. Garrison, B. H. Gere, David Gilbarg, B. P. Gill, Leonard Gillman, R. E. Gilman, A. M. Gleason, Casper Goffman, A. W. Goodman, R. O. Goodman, Saul Gorn, L. M. Graves, Leonard Greenstone, Lewis Greenwald, R. E. Greenwood, George Grossman, E. J. Gumbel, D. W. Hall, F. C. Hall, Marshall Hall, P. R. Halmos, Frank Harary, Philip Hartman, V. A. Harvey, Ernest Hawkins, K. E. Hazard, M. H. Heins, Olaf Helmer, T. H. Hildebrandt, Einar Hille, T. R. Hollcroft, Harold Hotelling, S. E. Hotelling, E. M. Hull, Witold Hurewicz, (L.) C. Hutchinson, S. A. Joffe, Fritz John, R. E. Johnson, Aida Kalish, Edward Kasner, M. E. Kellar, L. S. Kennison, H. S. Kieval, S. A. Kiss, J. W. Kitchens, J. R. Kline, Morris Kline, E. G. Kogbetliantz, E. R. Kolchin, B. O. Koopman, M. E. Ladue, V. V. Latshaw, J. R. Lee, Solomon Lefschetz, D. H. Lehmer, A. M. A. Lehr, M. E. Levenson, Howard Levi, Madeline Levin, Norman Levinson, J. V. Lewis, Marie Litzinger, E. R. Lorch, A. N. Lowan, C. I. Lubin, L. A. MacColl, G. W. Mackey, H. M. MacNeille, H. F. MacNeish, Murray Mannos, A. J. Maria, Imanuel Marx, S. P. Mead, H. L. Meyer, D. S. Miller, Deane Montgomery, T. W. Moore, Vladimir Morkovin, D. J. Morrow, Marston Morse, G. W. Mullins, F. J. Murray, D. S. Nathan, C. A. Nelson, O. E. Neugebauer, Morris Newman, G. E. Noether, P. B. Norman, L. R. Norwood, V. M. Noyes, C. O. Oakley, A. F. O'Neill, J. C. Oxtoby, J. S. Oxtoby, A. M. Peiser, A. J. Penico, Everett Pitcher, Harry Polachek, Harry Pollard, E. L. Post, William Prager, Walter Prenowitz, M. H. Protter, R. G. Putnam, H. W. Reddick, Mina Rees, R. W. Rempfer, Moses Richardson, R. G. D. Richardson, C. E. Rickart, John Riordan, J. F. Ritt, E. K. Ritter, J. H. Roberts, M. S. Robertson, Robin Robinson, S. L. Robinson, I. H. Rose, Ira Rosenbaum, J. H. Rosenbloom, P. C. Rosenbloom, Raphael Salem, John Salerno, H. E. Salzer, Hans Samelson, Arthur Sard, L. J. Savage, S. A. Shaaf, Robert Schatten, Henry Scheffé, Lowell Schoenfeld, Abraham Schwartz, J. B. Secrist, C. W. Seekins, I. E. Segal, Hyman Serbin, C. E. Shannon, C. B. Shapira, I. M. Sheffer, Max Shiffman, L. P. Siceloff, James Singer, D. M. Smiley, M. F. Smiley, Henry E. Smith, P. A. Smith, W. M. Smith, Herbert Solomon, V. E. Spencer, E. R. Stabler, E. P. Starke, Fritz Steinhardt, E. G. Straus, L. M. Straus, R. C. Strodt, W. C. Strodt, A. C. Sugar, Fred Supnick, Irving Sussman, Otto Szász, J. H. Taylor, Feodor Theilheimer, J. M. Thomas, L. F. Tolle, Hing Tong, Leonard Tornheim, C. A. Truesdell, Y. W. Tschen, H. F. Tuan, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, Annita Tuller, S. M. Ulam, W. R. Utz, H. S. Vandiver, H. E. Vansant, Oswald Veblen, R. J. Walker, A. D. Wallace, R. M. Walter, W. R. Wasow, Alan Wayne, J. V. Wehausen, Harry Weingarten, Alexander Weinstein, M. E. Wells, V. H. Wells, R. A. Wetzel, F. J. Weyl, George Whaples, A. P. Wheeler, G. W. Whitehead, S. S. Wilks, C. S. Williams, John Williamson, Clement Winston, Z. S. Wurtele, Bertram Yood, H. J. Zimmerberg, Leo Zippin, O. J. Zobel, M. F. Zucker, Antoni Zygmund.

The meeting opened Friday afternoon at 1:30 with an address on *The triangulation problem and its role in analysis* by Professor S. S. Cairns. Professor Solomon Lefschetz presided. President T. H. Hildebrandt presided at the Symposium in which Professor Marston Morse gave an address on *Topological methods in the theory of functions of a single complex variable.* Dr. M. H. Heins led the discussion.

On Saturday morning there were three sections for research papers: Analysis in which Dean R. G. D. Richardson presided, Geometry and Topology in which Professor A. D. Wallace presided, and Algebra, Statistics, etc., in which Professor Orrin Frink presided.

At 2:30 P.M. Saturday Professor Richard Brauer gave an address *On the arithmetics of algebras.* Professor Emeritus W. B. Fite presided.

The Council met on Friday at 4:30 P.M. in Havemeyer Hall.

The Secretary announced the election of the following one hundred and eighty persons to ordinary membership in the Society:

Professor Albert A. Aardal, Wartburg College, Waverly, Iowa;
Dr. James C. Abbott, Annapolis, Md.;
Professor William Franklin Adams, University of Alabama;
Mr. Merle M. Andrew, Massachusetts Institute of Technology;
Mr. Cloyd Payne Armbrister, Concord College, Athens, W. Va.;
Professor Homer Leon Arnold, Humboldt State College, Arcata, Calif.;
Professor Helen Barton, Woman's College, University of North Carolina;
Miss Ruby Baxter, Frances Shimer College, Mount Carroll, Ill.;
Professor Alice Knisely Bell, Fresno State College;
Mr. Edmund Callis Berkeley, Lieutenant Commander, U.S.N.R.;
Mr. Salvatore Dante Bernardi, University of Illinois;
Professor Arthur Bernhart, University of Oklahoma;
Mr. William Werner Boone, Princeton University;
Dr. Paul Boschan, The Econometric Institute, Inc., New York, N. Y.;
Mr. Nelson Allen Brigham, Arlington, N. J.;
Dr. Louis Francis Brown, Hollywood, Calif;
Professor A. C. Burdette, University of California, Davis, Calif.;
Mr. Robert C. Byers, Institute of Optics, University of Rochester;
Mr. George Overall Caldwell, Rust College, Holly Springs, Miss.;

Sister Rose Gertrude Calloway, Washington, D. C.;
Dr. Bernardo Guillermo Capó, Agricultural Experimental Station, Río Piedras, Puerto Rico;
Professor Francis L. Celauro, Lehigh University;
Professor Reynaldo Pedro Cesco, University of La Plata;
Mr. Chester Ray Clark, McKinley High School, Washington, D. C.;
Professor John R. Clark, Teachers College, Columbia University;
Mr. Charles Greenleaf Cloudman, Ebasco International Corp., New York, N. Y.;
Mr. Alberto Dias Coimbra, Lyceum de Coimbra, Coimbra, Portugal;
Professor William Atlee Conrad, U. S. Naval Academy;
Mr. E. Allen Cook, Office of Chief of Chemical Warfare Service, War Department;
Dr. Robert Cortell, College of the City of New York;
Dean Roger Flavius Cox, John Brown University, Siloam Springs, Ark.;
Miss Dorothy Irene Coyne, Pine Cobble School, Williamstown, Mass.;
Miss Kathleen Creagh, Waller Branch High School, Chicago, Ill.;
Professor James Thomas Culbertson, Southwestern University, Georgetown, Tex.;
Professor Charles Harold Douglas, Catawba College, Salisbury, N. C.;
Mr. Donald Gordon Duncan, University of British Columbia;
Mr. Walter George Dyer, Lieutenant (j.g.), U.S.C.G.;
Professor Anthony Joseph Eiardi, Boston College;
Mr. William Ralph Eikelberger, U. S. Naval Academy;
Miss Jane Lee Evans, Old Trail School, Akron, Ohio;
Miss Meta M. Ewing, Bay City Junior College, Bay City, Mich.;
Reverend Fred J. Fischer, DePaul University;
Professor Immanuel C. Fischer, University of Minnesota;
Mr. William Thompson Fishback, Radiation Laboratory, Massachusetts Institute of Technology;
Professor A. A. Fraenkel, Hebrew University, Jerusalem, Palestine;
Mr. Norman Saffrey Free, Mount Royal College, Calgary, Alberta, Canada;
Sister M. Elizabeth Frisch, Villa Madonna College, Covington, Ky.;
Professor Abel Gauthier, University of Montreal;
Miss Florence Evelyn Gerhardt, University of Wisconsin;
Mr. Robert Anthony Joseph Gildea, Brown University;
Mr. Malcolm Elisha Gillis, University of Tennessee;
Mr. Homer T. Gittings, Jr., Naval Ordnance Laboratory, Washington, D. C.;
Mr. Jacob Kopel Goldhaber, University of Wisconsin;
Dr. Louis Goldstein, Division of War Research, O.S.R.D., New York, N. Y.;
Mr. David Martin Good, Princeton University;
Professor Morris J. Gottlieb, Washington University;
Professor Pierre Lucien Gregoire, University of Montreal;
Professor Lino Gabriel Gutiérrez, University of Havana;
Professor William A. Hallam, West Virginia Wesleyan College;
Mr. Ray Irvine Hardin, New York, N. Y.;
Mr. Lyman John Harris, Aluminum Company of America;
Mr. Charles Edward Harrison, Technisonic Laboratory, St. Louis, Mo.;
Miss Bertha Irene Hart, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.;
Miss Veronica Andrea Harvey, St. Joseph's College for Women, Brooklyn, N. Y.;
Mr. Manuel Herschdorfer, Amherst College;
Professor Paul Otto Hoffmann, Newark College of Engineering, Newark, N. J.;

Mr. William Julian Jaffe, Newark College of Engineering, Newark, N. J.;
Mr. Lawrence Gabriel Jones, Radiation Laboratory, Massachusetts Institute of Technology;
Miss Margaret Joseph, Shorewood High School, Milwaukee, Wis.;
Dr. Charles Hillis Kaiser, Bennington College;
Miss Bruria Kaufman, Philadelphia, Pa.;
Mr. Peter John Kiernan, U. S. Military Academy;
Professor Allen Lewis King, Department of Physics, Dartmouth College;
Professor Lyman Coleman Knight, Muskingum College, New Concord, Ohio;
Mr. Paul J. Kopp, Major, Chemical Warfare Service, U.S.A.;
Dr. Waclaw Kozakiewicz, University of Saskatchewan;
Dr. Maria Zbigniew Krzywoblocki, Palo Alto, Calif.;
Miss Dorothy B. Landau, Junior High School 81, New York, N. Y.;
Miss Lucy Anne La Sala, O.S.R.D., New York, N. Y.;
Professor Lloyd Laurits Lassen, North Texas Agricultural College, Arlington, Tex.;
Mr. Oliver L. Lattimore, Texas College, Tyler, Tex.;
Dr. Nathan Lazar, Teachers College, Columbia University;
Mr. William John Lee, San Benito County High School and Junior College, Hollister, Calif.;
Mr. Walter Gilbert Leight, Captain, U.S.A.;
Professor W. H. Lipscombe, University of Akron;
Professor Brendan A. Lynch, Siena College, Loudonville, N. Y.;
Miss Ethelyne L. McBee, Bureau of Human Nutrition and Home Economics, U. S. Department of Agriculture, Washington, D. C.;
Miss Jessie Irene McClune, United Aircraft Corp., East Hartford, Conn.;
Mr. William Straight McCulley, University of Texas;
Professor Kenneth Madison McDonald, Judson College, Marion, Ala.;
Professor Gordon Richey Magee, University of Western Ontario;
Professor Dis Maly, Rensselaer Polytechnic Institute, Troy, N. Y.;
Mr. Alvin Stuart Mancib, West Somerville, Mass.;
Professor Armin William Manning, Concordia Collegiate Institute, Bronxville, N. Y.;
Professor Guy Ernest March, South Dakota School of Mines;
Reverend Norbert Martin, St. Francis College, Loretto, Pa.;
Mr. Paul Meier, Princeton University;
Mr. Paul Theodore Mielke, Wabash College, Crawfordsville, Ind.;
Dr. Abraham Miller, Lieutenant Commander, U.S.N.R.;
Mr. William Edward Miller, New Jersey State Teachers College, Trenton, N. J.;
Professor Johann Mokre, Barat College, Lake Forest, Ill.;
Professor Lawrence Joseph Monville, John Carroll University, Cleveland, Ohio;
Mr. Morris Morduchow, Polytechnic Institute of Brooklyn;
Mr. George F. Morecroft, Metropolitan Life Insurance Company, New York, N. Y.;
Mr. William Daniel Morgan, Saint Paul, Minn.;
Sister Marie Jesse Morrison, Nazareth College, Nazareth, Mich.;
Miss Catherine E. Moser, Penn Hall, Chambersburg, Pa.;
Mr. Frederick C. Mosteller, Princeton, N. J.;
Dr. Theodor Samuel Motzkin, Hebrew University, Jerusalem, Palestine;
Dr. M. R. Neifeld, Beneficial Management Corp., Newark, N. J.;
Dr. Horace Wakeman Norton, U. S. Weather Bureau;
Mr. Norman Harding Painter, Massachusetts Institute of Technology;
Professor Benjamin Wissler Partlow, Madison College, Harrisonburg, Va.;

Mr. Lester Charles Peach, Sergeant, U.S.A.;
Mr. Anthony Joseph Penico, University of Pennsylvania;
Mr. Phillip Irvine Peters, Union College, Barbourville, Ky.;
Miss Helen Bard Pettit, Mills College, Oakland, Calif.;
Mr. Robert I. Piper, So. California Telephone Company, Los Angeles, Calif.;
Professor Albert Roberts Poole, Montana State College, Bozeman, Mont.;
Professor Henry Jean Baptiste Putnam, Laval University;
Mr. Lloyd James Quaid, University of Minnesota;
Mr. Hugh Lawrence Quarles, Lieutenant Colonel, A.U.S.;
Mr. Russell Remage, Jr., Gassaway, W. Va.;
Miss Theresa M. Renner, Blackburn College, Carlinville, Ill.;
Mr. Franklin Victor Reno, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.;
Dr. Nicholas A. Renzetti, U.S.N.O.T.S., Inyokern, Calif.;
Mr. Rodolfo Alfredo Ricabarra, University of La Plata;
Sister Mary Edmund Rice, Our Lady of Cincinnati College;
Professor Toivo E. Rine, Illinois State Normal University;
Professor Lawrence A. Ringenberg, University of Maryland;
Professor Leland Frederick Samuel Ritcey, United College, Winnipeg, Manitoba, Canada;
Mr. Emerson Richard Rohrer, Electronic Technician's Mate 2d Class, U.S.N.;
Dr. Jerome Michael Sachs, Chicago City College;
Professor Alberto Enrique Sagastume Berra, University of La Plata;
Mr. John Salerno, U. S. Coast and Goedetic Survey;
Mr. William Salkind, U. S. Department of Agriculture;
Mr. Charles Saltzer, Brown University;
Miss Augusta Louise Schurrer, University of Wisconsin;
Miss Mary Lurline Scott, National Bureau of Standards, Washington, D. C.;
Sister M. Georgianne Segner, LeClerc College, Belleville, Ill.;
Mr. Harold Douglas Seielstad, Lieutenant Commander, U.S.C.G.;
Mr. Carl Bryce Seligman, Ensign, U.S.N.R.;
Professor Paul B. Selz, Parsons College, Fairfield, Iowa;
Miss Charlotte B. Shapira, McLaughlin-Cara Associates, Cambridge, Mass.;
Mr. William Henry Shields, Philadelphia, Pa.;
Mr. Walt R. Simmons, U. S. Employment Service, Arlington, Va.;
Mr. Maurice Singer, U.S.A.;
Mr. Benjamin Slepin, Philadelphia, Pa.;
Mr. Ralph Jeffery Slutz, Princeton University;
Mr. Oliver M. Smart, U. S. Navy Inspection Service, Westinghouse Electric and Manufacturing Company, Essington, Pa.;
Professor Harold E. Smith, Junior College of Connecticut, Bridgeport, Conn.;
Mr. Henry Elmore Smith, Dickinson College, Carlisle, Pa.;
Professor Allen K. Snyder, Valley Forge Military Academy and Junior College, Wayne, Pa.;
Mrs. Ruth Cottingham Sorrells (Mrs. C. C.), Highland Park Senior High School, Dallas, Tex.;
Mr. George Springer, Brown University;
Mr. Robert Clarence Stewart, Washington and Jefferson College, Washington, Pa.;
Mrs. Ruth G. Sumner, Oakland High School, Oakland, Calif.;
Mr. Arthur Frank Svoboda, DePaul University;

Mr. Taffee Tadashi Tanimoto, University of Chicago;
Mr. Benjamin Tannenbaum, Princeton University;
Mr. James Edgar Thompson, University of Minnesota;
Professor César Anselmo Trejo, University of La Plata;
Mr. Guy Z. Updike, College of the City of New York;
Professor Helen E. Vansant, Beaver College, Jenkintown, Pa.;
Professor Robert N. Varney, Department of Physics, Washington University;
Reverend Dunstan G. Veleaz, Quincy College, Quincy, Ill.;
Professor Helen M. Walker, Teachers College, Columbia University;
Mr. Earl Eugene Wallick, Senior High School, Lakewood, N. J.;
Mr. Camille Jerome Weber, The Chase National Bank of the City of New York;
Miss Alma Kohl Weinstein, Houston, Tex.;
Dr. Charles A. Welsh, U. S. Department of Justice;
Miss Edith Freda Whitmer, Stephens College;
Mr. Wilbur James Widmer, Gibbs and Cox, Inc., New York, N. Y.;
Mr. Arthur Strong Wightman, Ensign, U.S.N.R.;
Mr. Roger Williams, Jr., Belle Works, DuPont Company, Charleston, W. Va.;
Miss Marcella Ann Yarosheski, St. Genevieve of the Pines Junior College, Asheville, N. C.;
Professor Theodore Otte Yntema, School of Business, University of Chicago;
Mr. John Wesley Young, Jr., University of Pennsylvania;
Professor Paul McClure Young, Miami University;
Miss Lillian Bertha Zarling, North Dakota State School of Forestry, Bottineau, N. D.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

Brooklyn College: Professor Merle L. Bishop;
Brown University: Dr. Chi-Teh Wang;
University of California: Messrs. Donald Marvin Adelman, Bjarni Jónsson, and Samuel Wolfenstein;
University of California at Los Angeles: Messrs. Milton Drandell and William C. Hoffman;
California Institute of Technology: Messrs. Donald A. Darling, Lincoln Kearney Durst, Verne Gilbert Robinson, and James G. Wendel;
University of Chicago: Miss Hilde Ruth Marlin, Mr. August Newlander, Jr.;
College of the City of New York: Mr. Julius S. Dwork;
Columbia University: Miss Azelle Brown, Mr. David James Dickinson, Professor Pao Lu Hsu, Messrs. Morris Newman and John Bert Secrist, Jr., Miss Miriam Skinner;
Harvard University: Messrs. Benjamin Nelson Moyls and John Joseph Sopka, Jr.;
University of Illinois: Messrs. Delbert Ferrel Atkins, Richard William Ball, Bernard E. Howard, William A. Martin and Ernest A. Propes, Misses Mary Anice Seybold and Mary Frances Suter;
Indiana University· Messrs. Robert Earl MacKenzie and Samuel Schecter;
Institute for Advanced Study: Dr. Ky Fan;
Iowa State College: Mr. Russell Evan Carr;
State University of Iowa: Mr. Harlan Lowell Herrick;
Johns Hopkins University: Miss Fay Dalton Carpenter, Messrs. Raymond Huck, Stanley Leonard Isaacson, and Sylvan Wallach;
Massachusetts Institute of Technology: Mr. Burton Mills;

University of Michigan: Lieutenant William Boothby, Messrs. Arvid W. Jacobson, Edwin Henry Spanier, and Jesse Wright;
Michigan State College: Mr. Olan Trainer McMillan;
University of Minnesota: Miss Margaret Owchar;
Northwestern University: Mr. Richard Elton von Holdt and Mrs. Margaret Helen Stange;
University of Notre Dame: Reverend Henry F. DeBaggis;
Ohio State University: Messrs. H. Manfred Fliess and John Lewis Moll;
College of Saint Thomas: Mr. Donald John Lewis;
Stanford University: Mr. Andrew Heuer Van Tuyl;
Swarthmore College: Mrs. Elizabeth P. Allen;
Syracuse University: Mr. Helmut Aulbach;
University of Virginia: Mr. Victor La Rue Klee;
University of Washington: Mr. Harry Edward Kinerk;
Williams College: Dr. Arthur Smullyan;
University of Wisconsin: Messrs. Herman Jacob Cohen, Ottis William Rechard, Herbert John Ryser, Lee Robert White, and Russell Edward Wright;
Metropolitan Life Insurance Company: Mr. Edward A. Lew.

Professor T. G. Cowling of the University College of North Wales and Mr. Paul Vermes of The Grammar School, Farnborough, England, were admitted to the Society in accordance with the reciprocity agreement with the London Mathematical Society. Professor Alexander Markovich Ostrowski was admitted to the Society in accordance with the reciprocity agreement with the Swiss Mathematical Society.

The Secretary reported that, as of April 26, 1946, the total membership of the Society is 3,041.

The following appointments by the President were reported: as representative of the Society at the inauguration of Wilson Martindale Compton as President of The State College of Washington on December 11, 1945, Professor M. S. Knebelman; as representative of the Society at the inauguration of Byron Sharpe Hollinshead as President of Coe College on December 14, 1945, Professor E. N. Oberg; as representative of the Society at the inauguration of Arthur Holly Compton as Chancellor of Washington University on February 22, 1946, Professor L. M. Blumenthal; as representative of the Society at the Fiftieth Annual Meeting of The American Academy of Political and Social Science on April 5–6, 1946, Professor Arnold Dresden; as representative of the Society at the Sesquicentennial Celebration of the University of North Carolina on April 12–13, 1946, Professor J. M. Thomas; as representative of the Society at an educational conference celebrating the inauguration of James Lewis Morrill as President of the University of Minnesota on April 23–25, 1946, Professor R. W. Brink; as representative of the Society at the inauguration of John Philip Wernette as President of The University of New Mexico on May 10, 1946, Professor Lincoln LaPaz; as representative of the Society on the Committee on Definitions of Electrical Terms of the American Institute of Electrical Engineers, Dr. H. W. Bode; as auditors of the Society's accounts for 1946, Professors T. F. Cope and A. E. Meder, Jr.; as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for 1946–47, Professor R. P. Agnew (committee now consists of Professors J. R. Kline, Chairman, R. P. Agnew, and C. C. MacDuffee); as a member of the Committee to Select Hour Speakers for Eastern Sectional Meetings for 1946–47, Professor A. W. Tucker (committee now consists of Professors T. R. Hollcroft, Chairman, Oystein Ore, and A. W. Tucker); as a member of the Committee to Select Hour Speakers for Western Sectional Meetings for 1946–47, Professor N. E. Steenrod (committee now consists of Professors R. H. Bruck, Chairman, Tibor Radó, and N. E. Steenrod); as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for 1946–47, Professor Morgan Ward (committee now consists of Professors A. C. Schaeffer, Chairman, D. H. Lehmer, and Morgan Ward); as a Committee on Arrangements for the 1946 Summer Meeting at Cornell University, Professors B. W. Jones (Chairman), T. R. Hollcroft, W. T. Martin, J. B. Rosser, and R. J. Walker; as a committee to nominate representatives of the Society on the Policy Committee for Mathematics, Dean M. H. Ingraham (Chairman), Professors J. R. Kline and R. L. Wilder; as a Committee on Nomination of Officers and Members of the Council for 1947, Professors Saunders MacLane (Chairman), R. L. Jeffery, C. B. Morrey, Jr., Tibor Radó, and R. L. Wilder; as a member of the Committee on Printing Contracts, term to expire December 31, 1948, Professor J. M. Thomas (committee now consists of Professor R. M. Foster, Chairman, Dean M. H. Ingraham, and Professor J. M. Thomas); as a member of the Committee on Places of Meetings for 1946–48, Professor G. B. Price; as Chairman of the Committee on Places of Meetings for the year 1946, Professor C. R. Adams (committee now consists of Professors C. R. Adams, Chairman, G. B. Price, and W. T. Reid); as members of the Committee on Publicity, Professor R. J. Walker, 1946–48, Dr. J. M. Thompson and Professor J. B. Rosser, 1946–47 (committee now consists of Professors R. M. Foster, Chairman, J. B. Rosser, Dr. J. M. Thompson, and Professor R. J. Walker).

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1946: Bulletin—Professor R. E. Langer; Transactions—Professor A. A. Albert; Colloquium Publications—Professor J. F. Ritt; American Journal of Mathematics—Professor Hassler Whitney; Mathematical Reviews—Professor O. E. Neugebauer; Mathematical Surveys—Professor Nelson Dunford.

It was announced that Professor Will Feller is the representative of the Institute of Mathematical Statistics on the Policy Committee for Mathematics and that Professor Alonzo Church is the representative of the Association for Symbolic Logic. The representatives of the Society are Professors G. C. Evans, T. H. Hildebrandt, Marston Morse, and M. H. Stone. Professor J. R. Kline is an ex officio, nonvoting member of the committee and acts as secretary for the committee.

Professor J. L. Walsh was elected by the Council to fill the unexpired term of Professor J. D. Tamarkin as a member of the Mathematical Surveys Editorial Committee.

It was reported that Professor Nelson Dunford has been selected as the Chairman of the Mathematical Surveys Editorial Committee for the year 1946.

Certain invitations to deliver addresses were announced: Professor J. L. Doob for the 1946 Summer Meeting; Professor A. P. Morse for the 1946 Annual Meeting.

The Council adopted the following resolution on the death of Professor Harry Bateman:

> The Council of the American Mathematical Society records with sorrow the untimely death of Harry Bateman, Professor of Mathematics, Physics, and Aeronautics at the California Institute of Technology. Born at Manchester, England, and trained at Cambridge University (where he was Senior Wrangler in the Mathematical Tripos of 1905), Bateman had the breadth of outlook and diversity of interests characteristic of the British mathematicians of his generation. The applications of mathematics were as important to him as the refinements of pure mathematics. A year at Gottingen and Paris (1905–06) stimulated his interest in the then recent theory of integral equations, in which he did his earliest extensive work. He came to the United States in 1910 (Bateman was a naturalized U. S. citizen); and after appointments at Bryn Mawr College and The Johns Hopkins University, accepted the professorship, which he held till his death, at the California Institute of Technology on its organization in 1917.
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> Bateman was a rare combination of erudition and creativeness. In the field of the differential equations and special functions of mathematical physics, his knowledge and power were unrivalled. He had "instant recall" in any subject he had ever studied, and could direct an enquirer through the voluminous literature of half a century at a moment's notice. His own analysis, in his writings and in his lectures, was characterized by a brilliant technique that always, somehow, produced the right operator or the ingenious transformation to dispose of the problem in hand with a minimum of labor. If his advanced students had any criticism of his lectures, it was the complimentary remark that Bateman's difficulty was his inability to distinguish between an easy problem and a hard one.

Adequate notices of Bateman's research will appear elsewhere; it will suffice here to mention some of the topics which continued to interest him till the day of his death. Mathematical physics, particularly electrodynamics, fascinated him, with hydrodynamics, or perhaps sound, as a close second. Everything connected with wave motion, from sound to optics, was a lifelong interest. Here his knowledge was unequalled. When meteorology and seismology began to become mathematical Bateman added them to his repertory. In pure mathematics his interests centered in those subjects—geometrical transformations, definite integrals, partial differential and integral equations, probability—where his extraordinary technical facility might have free play. Although he was familiar with the more refined types of analysis of the past forty years, and appreciated them, he seldom cared to work in them himself. Two major treatises, on definite integrals and on all the special functions hitherto discussed in mathematics, were laid aside under the urgency of war work.

In this Society, Bateman was a member of the Council (1924–26, 1944–45), Vice President (1935–36), and Gibbs Lecturer (1943). His membership in learned societies included the National Academy of Sciences, the American Philosophical Society, the Royal Society, London, and several others in the fields of his many interests.

Bateman was an incessant worker. His one ambition was the advancement of mathematical science. He was equally distinguished for his great scientific attainments, his unfailing gentleness, and his extreme modesty.

The Council accepted an invitation to meet at Princeton University on November 2, 1946. The Secretary reported that the Council had, by mail vote, accepted an invitation from Swarthmore College for the 1946 Annual Meeting. The Council canceled the meeting scheduled for October 26, 1946, in New York City. Times and places for meetings of the Society in the midwest and far west, respectively, were set as follows: November 29–30, 1946, at Iowa State College; November 30, 1946, at the University of California at Los Angeles.

The Council nominated Professor John von Neumann as a representative of the Society in the Division of Physical Sciences of the National Research Council for a three-year period beginning July 1, 1946. The other representatives of the Society for this three-year period will be: Professors A. B. Coble, Saunders MacLane, M. H. Stone, Oswald Veblen, and Dr. Warren Weaver.

The Council authorized the Policy Committee for Mathematics to give public expression to opinion in support of the Kilgore-Magnuson Bill.

Titles and cross references to the abstracts of papers read at the meeting follow below. Papers 1–10 were read in the section for Analysis, 11–19 in the section for Geometry and Topology, 20–28 in the section for Algebra, Statistics, etc., and 29–67 whose abstract numbers are followed by the letter *t* were read by title. Paper 5 was read by Dr. Wasow, 9 by Dr. Szász, 13 by Professor Hall, 14 by Professor Kasner. Dr. Nachbin was introduced by Professor Antoni Zygmund, Professor Ringenberg by Professor D. W. Hall, Professor

Samuelson by Professor T. R. Hollcroft, and Professor Shen by Professor E. J. McShane.

1. Bernard Epstein: *Integral representation of solutions of the harmonic and wave equations.* (Abstract 52-5-144.)

2. R. C. Buck: *An extension of Carlson's theorem.* (Abstract 52-5-139.)

3. H. J. Zimmerberg: *A class of definite integral systems.* (Abstract 52-3-76.)

4. W. M. Chen: *Distortion theorems in the theory of pseudo conformal mappings.* (Abstract 52-5-142.)

5. K. O. Friedrichs and W. R. Wasow: *On singular perturbations of nonlinear oscillations.* (Abstract 52-5-149.)

6. Alexander Weinstein: *On a generalization of an axially symmetric potential.* (Abstract 52-5-171.)

7. Harry Pollard: *The mean convergence of orthogonal series of polynomials.* (Abstract 52-7-237.)

8. Herbert Federer: *The* (ϕ, k) *rectifiable subsets of* n*-space.* (Abstract 52-5-145.)

9. S. Minakshisundaram and Otto Szász: *On absolute convergence of multiple Fourier series.* (Abstract 52-5-156.)

10. R. P. Boas: *The rate of growth of an analytic function determined from its growth on a sequence of points.* (Abstract 52-7-230.)

11. Fred Supnick: *Rectilinear deformation.* (Abstract 52-5-213.)

12. P. R. Halmos: *Invariant measures.* (Abstract 52-5-151.)

13. D. W. Hall and J. W. T. Youngs: *Comments on the cores of certain classes of spaces.* (Abstract 52-5-202.)

14. Edward Kasner and John DeCicco: *Comparison of union-preserving and contact transformations in space.* (Abstract 52-5-187.)

15. Abraham Schwartz: *Higher normal spaces for spaces of class greater than one.* (Abstract 52-5-191.)

16. G. W. Whitehead: *A generalization of the Hopf invariant.* (Abstract 52-5-215.)

17. J. H. Roberts: *Open transformations and dimension.* (Abstract 52-5-210.)

18. Hing Tong: *Ideals of normed rings associated with topological spaces.* Preliminary report. (Abstract 52-7-241.)

19. I. E. Segal: *The spectrum of a commutative subalgebra of a group algebra.* Preliminary report. (Abstract 52-5-131.)

20. I. S. Cohen: *Groups and rings of finite rank.* Preliminary report. (Abstract 52-5-110.)

21. Marshall Hall: *The sum of continued fractions.* (Abstract 52-3-55.)

22. A. T. Brauer: *On a theorem of M. Bauer.* (Abstract 52-5-108.)

23. I. M. Sheffer: *Note on multiply-infinite series.* (Abstract 52-5-132.)

24. Will Feller: *The law of the iterated logarithm for identically distributed random variables.* (Abstract 52-5-196.)

25. Casper Goffman: *Measures of fluctuation of a variable mean.* Preliminary report. (Abstract 52-5-197.)

26. Ira Rosenbaum: *Hegel's observations on the differential and integral calculus and its foundations.* (Abstract 52-3-95.)

27. C. A. Truesdell: *On Sokolovsky's "momentless shells."* (Abstract 52-3-87.)

28. Garrett Birkhoff: *Symmetric Lagrangian systems.* (Abstract 52-7-242.)

29. Reinhold Baer: *Projectivities with fixed points on every line of the plane.* (Abstract 52-5-181-*t*.)

30. Richard Bellman: *Stability of difference equations.* (Abstract 52-7-228-*t*.)

31. Garrett Birkhoff and L. J. Burton: *A weakening of the Hölder conditions for Newtonian force fields.* (Abstract 52-5-137-*t*.)

32. A. T. Brauer and Gertrude Ehrlich: *On the irreducibility of certain polynomials.* (Abstract 52-5-109-*t*.)

33. B. L. Brown and N. H. McCoy: *Radicals and subdirect sums.* (Abstract 52-3-54-*t*.)

34. Nathaniel Coburn: *Pressure-volume relations and the geometry of the net of characteristics in two-dimensional supersonic flows.* (Abstract 52-5-174-*t*.)

35. M. A. Coler: Observations on primes. (Abstract 52-5-111-*t*.)

36. H. S. M. Coxeter: *A new extreme form in seven variables.* (Abstract 52-5-112-*t*.)

37. H. S. M. Coxeter: *The content of the general regular polytope.* (Abstract 52-5-182-*t*.)

38. Arnold Emch: *New properties of quadrics with umbilics based upon their stereographic projections.* (Abstract 52-5-185-*t*.)

39. Will Feller: *A limit theorem for random variables with infinite moments.* (Abstract 52-5-195-*t*.)

40. E. E. Floyd: *On the extensions of homeomorphisms on the interior of a two cell.* (Abstract 52-5-200-*t*.)

41. K. O. Friedrichs: *An inequality for potential functions.* (Abstract 52-5-146-*t*.)

42. K. O. Friedrichs: *On a theorem of Lichtenstein.* (Abstract 52-5-147-*t*.)

43. K. O. Friedrichs: *On the boundary value problems of the theory of elasticity and Korn's inequality.* (Abstract 52-5-148-*t*.)

44. R. A. Good: *On the construction of clusters.* (Abstract 52-5-116-*t*.)

45. W. H. Gottschalk: *Almost periodicity, equi-continuity and total boundedness.* (Abstract 52-5-201-*t*.)

46. G. W. Mackey: *On convex topological linear spaces.* (Abstract 52-7-235-*t*.)

47. Leopoldo Nachbin: *On linear expansions.* I. (Abstract 52-5-158-*t*.)

48. Isaac Opatowski: *Markoff chains with variable intensities: average duration of transition.* (Abstract 52-5-199-*t*.)

49. L. A. Ringenberg: *On the extension of interval functions.* (Abstract 52-5-161-*t*.)

50. Ira Rosenbaum: *Hegel, mathematical logic, and the foundations of mathematics.* (Abstract 52-5-194-*t*.)

51. H. E. Salzer: *Coefficients for facilitating the use of the Gaussian quadrature formula.* (Abstract 52-5-178-*t*.)

52. H. E. Salzer: *Further empirical results on tetrahedral numbers.* (Abstract 52-5-126-*t*.)

53. P. A. Samuelson: *A connection between the Bernoulli and Newton iterative processes.* (Abstract 52-3-84-*t*.)

54. P. A. Samuelson: *Computation of characteristic vectors.* (Abstract 52-3-85-*t*.)

55. P. A. Samuelson: *Generalization of the Laplace transform for difference equations.* (Abstract 52-3-86-*t*.)

56. Robert Schatten (National Research Fellow): *On projections with bound* 1. (Abstract 52-5-164-*t*.)

57. I. E. Segal: *Representation of certain commutative Banach algebras.* (Abstract 52-5-130-*t*.)

58. Y. C. Shen: *Interpolation to certain analytic functions by rational functions.* (Abstract 52-5-165-*t*.)

59. P. A. Smith: *Foundations of Lie groups.* (Abstract 52-5-166-*t*.)

60. A. C. Sugar: *The use of invariant inverted matrices for the simultaneous approximate solution of classes of boundary value problems.* Preliminary report. (Abstract 52-7-249-*t*.)

61. Fred Supnick: *A class of functions with associated circle manifolds.* (Abstract 52-5-167-*t*.)

62. Fred Supnick: *Equivalent rectilinear graphs.* (Abstract 52-5-214-*t*.)

63. Fred Supnick: *On certain limiting sequences of polygons.* (Abstract 52-5-168-*t*.)

64. Fred Supnick: *On Vitali's covering theorem.* (Abstract 52-5-169-*t*.)

65. Y. C. Wong: *Contributions to the theory of surfaces in a 4-space of constant curvature.* (Abstract 52-5-192-*t*.)

66. Y. C. Wong: *Scale hypersurfaces for conformal-Euclidean space.* (Abstract 52-5-193-*t*.)

67. H. J. Zimmerberg: *Two general classes of definite integral systems.* (Abstract 52-5-173-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The four hundred sixteenth meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, on Friday and Saturday, April 26–27, 1946. The attendance was about one hundred eighty, including the following one hundred fifty-four members of the Society:

A. A. Albert, E. S. Allen, E. W. Anderson, T. W. Anderson, B. H. Arnold, W. L. Ayres, Reinhold Baer, F. E. Baker, R. H. Bardell, Walter Bartky, G. E. Bates, H. W. Becker, J. H. Bell, P. O. Bell, S. F. Bibb, K. E. Bisshopp, H. L. Black, L. M. Blumenthal, D. G. Bourgin, R. H. Bruck, G. S. Bruton, E. L. Buell, R. H. Cameron, R. E. Carr, P. W. Carruth, W. B. Caton, E. W. Chittenden, R. V. Churchill, R. H. Cole, B. H. Colvin, Max Coral, J. J. Corliss, V. F. Cowling, M. M. Day, John DeCicco, Bernard Dimsdale, William H. Durfee, W F. Eberlein, Samuel Eilenberg, Martinus Esser, H P. Evans, H. S. Everett, G. M. Ewing, W. A. Ferguson, L. R. Ford, J. S. Frame, Evelyn Frank, M. R. Freundlich, J. W. Givens, E. L. Godfrey, H. H. Goldstine, L. W. Griffiths, V. G. Grove, V. C. Harris, W. L. Hart, G. E. Hay, C. T. Hazard, A. E. Heins, E. D. Hellinger, R. G. Heisel, M. R. Hestenes, Edwin Hewitt, E. H. C. Hildebrandt, D. L. Holl, B. E. Howard, H. K. Hughes, H. D. Huskey, T. J. Jaramillo, A. W. Jones, F. B. Jones, G. K. Kalisch, L. H. Kanter, Wilfred Kaplan, Irving Kaplansky, William Karush, J. L. Kelley, L. M. Kelly, Fred Kiokemeister, L. A. Knowler, Joseph Landin, E. P. Lane, R. E. Langer, C. G. Latimer, W. G. Leavitt, R. A. Leibler, Harry Levy, A. O. Lindstrum, M. I. Logsdon, A. T. Lonseth, Eugene Lukacs, C. C. MacDuffee, Saunders MacLane, W. S. Massey, A. E. May, E. J. Mickle, A. N. Milgram, H. J. Miser, C. W. Moran, E. J. Moulton, M. E. Munroe, S. B. Myers, August Newlander, C. V. Newsom, E. A. Nordhaus, Rufus Oldenburger, Isaac Opatowski, F. W. Owens, L. J. Paige, Gordon Pall, P. M. Pepper, H. P. Pettit, H. H. Pixley, A. L. Putnam, Tibor Radó, E. D. Rainville, Maxwell Reade, W. T. Reid, Haim Reingold, R. B. Rice, E. H. Rothe, J. M. Sachs, M. A. Sadowsky, R. G. Sanger, A. T. Schafer, R. D. Schafer, O. F. G. Schilling, G. E. Schweigert, M. E. Shanks, Seymour Sherman, H. A. Simmons, G. W. Smith, W. S. Snyder, Andrew Sobczyk, R. D. Specht, Abraham Spitzbart, C. E. Springer, N. E. Steenrod, R. R. Stoll, E. B. Stouffer, T. T. Tanimoto, H. P. Thielman, T. Y. Thomas, E. P. Vance, H. S. Wall, C. P. Wells, M. E. Wescott, A. L. Whiteman, L. R. Wilcox, R. L. Wilder, J. E. Wilkins, G. S. Young, J. W. T. Youngs, Oscar Zariski, Daniel Zelinsky.

On Friday and Saturday mornings, by invitation of the Committee for Invited Addresses at Western Sectional Meetings, Professor O. F. G. Schilling gave the Symposium lectures, under the title *Ideal theory on open Riemann surfaces*. The presiding officers at these sessions were Professors A. A. Albert and Saunders MacLane.

Sessions for the reading of short papers were held on Friday morning, with Professor Tibor Radó presiding, on Friday afternoon, with Professor C. C. MacDuffee presiding, and on Saturday morning, with Professor W. T. Reid presiding.

Titles and cross references for the abstracts of papers read at the meeting follow below: Papers 1 to 7 were read Friday morning, 8 to 17, Friday afternoon, and 18 to 24, Saturday afternoon. Papers 25 to 59, whose abstract numbers are followed by the letter *t*, were read by title. Paper 6 was read by Professor Schweigert, 12 by Professor MacLane, and 19 by Professor Cameron. Professor L. K. Hua was introduced by Professor A. A. Albert.

1. R. L. Wilder: *Certain topological properties in the large and their applications.* (Abstract 52-5-217.)

2. J. W. T. Youngs: *The topological theory of Fréchet surfaces (2-cell case).* (Abstract 52-5-221.)

3. R. G. Helsel: *A theorem on surface area.* (Abstract 52-5-203.)

4. G. S. Young: *Interior and border transformations on surfaces.* (Abstract 52-5-220.)

5. N. E. Steenrod: *Extensions of maps and products of cocycles.* (Abstract 52-5-212.)

6. G. E. Schweigert and G. S. Young: *Remarks concerning invariants for certain finite transformations.* (Abstract 52-5-211.)

7. E. H. Rothe: *Gradient mappings in Hilbert space.* (Abstract 52-5-162.)

8. Lois W. Griffiths: *Linear homogeneous diophantine equations.* (Abstract 52-5-118.)

9. H. W. Becker: *Combinatory interpretations of Bell's numbers.* (Abstract 52-5-106.)

10. Irving Kaplansky: *On a problem of Kurosh and Jacobson.* (Abstract 52-5-122.)

11. Grace E. Bates: *Free loops and nets and their generalizations.* (Abstract 52-5-105.)

12. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in abstract groups.* IIa. *Kernels and three-dimensional cohomology.* (Abstract 52-5-113.)

13. A. W. Jones: *Semimodular finite groups and the Burnside basis theorem.* (Abstract 52-5-120.)

14. F. B. Jones: *A characterization of a plane semi-locally-connected continuum.* (Abstract 52-3-101.)

15. Harry Levy: *The projective geometry of Riemannian spaces of two dimensions.* (Abstract 52-5-190.)

16. M. R. Hestenes: *An alternate sufficiency proof for the normal problem of Bolza.* (Abstract 52-7-233.)

17. Edwin Hewitt: *On rings of continuous functions.* Preliminary report. (Abstract 52-5-207.)

18. P. O. Bell: *Power series expansions for the equations of a variety in hyperspace.* (Abstract 51-9-173.)

19. R. H. Cameron and W. T. Martin: *The behavior of measure and measurability under change of scale in Wiener space.* (Abstract 52-5-140.)

20. J. E. Wilkins: *A note on the general summability of functions.* (Abstract 52-5-172.)

21. Maxwell Reade: *On averages of Newtonian potentials.* (Abstract 52-5-160.)

22. Wilfred Kaplan: *Qualitative analysis of physical systems.* (Abstract 52-5-176.)

23. W. F. Eberlein: *Wave equation in a stratified medium.* (Abstract 52-5-175.)

24. Andrew Sobczyk: *Stabilization of carrier-frequency servomechanisms.* (Abstract 52-5-180.)

25. A. A. Albert: The *Wedderburn principal theorem for Jordan algebras.* (Abstract 52-5-103-*t*.)

26. Reinhold Baer: *Absolute retracts in group theory.* (Abstract 52-5-104-*t*.)

27. H. W. Becker: *The general theory of rhyme.* (Abstract 52-5-107-*t*.)

28. R. H. Bing: *Converse linearity conditions.* (Abstract 52-5-136-*t*.)

29. L. M. Blumenthal: *Superposability in elliptic spaces.* (Abstract 52-7-250-*t*.)

30. D. G. Bourgin: *Approximate isometries.* (Abstract 52-5-138-*t*.)

31. W. B. Caton: *An inequality for analytic functions which are represented by their Fourier-Laguerre series.* (Abstract 52-5-141-*t*.)

32. John DeCicco: *Geodesic perspectivities upon a sphere.* (Abstract 52-5-183-*t*.)

33. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in abstract groups.* IIb. *Group extensions with a non-abelian kernel.* (Abstract 52-5-114-*t*.)

34. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in abstract groups.* III. *Non-associative systems and cohomology.* (Abstract 52-5-115-*t*.)

35. Arnold Emch: *Dissection of two equivoluminal parallelotops into two finite series of equal numbers of congruent pieces in ordinary and higher Euclidean spaces.* (Abstract 52-5-184-*t*.)

36. Evelyn Frank: *The location of the zeros of polynomials with complex coefficients.* (Abstract 52-3-67-*t*.)

37. R. A. Good: *On the theory of clusters.* (Abstract 52-5-117-*t*.)

38. M. R. Hestenes: *Theorem of Lindeberg in the calculus of variations.* (Abstract 52-5-152-*t*.)

39. L. K. Hua: *Geometries of matrices.* III. *Fundamental theorems in*

the geometries of symmetric matrices. (Abstract 52-5-119-*t*.)

40. G. B. Huff: *An arithmetic characterization of proper characteristics of linear systems.* (Abstract 52-5-186-*t*.)

41. G. B. Huff: *Rational-distance sets in the plane.* (Abstract 52-5-208-*t*.)

42. R. E. Johnson and Fred Kiokemeister: *The endomorphisms of the total operator domain of an infinite module.* (Abstract 52-3-57-*t*.)

43. Edward Kasner and John DeCicco: *Conformal perspectivities upon a sphere.* (Abstract 52-5-188-*t*.)

44. Edward Kasner and John DeCicco: *The distortion of angles in general cartography.* (Abstract 52-5-189-*t*.)

45. Fred Kiokemeister: *A note on the Schmidt-Remak theorem.* (Abstract 52-5-123-*t*.)

46. Fred Kiokemeister: *A theory of normality for quasigroups.* (Abstract 52-5-124-*t*.)

47. Fred Kiokemeister: *Reducibility in a class of homogeneous polynomials.* (Abstract 52-5-125-*t*.)

48. M. E. Munroe: *A second note on weak differentiability of Pettis integrals.* (Abstract 52-5-157-*t*.)

49. Isaac Opatowski: *Average duration of transition in Markoff chains.* (Abstract 52-5-198-*t*.)

50. George Piranian: *A summation matrix with a governor.* (Abstract 52-5-159-*t*.)

51. Ida Roettinger: *On certain finite integral transformations.* (Abstract 52-5-177-*t*.)

52. R. D. Schafer: *Equivalence in a class of division algebras of order* 16. (Abstract 52-5-127-*t*.)

53. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* I. (Abstract 52-5-128-*t*.)

54. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* II. (Abstract 52-5-129-*t*.)

55. Bernard Vinograde: *Sfield (division ring) composites.* (Abstract 52-5-134-*t*.)

56. H. S. Wall: *Bounded J-fractions.* (Abstract 52-5-170-*t*.)

57. R. L. Wilder: *A generalization of local co-connectedness and its applications.* (Abstract 52-5-216-*t*.)

58. R. L. Wilder: *Concerning generalized n-cells.* (Abstract 52-5-218-*t*.)

59. R. L. Wilder: *Homology groups of perfectly normal spaces.* (Abstract 52-5-219-*t*.)

R. H. Bruck,
Associate Secretary

THE APRIL MEETING IN BERKELEY

The four hundred seventeenth meeting of the American Mathematical Society was held at the University of California on Saturday, April 27, 1946. The attendance totaled a little over sixty and included the following fifty-four members of the Society:

H. M. Bacon, E. W. Barankin, E. M. Beesley, Z. W. Birnbaum, J. L. Brenner, F. A. Butter, G. B. Dantzig, A. C. Davis, E. A. Davis, R. P. Dilworth, L. K. Durst, A. L. Foster, J. W. Green, W. S. Gustin, S. M. Hallam, J. G. Herriot, W. C. Hoffman, Alfred Horn, C. G. Jaeger, R. D. James, Bjarni Jónsson, E. L. Lehmann, Hans Lewy, G. F. McEwen, J. C. C. McKinsey, W. A. Mersman, E. D. Miller, W. E. Milne, K. A. Morgan, C. B. Morrey, A. P. Morse, Jerzy Neyman, J. W. Odle, C. D. Olds, O. G. Owens, George Polya, W. C. Randels, J. B. Robinson, R. M. Robinson, A. C. Schaeffer, Abraham Seidenberg, T. W. Simpson, D. C. Spencer, Pauline Sperry, J. J. Stoker, J. D. Swift, L. H. Swinford, Gabor Szegö, Alfred Tarski, R. K. Wakerling, Morgan Ward, A. R. Williams, František Wolf, Max Zorn.

Contributed papers were presented during the morning session with Professor W. E. Milne presiding. In the afternoon session the remaining contributed papers were presented after which Professor R. P. Dilworth delivered the invited hour address entitled *The combinatorial methods of lattice theory*. Professor C. B. Morrey presided during the afternoon session.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Paper number 2 was read by Professor James and paper number 7 by Professor Spencer.

1. E. W. Barankin: *On the eigen-values of infinite matrices and of linear integral equations*. Preliminary report. (Abstract 52-1-10.)

2. R. D. James and W. H. Gage: *A generalized integral*. Preliminary report. (Abstract 52-5-155.)

3. J. D. Swift: *Certain power series with coefficients in a finite field*. Preliminary report. (Abstract 52-5-133.)

4. Alfred Horn: *The asymptotic behavior of solutions of integral equations*. Preliminary report. (Abstract 52-5-154.)

5. W. S. Gustin: *Convex boundary sets*. (Abstract 52-5-150.)

6. Morgan Ward: *Note on the order of the free distributive lattice*. (Abstract 52-5-135.)

7. A. C. Schaeffer and D. C. Spencer: *The coefficients of schlicht functions*. III. Preliminary report. (Abstract 52-5-163.)

8. G. F. McEwen: *Diffusion from an instantaneous circular area source*. Preliminary report. (Abstract 52-7-244.)

9. Z. W. Birnbaum: *Tshebysheff inequality for two dimensions*. (Abstract 52-7-255.)

10. H. W. Eves: *Arc chains and arc necklaces*. Preliminary report. (Abstract 52-3-91-*t*.)

A. C. SCHAEFFER,
Associate Secretary

STEFAN BANACH
1892-1945

News came recently that Stefan Banach died in Europe shortly after the end of the war. The great interest aroused in this country by his work is well known. In fact, in one of Banach's main fields of work, the theory of linear spaces of infinitely many dimensions, the American school has developed and continues to contribute very important results. It was a rather amazing coincidence of scientific intuition which focused the work of many mathematicians, Polish and American, on this same field, a field which grew so richly in the period between the two world wars. Actually, it was E. H. Moore who was the precursor of the doctrine of applying abstract algebraic methods to many concrete problems of analysis. It is not necessary to describe the achievements of Banach and his students which center around his monograph on the theory of linear operations.

Banach's work brought out for the first time in the *general* case the success of the methods of geometrical and algebraic approach to problems in linear analysis—far beyond the more formal discoveries of Volterra, Hadamard, and their successors. His results embraced more general spaces than the work of such mathematicians as Hilbert, Schmidt, Riesz, von Neumann, Stone, and others.[1] Many mathematicians, especially the younger ones in the United States, took up this idea of geometrical and algebraic study of linear function spaces, and the work is still going on vigorously and producing rather important results. Also, it is not necessary to stress the possibilities of the application of similar methods of approach to problems of nonlinear analysis, at present so numerous and important for immediate applications. Banach had a number of results (unfortunately unpublished) on the theory of polynomials and analytical operators in a class of suitably defined infinitely-dimensional spaces.

He had in preparation a sequel to his volume on linear operators dealing with spaces on which one could study nonlinear transformations. Various approaches to the study of "nonlinear analysis" have of course been made before by mathematicians. In the opinion of the writer the proper definition of a natural class of spaces on which these more general operators should be studied has not yet been found. Very often in the history of mathematical theories one of the factors deciding the success of a new discipline is the proper amount of generality for the class of objects under discussion. It should contain a large class of examples already existing in mathematics and not be overburdened by too many arbitrary constructions.

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[1] The initial general formulation was published, almost simultaneously with that of Banach, by Norbert Wiener who, however, changed his field of interest very shortly afterward.

Banach's contributions to fields of mathematics grouped around the theory of functions of real variables, set theory, and general groups are equally important and include some of the most elegant and final results in these fields. To give only a few examples, the specialist will remember his theorems on functions of bounded variation, the theorems on differentiability of functions, and his paper on the length of curves and the area of surfaces. His work of the early twenties was characterized by extreme elegance and perspicacity of proofs. One remembers the originality of his proof of Vitali's decomposition theorem and the beautiful set-theoretical investigations on one-to-one transformations of abstract sets. His proof of the theorem stating that a set which is locally of first category is of first category in the large, even for the non-separable case, is an example of his ingenuity.[3] Of course, most mathematicians know Banach's results on measure in Euclidean spaces and in abstract sets (containing the results obtained jointly with Tarski on the paradoxical decomposition of two spheres of different radii into an equal number of pairwise congruent sets). Banach's theories of generalized limits for sequences of numbers and of functions, a joint work with Mazur, find a growing number of applications. He wrote also two interesting papers on general metric groups.

When we survey his work, the role played by the set-theoretical and topological methods in more classical mathematical disciplines becomes obvious. For instance, the very important work of Schauder (a student of Banach who was killed during the war) in which he establishes existence theorems in the theory of differential equations uses the method of general function spaces. As other examples of the success of these methods of analysis one has only to point out Banach's work in the theory of general orthogonal series or general integrals.

Among Banach's writings, in addition to his well known monograph on linear operations, are a university textbook on differential and integral calculus in two volumes and a university text on mechanics—the latter, in the opinion of the writer, a masterful presentation, extremely worth translating into English.

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[3] This theorem was still further generalized in an interesting paper by Deane Montgomery.

Banach was born in 1892. His family was poor and he had very little conventional schooling at first. Thus he was mostly self-educated when he went to the Polytechnical Institute in Lwów. It is said that it was Steinhaus who accidentally discovered Banach's talent by overhearing a mathematical conversation between Banach and another student on a park bench in Cracow. Banach and Steinhaus were to become later the closest collaborators and the senior members of the Lwów school of mathematics. Banach was a student at the Polytechnical Institute between 1910 and 1914. He obtained his Ph.D. in 1920, and was instructor at the Polytechnical Institute between 1920 and 1922, became a docent and an "extraordinary professor" at the Lwów University in 1922, and a professor in 1929. He was a member of the Polish Academy of Sciences and won several prizes including the prize of the Academy and a scientific prize of the City of Lwów. The writer has no precise knowledge of his life and work from the outbreak of the war to his premature death in the fall of 1945.

Banach worked in periods of great intensity separated by stretches of apparent inactivity. During the latter, however, his mind kept working on the selection of statements that would best serve as focal theorems in the next field of study. He liked constant mathematical discussion with friends and students. The writer remembers a mathematical session with Mazur and Banach lasting seventeen hours without interruption except for meals. In general both the Lwów and Warsaw mathematical schools were fond of and successful in collaboration. A large proportion of the papers published were written by more than one author. It was Banach and, in Warsaw, Sierpiński who were mainly responsible for this development.

Much of the mathematical work was carried on in a way not usual in American mathematical centers. The mathematicians in Lwów met not only in their classrooms and offices but spent long hours every day in two coffee houses which served as informal meeting places. They discussed problems over coffee or beer, and marble table tops and napkins took the place of blackboards. It was hard to outlast or outdrink Banach during these sessions. He was always there. The weekly (Saturday night!) meetings of the mathematical society, where papers were presented, provided the more formal discussions.

Banach's faculty for proposing problems illuminating entire sections of mathematical disciplines was very great. These, properly formulated, can often play the role of *experimenta crucis* in a physical theory. The constant challenge of these questions, proposed by mathematicians of the group over which Banach presided, stimulated the gradual progress of the theories, now so well known, in functional analysis and real variables. He took up eagerly the challenge of any important question connected with the domain in which he was working at the time, and he could spend months on it.

Banach's publications reflect only a part of his mathematical powers. The diversity of his interests in mathematics surpasses by far that shown in his papers. His personal influence on other mathematicians in Lwów, and in Poland, was in many cases decisive. It stands out as one of the main traits of this remarkable period of twenty years when so much mathematical work was accomplished and so many high hopes were raised—all to be interrupted by World War II.

All this is now a rather distant memory. Of the many outstanding mathematicians who were Banach's students, Schauder, Auerbach, and Kaczmarz are dead. All the others have left Lwów. But some members who survived the war are trying to reconstitute the group in the new surroundings of Breslau to which Lwów's university has been transferred.

S. ULAM

BOOK REVIEW

Jacobian elliptic functions. By E. H. Neville. Oxford University Press, 1944. 16+331 pp. $7.50.

In the development of the theory of elliptic functions it is shown at an early stage that, as far as singularities are concerned, the simplest elliptic functions other than constants are those of order 2. This naturally leads to the classification of elliptic functions of order 2 into those with one double pole (of zero residue) and those with two simple poles in the parallelogram of periods. The Weierstrassian theory of doubly periodic functions, the theory which is still frequently included in a first course of complex variables, starts with a function $\wp(z)$ of the first kind which has the double pole at the origin. By using Liouville's general theorems, elliptic functions with singularities, arbitrary within the permissible limits, are constructed.

The Jacobian theory starts out basically with functions of the second kind, and it is Professor Neville's merit to lay particular stress on this purely function-theoretic classification. Historically, the Jacobian functions arose in connection with the inversion of the Legendre integral

$$u = \int_0^{\phi} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta$$

as $\operatorname{sn} u = \sin \phi$, $\operatorname{cn} u = \cos \phi$, $\operatorname{dn} u = d\phi/du$. While this manner of introduction of the Jacobian functions is understandable enough in view of the fact that the study of elliptic functions was prompted by the occurrence in physical and geometrical problems of certain integrals which could not be evaluated by elementary functions, it certainly lends a character of arbitrariness to the selection of the standard functions of the theory. The other procedure followed in the past has been to postpone the theory of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ until the study of theta functions. The function $\operatorname{sn} u$ is then introduced by means of the relation

$$\operatorname{sn} u = \frac{\theta_3}{\theta_2} \cdot \frac{\theta_1(u/\theta_3^2)}{\theta_4(u/\theta_3^2)},$$

which is hardly calculated to inspire a reader with a sense of the importance and necessity of studying Jacobian functions.

In view of this fact, the author's procedure certainly deserves great praise. After a brief résumé of the Weierstrassian theory, he introduces the three primitive functions, as he calls them. Denote the half-periods of the function $\wp(z)$ by ω_f, ω_g, and introduce for purposes of symmetry ω_h, defined by the relation $\omega_f + \omega_g + \omega_h = 0$. Setting $\wp(\omega_i) = e_i$, $i = f, g, h$, the three primitive functions fjz, gjz, hjz are defined by means of the equations

$$fjz = (\wp(z) - e_f)^{1/2}, \quad gjz = (\wp(z) - e_g)^{1/2}, \quad hjz = (\wp(z) - e_h)^{1/2},$$

where the radical in each case is chosen so that, as $z \to 0$,

$$zfjz \to 1, \qquad zgjz \to 1, \qquad zhjz \to 1.$$

Due to the fact that the radicand in each case has all its zeros of order 2, the above definition yields single-valued functions. Except for an unessential, although significant, change of notation, these functions were also considered by many other authors, such as Jordan and Tannery and Molk. The functions fjz, gjz, hjz are of the second kind, with simple poles at the lattice points and simple zeros at the points congruent to ω_f, ω_g, ω_h, respectively. The periods of a function such as fjz are now $2\omega_f$, $4\omega_g$, $4\omega_h$, and ω_f, ω_g, ω_h are now referred to as quarter-periods. To secure a comprehensive notation, ω_j is introduced as an alternative symbol for the origin, and nine new functions are adjoined to the former three, so as to form a set of twelve functions, each of which has simple zeros congruent with one of the four points ω_j, ω_f, ω_g, ω_h, and simple poles congruent with another of these points. The function which has a zero at ω_p and a pole at ω_q is denoted by pqz, where p and q are distinct and may assume any of the values j, f, g, h. These functions are defined by means of expressions such as

$$jfz = fj(z - \omega_f), \qquad hfz = gj(z - \omega_f), \qquad gfz = hj(z - \omega_f),$$

and so forth, together with their reciprocals. The author calls this set of functions the twelve elementary functions, and the notation is a great aid to memory in bringing directly into evidence the functional structure.

After a discussion of the simpler properties of the elementary functions and their addition theorems, the author embarks on a solution of the difficult inversion problem, which appealed to this reviewer as the most original and interesting part of the book. As is well known, it is easy to show that the relation $\zeta = \wp(z)$ is equivalent to the equation

$$z = \int^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} dt, \tag{1}$$

where the invariants g_2 and g_3 are expressible in terms of periods of the function. The converse problem of inverting an integral of the form (1) by means of a function $\wp(z)$, taken for suitable periods, the so-called problem of inversion, is far more difficult. This problem is frequently solved by making use of the theory of theta functions. Exactly the same type of problem arises in the study of the twelve elementary elliptic functions. The author examines in detail the elliptic integral

$$z = I(w) = \int_w^\infty (R(w))^{-1/2}dw,$$

where $R(w) = (w^2 - b^2)(w^2 - c^2)$, and the radical is chosen so as to be asymptotic to w^2 as $w \to \infty$ along the path of integration. The resulting discussion is extremely enlightening and shows clearly the important role that topology plays in the problem, a fact which is completely obscured when the problem is solved by means of theta functions. As the author shows, it is relatively easy to establish that the inverse function $w(z)$ is single-valued, analytic, and has no poles other than simple ones. On the other hand, the possibility of the analytic continuation of $w(z)$ over the whole finite plane, which the author picturesquely describes as the ubiquity of the function, is considerably more difficult. To put it a little differently, it has to be shown that, to every complex value z_0, there exists a point w_0 and a path from w_0 to infinity such that

$$z_0 = \int_{w_0}^\infty (R(w))^{-1/2}dw.$$

The author gives two proofs of this: the first, his own, which makes use of set-theoretic considerations, and the second, due to Goursat, which, while shorter, depends on somewhat elaborate algebraic manipulations. In this discussion of the inversion problem the reviewer can not entirely agree with the author in completely ignoring Riemann surfaces. At least as far as the American student is concerned, the notion of Riemann surfaces is sufficiently well known from a first course in complex variables to be used in the relatively simple situations that the inversion problem requires, and would add to the geometric understanding of the problem. In general, the geometric aspects of the theory of elliptic functions, such as the problems of conformal mapping associated with the Jacobian functions, have been allotted an unduly minor place in the book, being confined for the most part to problems and casual references.

The second half of the book is devoted to the study of the Jacobian functions proper. This merely requires a suitable specialization of the preceding theory. Here the author introduces the notion of a Jacobian lattice. A lattice is called Jacobian if $gj\omega_f = 1$. It follows that there is one, and only one, Jacobian lattice similar to a given lattice. The Jacobian elliptic functions are functions with simple poles constructed on a Jacobian lattice. Since this method of construction determines the functions to within a multiplicative constant, the functions are multiplied by such constants as will make the leading coefficient at the origin in each case equal to unity. The previous notation, pqz, is now modified so as to give the classical notation. This is done by writing the quarter-periods ω_f, ω_g, ω_h as K_c, K_n, K_d, respectively, and using K_s, as ω_f before, to denote the origin. We then obtain the classical functions sn u, cn u, dn u as well as the nine Glaisher functions.

Much of the following discussion of the Jacobian functions is to a large extent a specialization of the earlier developments. The elliptic integrals which represent the inversion of the standard Jacobian and Glaisher functions are exhibited and the addition theorems are established. An elegant discussion of the Jacobi and Landen transformations follows. A chapter is devoted to integrals of functions of the type pq^2u, another chapter to the dependence of the functions on the parameter, containing some little known results of Hermite. There is a brief introduction to the theta functions, and the book concludes with the application of the preceding theory to the computation of real integrals and the determination of some conformal maps.

In the introduction the author states that his aim in writing the book was the restoration of the study of Jacobian functions to the elementary curriculum. The reviewer feels sceptical, for various reasons, of the possibility of such a reform in this country. Aside from this issue, however, the book is an excellent one and constitutes a valuable and important addition to the large and distinguished literature on the subject. The style of the author is vivid and picturesque, and the relatively few misprints cause no real difficulty in reading. The reviewer can heartily recommend it to any one with some background in complex variables who wishes to learn the Jacobian theory.

W. SEIDEL

NOTES

The Keith prize of the Royal Society of Edinburgh has been awarded to Dr. W. L. Edge of the University of Edinburgh for his work in geometry.

Professor Sydney Chapman of the Imperial College of Science and Technology, London, has been elected a foreign associate of the National Academy of Sciences.

Professor A. C. Offord of the University of Durham and Dr. A. G. Walker of the University of Liverpool have been elected ordinary fellows of the Royal Society of Edinburgh.

Dr. W. G. Penney of the Imperial College of Science and Technology, Professor A. R. Richardson of Cape Town, and Professor L. Rosenhead of the University of Liverpool have been elected fellows of the Royal Society.

Dr. J. H. Curtiss of Cornell University has received a Commendation Ribbon from the Secretary of the Navy for his work in statistical engineering for the Bureau of Ships and the Office of the Commander-in-Chief.

Assistant Professor Jesse Douglas of Brooklyn College has been elected a member of the National Academy of Sciences.

Associate Professor V. C. Poor of the University of Michigan has retired with the title professor emeritus.

The following have been granted post-service fellowships by the John Simon Guggenheim Memorial Foundation: T. W. Anderson, C. L. Pekeris, G. B. Price, H. E. Robbins, Henry Scheffé, I. E. Segal, A. H. Taub, and J. W. T. Youngs.

The following have received John Simon Guggenheim Memorial Fellowships: R. H. Bruck, Mark Kac, and P. C. Rosenbloom.

Associate Professor C. B. Allendoerfer of Haverford College has been promoted to a professorship.

Professor L. D. Ames of the University of Southern California has retired.

Dr. B. H. Arnold of Purdue University has been appointed to an assistant professorship at Montana State College.

Professor H. F. Bohnenblust of Indiana University has been appointed to a professorship at California Institute of Technology.

Mr. F. E. Carner has accepted a position as junior mathematician with Franklin Institute.

Associate Professor K. S. Cole of the College of Physicians and Surgeons of Columbia University has been appointed to a professorship of biophysics at the University of Chicago.

Dr. J. H. Curtiss, whose leave of absence from Cornell University will be continued for the academic year 1946–1947, has been appointed Assistant to the Director of the National Bureau of Standards.

Dr. C. H. Dowker has been appointed to an assistant professorship at Tufts College, Medford, Massachusetts.

Dr. D. C. Duncan of Los Angeles City College has accepted a position at East Los Angeles Junior College.

Dr. J. H. Giese has been appointed mathematician at the Ballistic Research Laboratory at Aberdeen Proving Ground.

Mr. E. L. Godfrey has been appointed to an assistant professorship at Defiance College, Defiance, Ohio.

Dr. Saul Gorn has accepted a position as principal research mathematician at the Aircraft Radiation Laboratory, Wright Field, Dayton, Ohio.

Dr. M. H. Heins of Harvard University has been appointed to an associate professorship at Brown University.

Dr. J. F. Heyda of the University of Nebraska has been appointed to an assistant professorship at Franklin and Marshall College, Lancaster, Pa.

Assistant Professor Withold Hurewicz of the University of North Carolina has been appointed to an associate professorship at the Massachusetts Institute of Technology.

Dr. G. K. Kalisch of Cornell University has been appointed to an assistant professorship at the University of Minnesota.

Dr. Joseph Lehner has accepted a position as mathematician with Hydrocarbon Research, Inc., New York, New York.

Dr. R. J. Levit has been appointed to an assistant professorship at the University of Georgia.

Professor W. T. Martin of Syracuse University has been appointed to a professorship at Massachusetts Institute of Technology.

Associate Professor G. M. Merriman of the University of Cincinnati has been promoted to a professorship.

Dr. D. D. Miller has been appointed to an associate professorship at the University of Tennessee.

Professor Deane Montgomery of Smith College has been appointed to an associate professorship at Yale University.

Dr. J. E. Morton of the National Bureau of Economic Research has been appointed to an associate professorship at Cornell University.

Assistant Professor Ivan Niven of Purdue University has been promoted to an associate professorship.

Mr. P. B. Norman of Columbia University has been appointed to an assistant professorship in the School of Engineering, Pratt Institute, Brooklyn, New York.

Dr. A. C. Olshen has been appointed comptroller and actuary by the West Coast Life Insurance Company, San Francisco, California.

Dr. Maxwell Reade of Purdue University has been promoted to an assistant professorship.

Dr. L. A. Ringenberg of Ohio State University has been appointed to an assistant professorship at the University of Maryland.

Dr. Max Shiffman of Princeton University has been appointed to an assistant professorship at New York University.

Assistant Professor A. H. Smith of Purdue University has been promoted to an associate professorship.

Professor M. H. Stone of Harvard University has been appointed to a distinguished service professorship at the University of Chicago.

Mr. C. S. Sutton of the University of Delaware has been appointed to an assistant professorship at The Citadel, Charleston, South Carolina.

Professor J. L. Synge of Ohio State University has been appointed to a professorship at Carnegie Institute of Technology.

Dr. E. P. Vance of Oberlin College has been promoted to an assistant professorship.

Dr. R. W. Wagner of Oberlin College has been promoted to an assistant professorship.

Professor J. L. Walsh of Harvard University has been appointed to the Perkins professorship.

Dr. L. F. Walton has been appointed lecturer at Santa Barbara College of the University of California.

Dr. J. A. Ward has been appointed to an associate professorship at the University of Georgia.

Associate Professor D. L. Webb of Texas Technological College has been appointed to an assistant professorship at the University of Arizona.

Associate Professor Hassler Whitney of Harvard University has been promoted to a professorship.

Dr. R. H. Wilson has been appointed to an assistant professorship at Temple University.

Associate Professor Jacob Wolfowitz of North Carolina State College has been appointed to an associate professorship of mathematical statistics at Columbia University.

Dr. G. S. Young of Purdue University has been promoted to an assistant professorship.

The following appointments to instructorships are announced:

University of Buffalo: Mr. H. W. Baeumler, Miss Ruth A. Brendel, Miss Irene L. Fisher, Mrs. Magdalene M. Hettler, Mr. C. E. Maley, Miss Audrey L. Strabel, Mrs. Rosemary R. Trautman; College of the City of New York: Dr. Lee Lorch; Columbia University: Mr. G. N. Raney; University of Delaware: Mr. Russell Remage; Illinois Institute of Technology: Dr. A. B. Carson, Dr. Anatol Rapoport, Miss Beulah I. Shoesmith; Iowa State College: Mr. Ralph M. Robinson; Michigan College of Mining and Technology: Mr. L. C. Dawson; University of Michigan: Dr. Samuel Kaplan; Montana State University: Dr. Andrewa R. Noble; North Carolina State College: Mr. S. R. Baker; Northwestern University: Mr. R. S. Wolfe; School of Engineering, Pratt Institute, Brooklyn, New York: Mr. G. C. Helme; Purdue University: Dr. John Dyer-Bennet, Mr. H. L. Harter, Mr. G. L. Kinnett, Dr. Sam Perlis, Mr. G. C. Sipple, Mr. J. P. Wayne; University of Rochester: Mr. Norman Gunderson; Rutgers University: Mr. Simon Lopata; University of Southern California: Dr. P. J. Kelly; Syracuse University: Miss Frances M. Wright, Mr. H. (H.) Zatzkis; Temple University: Dr. Lowell Schoenfeld; University of Tennessee: Mrs. May C. Blackstock; Tulane University: Mr. E. P. Miles; Yale University: Mr. P. T. Bateman; University of Washington: Dr. Fumio Yagi.

Professor Leonida Tonelli of the University of Pisa died March 12, 1946. He had been a member of the Society since 1932.

Professor Emeritus A. R. Crathorne of the University of Illinois died March 7, 1946, at the age of seventy-two years. He had been a member of the Society since 1900.

Dr. Josephine R. Roe died April 29, 1946, at the age of eighty-eight years. She had been a member of the Society since 1910.

The information about Professor Emeritus E. E. Whitford which appeared in the May 1946 issue of the BULLETIN was in error. Professor Whitford died May 3, 1946, at the age of eighty-one years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

222. H. S. M. Coxeter: *A simple proof of the eight square theorem.*

It is shown, by direct comparison of terms, that the conjugates of Cayley numbers have the properties $\overline{ab}=\bar{b}\bar{a}$ and $\bar{a}a\cdot b=\bar{a}\cdot ab$. Defining $a^{-1}=(Na)^{-1}\bar{a}$, it is deduced that $b=a^{-1}\cdot ab$ and similarly $ab\cdot b^{-1}=a$. As Ruth Moufang remarked, these relations imply $b^{-1}a^{-1}=(ab)^{-1}$. One now has $N(ab)=\overline{ab}\cdot ab=b\bar{a}\cdot ab=Nb\cdot Na\cdot b^{-1}a^{-1}\cdot ab=Na\ Nb$. (Cf. Dickson, Ann. of Math. (2) vol. 20 (1919) p. 164.) (Received May 29, 1946.)

223. H. S. M. Coxeter: *Integral Cayley numbers.*

The Cayley numbers $a_0+a_1i+a_2j+a_3k+(c_4+a_5i+a_6j+a_7k)h$, where $h=2^{-1}(i+j+k+e)$ and the a's are arbitrary integers, are shown to be a set of integral elements in the sense of Dickson (*Algebras and their arithmetics*, Chicago, 1923, pp. 141–142). The integral Cayley numbers of norm 1, 2 and 4 are represented by the vertices of the eight-dimensional polytopes 4_{21}, 2_{41} and 1_{42}. (Received May 29, 1946.)

224. Roy Dubisch: *On the direct product of rational generalized quaternion algebras.*

Utilizing the results of Albert (Bull. Amer. Math. Soc. vol. 40 (1934) pp. 164–176) on the integral domains of rational generalized quaternion algebras and the results of Latimer (Duke Math. J. vol. 1 (1935) pp. 433–435) on the fundamental number of a rational generalized quaternion algebra, the author proves that any finite number of such algebras $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_n$ contain a common quadratic subfield $\mathfrak{Z}$ and $\mathfrak{Q}_i=(\mathfrak{Z}, S, -d_i)$ $(i=1, \cdots, n)$ where d_i is the fundamental number of $\mathfrak{Q}_i$. Then $\mathfrak{Q}_1\times\cdots\times\mathfrak{Q}_n\sim\mathfrak{B}=(\mathfrak{Z}, S, \pm\Pi d_i)$. (Received May 27, 1946.)

225. Orrin Frink: *Complemented modular lattices and projective spaces of infinite dimension.*

Garrett Birkhoff (Ann. of Math. vol. 36 (1935) pp. 743–748) showed that every complemented modular lattice of finite dimension is the direct union of the lattices of subspaces of projective geometries. In this paper complemented modular lattices in general, without restriction on the dimension, are characterized as subdirect unions of the subspace lattices of projective planes and irreducible projective coordinate spaces of possibly infinite dimension. It is shown that every complemented modular lattice determines a unique projective space whose points are the maximal dual ideals of the lattice. Conversely, the lattice of subspaces of every projective space is complete, atomic, complemented, and modular. Every irreducible projective space not a projective plane is a coordinate space, and is determined by a division ring and by a cardinal number which may be infinite, namely its dimension. In particular, points, lines, planes and hyperplanes may be added to continuous geometries by the use of lattice ideals. This may permit the introduction of coordinates into continuous geometries by the method of von Staudt. (Received May 11, 1946.)

226. Irving Kaplansky: *Topological rings.*

The paper is devoted to the study of (i) the radical (in the sense of Jacobson, Amer. J. Math. vol. 67 (1945)) of a topological ring, (ii) the structure of compact rings, (iii) rings of functions, (iv) rings of endomorphisms. The radical is closed if the quasi-regular elements are either open or closed, but an example is given of a ring whose radical is not closed. The radical of a compact ring is the union of all (topological) nil ideals. A compact semisimple ring is continuously isomorphic to a complete direct sum of finite simple rings. Compact integral domains are local rings—possibly without chain condition. The ring of functions from a topological space S to a topological ring A is studied; results known when A is the reals or complexes are proved for any simple ring A provided S is totally disconnected. Ideal theory in the ring is also examined. The regular representations of a topological ring and the structure of the ring of continuous endomorphisms of certain topological groups comprise the final section. (Received May 31, 1946.)

227. J. C. C. McKinsey: *On the representation problem for projective algebras.*

C. J. Everett and S. Ulam (*Projective algebra* I, Amer. J. Math. vol. 68 (1946) pp. 77–88) have recently given postulates for projective algebras, and have solved the representation problem for all complete atomic projective algebras. The author shows that every projective algebra is isomorphic to a subalgebra of a complete atomic projective algebra; this result, combined with that of Everett and Ulam, provides a solution of the representation problem for all projective algebras. The method of proof is similar to the method used by M. H. Stone to establish a representation theorem for Boolean algebras. If A is an additive prime ideal, then A_1 is defined to be the additive prime ideal which contains all elements of the form a_1, for a in A. If A and B are additive prime ideals, then $A\square B$ is defined to be the class of all additive prime ideals C which contain all elements of the form $a\square b$, where a is in A and b is in B. These definitions are extended in an obvious way so as to permit operations on classes of prime ideals. (Received April 29, 1946.)

ANALYSIS

228. Richard Bellman: *Stability of difference equations.*

Using methods developed in the study of stability of differential equations (cf. the author's paper, *The stability of solutions of linear differential equations*, Duke Math. J. vol. 10 (1943)), the matrix difference equation $X_{n+1}=(A+B)X_n$, where B is a perturbation matrix, is treated, and various hypotheses are made as to the solutions of $X_{n+1}=AX_n$ and the nature of B. Analogues of results of Liapounoff for differential equations are also obtained. (Received April 10, 1946).

229. Stefan Bergman: *On functions which satisfy certain systems of partial differential equations.*

Using methods of previous papers, the author studies flows in the pseudo-logarithmic (θ, λ)-plane. He introduces flows $\mathfrak{F}^{(L,k)}(\phi^{(L,k)}, \psi^{(L,k)})$, $k=1, 2$, in which ϕ is the potential function and $\phi^{(L,2)}$ a fundamental solution of $\Delta\phi - N(\partial\phi/\partial\lambda)=0$, ψ is the stream function and $\psi^{(L,1)}$ a fundamental solution with singularity at (θ_0, λ_0) of $\Delta\psi + N(\partial\psi/\partial\lambda)=0$. The flows $\mathfrak{F}^{(n,k)}(\phi^{(n,k)}, \psi^{(n,k)})$, $n=1, 2, 3, \cdots$; $k=1, 2$, $\psi^{(n,1)} = (\partial^n\psi^{(L,1)}/\partial\theta^n)$, $\phi^{(n,2)} = (\partial^n\phi^{(L,2)}/\partial\theta^n)$, are infinite of the nth order at (θ_0, λ_0), have *both* components single-valued in the schlicht (θ, λ)-plane. By $x = x(\theta, \lambda)$, $y = y(\theta, \lambda)$ transition to the physical plane is defined. The author computes for the flows $\mathfrak{F}^{(L,k)}$, $\mathfrak{F}^{(n,k)}$ pairs of numbers, $(X^{(L,k)}, Y^{(L,k)})$, $(X^{(n,k)}, Y^{(n,k)})$ respectively, called T-periods which depend only upon θ_0 and λ_0. Let $\mathfrak{F}$ be a flow whose stream function is $\alpha\psi^{(L,1)} + \beta\psi^{(1,1)} + \gamma\psi^{(1,2)} + \psi_1$, where ψ_1 is a regular function. The necessary and sufficient conditions that $\mathfrak{F}$ represent in the physical plane a flow around a closed curve are $\alpha X^{(L,1)} + \beta X^{(1,1)} + \gamma X^{(1,2)} = 0$ and $\alpha Y^{(L,1)} + \beta Y^{(1,1)} + \gamma Y^{(1,2)} = 0$. (Received May 31, 1946.)

230. R. P. Boas: *The rate of growth of an analytic function determined from its growth on a sequence of points.*

Theorems of V. Bernstein and of Levinson (*Gap and density theorems*, New York, 1940, chap. 7) are given simplified proofs and still further sharpened. The proofs depend on the following lemma of Phragmén-Lindelöf type: let $\phi(z)$ be analytic in $x \geqq 0$, $|\phi(z)| \leqq e^{e^{|z|}}$, and $|\phi(re^{i\theta})| \leqq A \exp\{r\cos\theta - r\delta(r)\} + B$ for $\pi/4 \leqq |\theta| \leqq \pi/2$, where $\delta(r) \geqq 0$, $\delta(r) \to 0$, and $\int_1^\infty r^{-1}\delta(r)dr = \infty$. Then $\phi(z)$ is bounded in $x \geqq 0$. (Received April 2, 1946.)

231. W. F. Eberlein: *Weak sequential compactness and regularity of Banach spaces.*

Although it is well known that the unit sphere of a reflexive Banach space is weakly sequentially compact, the converse theorem has been obtained only under such additional restrictions as separability (Banach), strict convexity (Milman), or compactness in the sense of directed systems (Goldstine and others). It is shown that these restrictions can be dropped: A Banach space is reflexive if and only if its unit sphere is weakly sequentially compact. The proof involves a theorem of Mazur and properties of the weak neighborhood topologies. (Received May 31, 1946.)

232. Evelyn Frank: *The real parts of the zeros of a complex polynomial.*

Let $P(z)$ be a polynomial of degree n with complex coefficients and $P^*(z) = \bar{P}(-z)$, where $\bar{P}(z)$ is obtained by replacing the coefficients of $P(z)$ by their complex conjugates. If ξ is a complex constant such that $|P^*(\xi)| > |P(\xi)|$, then the polynomial of order $n-1$, $P_1(z) = [P^*(\xi)P(z) - P(\xi)P^*(z)]/(z-\xi)$, has one zero less than $P(z)$ with real part of the same sign as $\text{Re}(\xi)$ and the same number of zeros as $P(z)$ with real parts of opposite sign to $\text{Re}(\xi)$. By repetition of this reduction theorem, after $n-1$ steps the number of zeros of $P(z)$ with positive real parts and the number with negative real parts are found. By the use of this method and successive approximations, the real and imaginary parts of the zeros of $P(z)$ may actually be computed. (Received May 31, 1946.)

233. M. R. Hestenes: *An alternate sufficiency proof for the normal problem of Bolza.*

The purpose of the present paper is to show that a sufficiency theorem for the problem of Bolza can be obtained from the sufficiency theorem for the simpler case in which there are no differential side conditions. More precisely it is shown that if an arc C_0 is a normal arc satisfying the usual sufficient conditions for the problem of Bolza, one can add to the integrand of the function $I(C)$ to be minimized a linear combination of the functions defining the differential side conditions so as to obtain a function $J(C)$ such that C_0 satisfies the usual sufficient conditions for the problem of minimized $J(C)$ without restricting the arcs to satisfy the differential side conditions. The arc C_0 will afford a relative minimum to $J(C)$ in the class of all neighboring arcs satisfying the end conditions and therefore will minimize $I(C)$ in case the arcs are restricted to satisfy the differential equations. (Received April 17, 1946.)

234. William Karush: *A semi-strong minimum for a double integral problem in the calculus of variations.*

The author considers the problem of minimizing an integral $I(S)=\iint_A f(x, y, z, z_x, z_y)dxdy$ in a class of surfaces S: $z=z(x, y)$ defined by functions which are Lipschitzian on A and its boundary, and which have common boundary values. Define $D^2(z)=\iint_A(z^2+z_x^2+z_y^2)dxdy$. The following theorem is proved. Let S_0: $z=z_0(x, y)$ be a surface of class C'' satisfying the conditions: (i) S_0 is an extremal surface, (ii) f E-dominates $L=(1+z_x^2+z_y^2)^{1/2}$ near S_0 (cf. Hestenes, Trans. Amer. Math. Soc. vol. 60), (iii) there exists $\theta>0$ such that $I_2(\zeta)\geqq\theta D^2(\zeta)$, where I_2 is the second variation of I along S_0, and ζ is an arbitrary variation vanishing on C. Then for every finite $M>0$ there exists $\epsilon>0$ and a neighborhood U of S_0 in (x, y, z)-space such that $I(S)-I(S_0)>\min[\epsilon, \epsilon D^2(z-z_0)]$ for every S in U coinciding with S_0 on C and having $|z_x|+|z_y|<M$ almost everywhere on A. It is shown that E-dominance is equivalent to nonsingularity and the Weierstrass condition II_N. For special integrands a strong relative minimum is obtained (cf. F. G. Myers, Duke Math. J. (1943)). The proof is based on a convergence theorem analogous to one of Morrey (Duke Math. J. (1940)). (Received May 29, 1946.)

235. G. W. Mackey: *On convex topological linear spaces.*

In this paper the theory of linear systems developed in an earlier article (Trans. Amer. Math. Soc. vol. 57 (1945) pp. 155–206) is applied to the study of convex topological linear spaces. A linear system is associated with every convex topological linear space and the properties of the latter correlated with those of the linear system and with the strength of its topology relative to the strengths of the other convex topologies associated with the same linear system. A more detailed description of the contents of this paper will be found in Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) pp. 315–319. (Received April 4, 1946.)

236. G. R. MacLane: *Concerning the uniformization of certain Riemann surfaces allied to the inverse-cosine and inverse-gamma surfaces.*

Any Riemann surface F_ω obtained from the inverse-cosine surface by displacing the branch-points along the real axis is parabolic. The corresponding entire function $w=f(z)$ is such that $f'(z)=e^{-az^2}+bz\prod_{-\infty}^{\infty}\{(1-z/c_n)e^{z/c_n}\}$, with $a\geqq 0$, b real, c_n real, $c_n\rightarrow\pm\infty$ as $n\rightarrow\pm\infty$. Conversely the image of the z-plane by any entire function of this form is a distorted inverse-cosine surface. Analogous results are obtained for the surfaces "like" those defined by $w=\cos z^{1/2}$ and $w=1/\Gamma(z)$. These results follow from a study of the polynomial surfaces analogous to the surface in question. (Received May 23, 1946.)

237. Harry Pollard: *The mean convergence of orthogonal series of polynomials.*

The literature concerning series of orthogonal polynomials is devoted chiefly to the questions of ordinary convergence or summability. In the present paper the problem of mean convergence is discussed for the classical polynomials. For example it is established that if $f(x)\in L^p$, its Legendre series converges to it in pth mean if $4/3<p<4$, but not necessarily otherwise. A more complete abstract appears in Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 8–10. (Received April 8, 1946.)

238. M. H. Protter: *Generalized spherical harmonics.*

A class of solutions, $\{\overline{R}_{n,\nu}(x, y, z)\}$, of the equation $Lu=\{\sigma_1(x)\tau_1(y)\pi_1(z)u_x\}_x+\{\sigma_2(x)\tau_2(y)\pi_2(z)u_y\}_y+\{\sigma_2(x)\tau_1(y)\pi_2(z)u_z\}_z=0$ is considered which possesses the following properties: (1) For each n and ν, $\overline{R}_{n,\nu}(x, y, z)$ is a solution of $Lu=0$ regular in the entire plane. (2) The functions $\overline{R}_{n,\nu}(x, y, z)$ can be computed by quadratures. (3) Every solution of $Lu=0$ can be expanded uniquely in an absolutely convergent series in terms of the functions $\overline{R}_{n,\nu}(x, y, z)$ where the coefficients of the expansion are given by simple differential formulas. In the special case that $Lu=0$ reduces to the Laplace equation the functions $R_{n,\nu}(x, y, z)$ become homogeneous, harmonic polynomials of degree n. Relations are found between these polynomials, Bessel functions, and associated Legendre functions. (Received May 10, 1946.)

239. I. E. Segal: *Irreducible representations of operator algebras.*

It is shown that a uniformly closed self-adjoint algebra of bounded operators on a Hilbert space has a complete set of irreducible, adjoint-preserving, continuous representations by bounded operators on (not necessarily separable) Hilbert spaces, irreducibility signifying that there exists no nontrivial closed invariant subspace. In the case of a certain operator algebra on the space of complex-valued functions square-integrable relative to Haar measure over a locally compact group, such a representation induces in a natural way an irreducible, strongly continuous, unitary representation of the group. A recent theorem of Gelfand and Rykov (Rec. Math. (Mat. Sbornik) N.S. vol. 13 (1943) pp. 301–316) is a corollary. The proofs exploit devices previously employed by Gelfand and Neumark (ibid. vol. 12 (1943) pp. 197–213), Gelfand and Rykov, and the author. (Received April 25, 1946.)

240. Otto Szász: *On the Möbius inversion formula and closed sets of functions.*

The author gives a generalization of the Möbius inversion formula and conditions for its validity. Application is made to establish completeness of certain sequences of functions of the type $f(nt)$, $n=1, 2, 3, \cdots$, where $t\geqq 0$, and $f(t)\in L_r$, or $f(t)$ is continuous in some interval. Some particular results are: (1) $f(t)=(\sin t)^\alpha$ for $0\leqq t\leqq\pi$, $f(-t)=-f(t)=f(2\pi-t)$. In this case the span of the sequence $\{f(nt)\}$ is identical with the span of the sequence $\{\sin nt\}$ in $C(0, \pi)$ for $\alpha>1/3$; for $\alpha=1/3$ the sequence $\{f(nt)\}$ is complete in $L_\infty(0, \pi)$. (2) If $f(t)=\sin t+\sum_2^\infty a_n \sin nt$ and if $\sum_2^\infty|a_n|\leqq 1$, then the sequence $f(nt)$ is complete in $L_\infty(0, \pi)$. (3) Let $f(t) = t - [t] - 1/2$, $f(n) = 0$; the sequences $\{f(nt)\}$ and $\{\sin 2n\pi t\}$ have the same span in $C(0, 1/2)$. (4) $f(t) = \operatorname{sgn} \sin n\pi t$; the sequence $\{f(nt)\}$ is complete in $L_r(0, 1)$ for all $r > 1$. (5) Each of the sequences $1, e^{-t}, t/(e^{nt}-1)$; $1, e^{-t}, t^2/(e^{nt}-1)^2$, $n = 1, 2, 3, \cdots$, is complete in $C(0, \infty)$. (Received May 31, 1946.)

241. Hing Tong: *Ideals of normed rings associated with topological spaces*. Preliminary report.

Let R be a perfectly normal bicompact space, $\mathfrak{R}$ the normed abelian ring of complex-valued functions continuous over R (for $f \in \mathfrak{R}$, $\|f\| = \max_x |f(x)|$), $\mathfrak{J}$ any (closed) ideal in $\mathfrak{R}$. Then $\mathfrak{J}$ is a principal ideal. If $\prod_{f \in \mathfrak{J}} \mathfrak{N}_f = N_{\mathfrak{J}}$ ($\mathfrak{N}_f$ denotes the set of zeros of f) corresponds to $\mathfrak{J}$, the correspondence is an isomorphism (that is, $\mathfrak{J} \leftrightarrow N_{\mathfrak{J}}$) such that ($\alpha_1$) $\mathfrak{J}_1 \cup \mathfrak{J}_2 \leftrightarrow N_{\mathfrak{J}_1} \cdot N_{\mathfrak{J}_2}$, ($\alpha_2$) $\mathfrak{J}_1 \cap \mathfrak{J}_2 \leftrightarrow N_{\mathfrak{J}_1} + N_{\mathfrak{J}_2}$, ($\alpha_3$) $\mathfrak{R} \leftrightarrow \{0\}$, ($\alpha_4$) $\sum_\alpha \mathfrak{J}_\alpha \leftrightarrow \prod_\alpha N_{\mathfrak{J}_\alpha}$, ($\alpha_5$) $\prod_\alpha \mathfrak{J}_\alpha \leftrightarrow$ closure of $\sum_\alpha N_{\mathfrak{J}_\alpha}$. Conversely, let $\mathfrak{R}$ be the normed ring of bounded complex-valued functions continuous over a topological space R. The conditions $\mathfrak{J} \leftrightarrow N_{\mathfrak{J}}$, ($\alpha_1$) holds, and every $\mathfrak{J} \subseteq \mathfrak{R}$ is principal imply that R is a perfectly normal bicompact space. If R is a T_1' (a space $\ni y \in x \rightarrow x = y$) bicompact normal space, the above results hold providing "principal ideal" is replaced by "ideal." The results also hold for bicompact normal spaces if isomorphism is relaxed to homomorphism. The ring of continuous mappings (continuous in the strong topology) of R into the ring of bounded linear operators over a Banach space with a basis leads to the same results if an ideal means a two-sided ideal. The results do not hold for non-separable Banach spaces. (Received April 10, 1946.)

APPLIED MATHEMATICS

242. Garrett Birkhoff: *Symmetric Lagrangian systems*.

Let Ω be any Lagrangian dynamical system with no potential energy and kinetic energy function $L = 2^{-1}E_{jk}\dot{q}_j\dot{q}_k$. Suppose that L is invariant under a simply transitive group G of rigid motions on its configuration space. Then the "generalized force" required to maintain motion along a one-parameter subgroup in the jth coordinate direction has the components $Q_i = c^{ij}_k E_{jk}$, where the c^{ij}_k are the structure constants of G. It is a corollary that the d'Alembert paradox would take in non-Euclidean geometry the following form. A rigid body moving under translation, rotation, or screw motion along an axis in an incompressible, nonviscous fluid without circulation will experience no thrust along or torque about the axis. However cross-force is possible. (Received May 13, 1946.)

243. R. J. Duffin: *Nonlinear networks*. III.

A system of n nonlinear differential equations is shown to have a periodic solution. The interest of these equations is that they describe the vibrations of electrical networks under a periodic impressed force. Consider an arbitrary linear network of inductors, resistors and capacitors which does have a solution for a given periodic impressed force. The main result of this note states that the existence of a periodic solution is still guaranteed if the linear resistors of such a network are replaced by *quasi-linear* resistors. A quasi-linear resistor is one whose differential resistance lies between positive limits. No other sort of nonlinearity besides this type of nonlinear damping is considered. The proof rests on the closure properties of nonlinear transformations in Hilbert space. (Received May 7, 1946.)

244. G. F. McEwen: *Diffusion from an instantaneous circular area source.* Preliminary report.

Corresponding to an initial concentration S_1 within a circular area of radius r_1 and constant coefficient of diffusion a^2, the partial differential equation $\partial S/\partial t = a^2(\partial^2 S/\partial r^2 + r^{-1}\,\partial S/\partial r)$ has the formal solution $S = S_1 r_1 \int_0^\infty J_1(\lambda r_1) J_0(\lambda r) \exp(-a^2 t\lambda^2) d\lambda$, which reduces to $S = S_1[1 - \exp(-r_1^2/4a^2t)]$ for $r = 0$. Differentiation of the formal solution with respect to r yields an expression whose integral with respect to λ, from 0 to ∞, is known. The general solution in a form adapted to computation is found by integrating this expression from 0 to r, $\int_0^r I_1(rr_1/2a^2t)\exp(-r^2/4a^2t)dr$, and adding the solution for $r = 0$. Repeated integration by parts in two ways reduces this integral to corresponding converging series and leads to the solutions: $S = S_1\{1 - \exp[-(r-r_1)^2/4a^2t]\sum_{n=0}^{\infty}(r/r_1)^n f_n(rr_1/2a^2t)\}$ for $r \leq r_1$, and $S = S_1 \exp[-(r-r_1)^2/4a^2t] \cdot \sum_{n=1}^{\infty}(r_1/r)^n f_n(rr_1/2a^2t)$ for $r \geq r_1$, where $f_n(s) = e^{-s} I_n(s) \leq 1$. For a diffusion coefficient $a^2 r$ the equation $\partial S/\partial t = a^2(r\partial^2 S/\partial r^2 + 2\partial S/\partial r)$ has, for the same boundary conditions, the formal solution, $S = 2S_1 r_1/r^{1/2}\int_0^\infty J_1(2(r_1)^{1/2}\lambda) J_1(2r^{1/2}\lambda)\exp(-a^2 t\lambda^2)d\lambda$, which yields two solutions: $S = S_1\{1 - (r_1/r)^{1/2}\exp[-(r^{1/2} - (r_1)^{1/2})^2/a^2t] f_1(2(rr_1)^{1/2}/a^2t) - \exp[-(r^{1/2} - (r_1)^{1/2})^2/a^2t]\sum_{n=0}^{\infty}((r/r_1)^{1/2})^n f_n(2(rr_1)^{1/2}/a^2t)\}$, and $S = S_1 \exp[-(r^{1/2} - (r_1)^{1/2})^2/a^2t]\sum_{n=1}^{\infty}((r_1/r)^{1/2})^n f_n(2(rr_1)^{1/2}/a^2t)$. (Received May 10, 1940.)

245. H. E. Salzer: *Alternative formulas for direct interpolation of a complex function tabulated along equidistant circular arcs.*

An improved method of complex Lagrangian interpolation for an analytic function that is tabulated along the arc of a circle at equidistant intervals is obtained by generalization of a scheme recently described by W. J. Taylor. In place of Taylor's binomial coefficients, for n-point direct interpolation, auxiliary quantities $A_k^{(n)}$, where $n = 3, 4, \cdots, 11$ and $k = -[(n-1)/2]$ to $[n/2]$, are given as functions of the angular interval θ, so that for extensive use for some fixed θ, they can be found as fixed complex numbers. In this method of complex interpolation the number of operations increases linearly with n, whereas in the author's previous method (which employed Lagrangian interpolation polynomials explicitly and was cumbersome even for the 5-point case) the number of operations increased quadratically with n. Advantage is taken of the fact that, for each n, all the quantities $A_k^{(n)}$ can be multiplied by any nonzero factor in order to simplify their expression. It is proved that for odd n, $A_{-k}^{(n)}$ equals conjugate of $A_k^{(n)}$, and for even n, $A_{k+1}^{(n)} = -\exp(i\theta\{k(2-n) - n/2 + 1\})A^*_{-k}$. (Received May 14, 1946.)

246. S. A. Schaaf: *On the superposition of a heat source and a contact resistance.*

The Laplace transform is used to determine the temperature distributions $T_i(x, t)$ $(i = 1, 2)$ in two semi-infinite heat conductors, initially at zero temperature and in contact along the interface plane $x = 0$ where there is superposed a heat source of strength $S(t)$ and a contact resistance. It is shown that such a superposition leads in many cases (including those in which the heat source is caused by friction as the two solids slide against each other along the interface) to a boundary condition at $x = 0$ of the type $T_1 - T_2 = R(C_1 S + K_1 \partial T_1/\partial x) = -R(C_2 S + K_2 \partial T_2/\partial x)$, where R is the resistance constant, K_i the heat conductivity and C_i a constant whose interpretation depends on the physical model of the interface. (Received May 16, 1946.)

247. A. C. Sugar: *An elementary exposition of the relaxation method.*

This is, as labeled, an elementary exposition. The purpose of this paper is to exhibit the simplicity and power of the relaxation method, to complete and explain previous sketchy or obscure discussions of this subject. It is also intended to direct attention to some mathematical, physical, and philosophical questions that may be raised in connection with this method. (Received May 29, 1946.)

248. A. C. Sugar: *On the relaxation-matrix method of solving boundary value problems.*

This is a continuation of the study of the use of relaxation and iterative methods of inverting the matrices of the systems of equations obtained from boundary value problems by finite difference methods. In a previous paper, entitled *The use of invariant inverted matrices for the approximate solution of classes of boundary value problems,* the writer inverted matrices by relaxation methods and applied them to the solution of simple illustrative problems. In the present paper, this work is continued and applications are made to some of the typical boundary value problems of mechanics. (Received May 29, 1946.)

249. A. C. Sugar: *The use of invariant inverted matrices for the simultaneous approximate solution of classes of boundary value problems.* Preliminary report.

The writer considers the simultaneous approximate solution of classes of boundary value problems. Each class consists of the totality of boundary value problems having the same differential equation and the same boundary but different boundary conditions. Using finite difference methods it is shown that the derived system of equations will have an inverse matrix M, invariant over the class, which may be determined by relaxation methods. A solution of any member of the class may then be obtained by multiplying M by a column matrix defined by the corresponding boundary values. This paper will be primarily concerned with the application of this method to Laplace's equation. The effect of modifications of the boundary on M will be considered. This technique may be applied to many other types of differential equations. This is true, in particular, of Poisson's equation and of nonlinear equations containing the Laplace operator, since, as far as the algebraic treatment is concerned, these two types may be treated as Laplace equations with altered boundary conditions. Finally, the possibility of considering M as an approximation or an analogue of Green's function is studied. (Received April 16, 1946.)

GEOMETRY

250. L. M. Blumenthal: *Superposability in elliptic spaces.*

Two subsets of a metric space M are *superposable* provided a congruent (that is, one-to-one, distance-preserving) mapping of M onto itself exists which maps one subset onto the other. In spaces most studied (euclidean, spherical, and so on) congruence of subsets implies that they are superposable, but this is not the case in elliptic spaces $E_{n,r}$ for $n>1$, and hence this property cannot be expressed in metric terms alone. This paper shows that congruent but not superposable subsets of $E_{n,r}$ fall into two classes (a) congruent subsets contained irreducibly in different dimensional subspaces and (b) those contained irreducibly in subspaces of the same dimension. By means of attaching to each elliptic m-tuple a class of equivalent matrices, necessary and sufficient conditions for superposability of elliptic subsets are obtained in terms of equivalence of the corresponding classes of attached matrices. These conditions are applied to yield new properties of elliptic spaces and quadratic forms. (Received April 16, 1946.)

251. Herbert Busemann: *Intrinsic area.*

Recent examples of Besicovitch have turned the attention to the geometric concept of area. Although the requirement that area should be intrinsic (that is, depend only on the geodesic distances on the surface and not on the way the surface is imbedded) is fundamental for all applications in geometry, it has never entered the modern investigations on area. An intrinsic area can be defined for a general class of surfaces in metric spaces in such a way that it has the standard value for euclidean polyhedra and smooth surfaces in Riemann spaces. Moreover the definition attributes an area to surfaces in Finsler spaces. This value is for smooth surfaces the only area which coincides with the customary area for elementary surfaces, and for which the surface with the greater intrinsic distances has always the greater area. The paper will appear in the Annals of Mathematics. (Received May 27, 1946.)

252. N. A. Court: *On the biratio of the altitudes of a tetrahedron.*

The three pairs of opposite edges a, a'; b, b'; c, c' of a tetrahedron (T) are the axes of three orthogonal hyperboloids (aa'), (bb'), (cc') belonging to the same pencil of quadrics which also includes the hyperboloid (H) formed by the altitudes of (T) (Bull. Amer. Math. Soc. vol. 51 (1945) p. 663; Duke Math. J. vol. 13 (1946) pp. 123–128). The biratio (that is, the anharmonic ratio) of the four hyperboloids considered is equal to the biratio ψ of the four altitudes of (T). If the three vertices of (T) are fixed, for a given value of ψ, the fourth vertex lies in a plane perpendicular to the plane formed by the fixed vertices. The biratio of the tetrad of points in which an altitude of (T) meets the faces of the tetrahedron formed by the orthocenters of the faces of (T) is equal to ψ. (Received May 29, 1946.)

253. Tibor Radó: *The isoperimetric inequality and the Lebesgue definition of surface area.* I.

Let U^* denote the solid unit sphere $u^2+v^2+w^2\leq 1$, and let $p=T(p^*)$, $p^*\in U^*$, be a topological mapping from U^* into Euclidean three-space. S, D will denote the images, under T, of the boundary and of the interior of U^*, and $|S|$, $|D|$ will denote the three-dimensional measures of S, D. Let A be the Lebesgue area of S, and consider the isoperimetric inequality $(*)$ $V^2\leq A^3/36\pi$, where V is the volume enclosed by S. In case $|S|>0$, a decision must be made whether the interior volume $V_i=|D|$ or the exterior volume $V_e=|D|+|S|$ is to be used in the inequality $(*)$. An example of Besicovitch shows that $(*)$ generally fails to hold for V_e. In this paper, it is shown that $(*)$ holds in the Besicovitch example for V_i, and further examples, based on an observation of Geöcze made in 1913, are given which indicate that beyond the elementary level the concept of "enclosed" volume must be properly defined if the isoperimetric inequality is to hold. (Received May 27, 1946.)

254. Tibor Radó: *The isoperimetric inequality and the Lebesgue definition of surface area.* II.

Let S^* denote the unit sphere $u^2+v^2+w^2=1$, and let $p=T(p^*)$, $p^*\in S^*$, be a continuous mapping from S^* into Euclidean xyz space. Then T determines a (not necessarily simple) closed surface S. Define an index-function n (x, y, z) as follows: if the point (x, y, z) lies on S, then $n=0$; if (x, y, z) does not lie on S, then n is equal to the topological index of the point (x, y, z) with respect to S. Then $n(x, y, z)$ vanishes outside of a sufficiently large sphere K. Define $V(S)$, the volume enclosed by S, as the integral of $|n(x, y, z)|$ in K if this integral exists, and let $V(S)=\infty$ otherwise. The purpose of the paper is to establish the isoperimetric inequality $V(S)^2 \leqq A(S)^3/36\pi$, where $A(S)$ is the Lebesgue area of S, as a generalization of previous results of Tonelli and Bonnesen. (Received May 29, 1946.)

STATISTICS AND PROBABILITY

255. Z. W. Birnbaum: *Tshebysheff inequality for two dimensions.*

For independent random variables X, Y with expectations zero and variances σ_X^2, σ_Y^2 the trivial inequality $P(X^2+Y^2 \geqq T^2) \leqq (\sigma_X^2+\sigma_Y^2)/T^2$ is replaced by a sharp inequality. (Received April 5, 1946.)

256. Mark Kac: *A discussion of the Ehrenfest model.* Preliminary report.

A particle moves along a straight line in steps Δ, the duration of each step being τ. The probabilities that the particle at $k\Delta$ will move to the right or left are $(1/2)(1-k/R)$ and $(1/2)(1+k/R)$ respectively. R and k are integers and $|k| \leqq R$. M. C. Wang and G. E. Uhlenbeck in their paper *On the theory of Brownian motion.* II (Review of Modern Physics vol. 17 (1945) pp. 323–342) discuss this random walk problem and state several unsolved problems. In answer to some of the questions raised the following results are obtained: Let $(1-z)^{R-j}(1+z)^{R+j}=\sum C_k^{(j)} z^k$ (j an integer), then the probability $P(n, m|s)$ that a particle starting from $n\Delta$ will come to $m\Delta$ after time $t=s\tau$ is equal to $2^{-2R}(-1)^{R+n}\sum (j/R)^s C_{R+j}^{(-n)} C_{R+m}^{(j)}$, where the summation is extended over all j such that $|j| \leqq R$. Also, if R is even the probability $P'(n, 0|s)$ that the particle starting from $n\Delta$ will come to 0 at $t=s\tau$ for the first time is calculated. For $n=0$ this gives a solution of the so-called recurrence time problem first studied on simpler models by Smoluchowski. Through a limiting process in which $\tau \rightarrow 0$, $\Delta \rightarrow 0$, $\Delta^2/2\tau \rightarrow D$, $1/R\tau \rightarrow \beta$, $n\Delta \rightarrow x_0$, $m\Delta \rightarrow x$, $s\tau = t$, one is led to fundamental distributions concerning the velocity of a free Brownian particle. In particular, $P(n, m|s)$ approaches the well known Ornstein-Uhlenbeck distribution. (Received May 23, 1946.)

257. Howard Levene: *A test of randomness in two dimensions.*

A square of side N is divided into N^2 unit cells, and each cell takes on the characteristics A or B with probabilities p and $q=1-p$ respectively, independently of the other cells. A cell is an "upper left corner" if it is A and the cell above and cell to the left are not A. Let V_1 be the total number of upper left corners and let V_2, V_3, V_4 be the number of similarly defined upper right, lower right, and lower left corners respectively. Let $V=(V_1+V_2+V_3+V_4)/4$. It is proved that V is normally distributed in the limit with $E(V)=p(Nq+p)^2$ and $\sigma^2(V)=N^2pq^2(2-10p+22p^2-13p^3)/2$. The conditional limit distribution of V, when p is estimated from the data, and the limit distribution of a related quadratic form are also obtained. These statistics are in a sense a generalization of the run statistics used for testing randomness in one dimension. (Received May 28, 1946.)

Topology

258. R. F. Arens and J. L. Kelley: *Characterization of the space of continuous functions.*

If B is a Banach space then either of the following sets of conditions is necessary and sufficient that B be the space of continuous real-valued functions over some compact Hausdorff space. I: (a) If E is the unit sphere of B then E has at least one extreme point. (b) If C is a maximal, convex subset of the surface S of E, then C and $-C$ span, that is, if $f\in E$, then $f=tf_1+(t-1)f_2$, $0\leq t\leq 1$, $f_1, f_2\in C$. (c) If a set $\mathcal{A}$ of maximal convex subsets of S has a void set theoretic intersection, then one may find directed sets C_μ, C_ν in $\mathcal{A}$, such that for each $f\in E$, dist $(f, C_\mu)+$ dist $(f, C_\nu)\to 2$. II: (a) If Σ is the unit sphere in B^*, then there is an f in B such that for any extreme point ξ of Σ, $|\xi(f)|=1$. (b) If $\mathcal{A}$ is any collection of extreme points of Σ, and if, in the topology for B^* determined by the elements of B, $\mathcal{A}$ has no pair of diametric limit points, then there is an f in B such that, for $\xi\in\mathcal{A}$, $\xi(f)=1$. (Received May 27, 1946.)

259. Felix Bernstein: *A new four-color problem.*

Let S be a partial set of the set of all points of the lattice whose coordinates are all the positive and negative integers, and T the complementary set. Two points of S, A and B, are called neighbors if the circle over the diameter AB in its interior (or in its interior and on its periphery) does not contain a point of the set T. It is asked how many colors are needed, coloring all points of S so that no two neighbors have the same color. The answer for $T=0$ obviously is four. It is suggested that four is the general answer. It seems impossible to prove the theorem by complete induction from n to $n+1$ unless the definition of neighborhood is much weakened. In order to clarify this further, the inequality $(1+x)^n>nx^{n-1}+(n(n-1)/2)x^{n-2}+\cdots+1$ is discussed as an example of a theorem whose direct proof by complete induction seems not possible while the more inclusive binomial theorem itself from which this inequality follows can be derived by complete induction. By analogy the four-color theorem may be a simple consequence of a more inclusive theorem which can be proved by complete induction. (Received May 17, 1946.)

260. W. H. Gottschalk and G. A. Hedlund: *Recursive properties of transformation groups.*

Let the mapping $f: X\times T\to X$ define a transformation group, where X is a topological space and T is an additive abelian locally compact topological group. Let there be distinguished in T certain nonvacuous sets, called *admissible*, which satisfy this condition: If A is an admissible set and if B is a set in T such that for some compact set C in T we have $a\in A$ implies $B\cap(a+C)\neq\emptyset$, then B is an admissible set. If $x\in X$ and if S is a set in T, then x is said to be *recursive* with respect to S provided that if U is a neighborhood of x, then there exists an admissible set A such that $A\subset S$ and $f(x, A)\subset U$. Let G be a subgroup of T such that $T=F+G$ for some compact set F in T. Define H to be the set of all elements t of T such that $x\in\bar{f}(x, t+G)$. It is proved that if H is a group and if x is recursive with respect to T, then x is recursive with respect to G. Twelve results concerning almost periodic, recurrent and strongly recurrent points follow immediately. (Received April 9, 1946.)

261. R. H. Sorgenfrey: *On the topological product of paracompact spaces.*

A space S is defined by Jean Dieudonné (J. Math. Pures Appl. vol. 23 (1944) pp. 65–76) to be *paracompact* provided that every covering of S by open sets has a neighborhood-finite refinement which covers S. In this note a negative answer is given to the question: If S is paracompact, is $S \times S$ paracompact? The example given also provides a simple instance of a normal Hausdorff space S such that $S \times S$ is not normal. (Received April 16, 1946.)

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A GEOMETRICAL CHARACTERIZATION FOR THE AFFINE DIFFERENTIAL INVARIANTS OF A SPACE CURVE

L. A. SANTALÓ

1. **Introduction.** Let $x=x(s)$ be the vector equation of a space curve C with the affine arc length s as parameter. It is known that $x(s)$ satisfies a differential equation of the following form [1, p. 73; 3, p. 76][1]

$$x'''' + kx'' + tx' = 0, \tag{1.1}$$

where the primes represent derivatives with respect to s. The vector x' is the *tangent vector* and the vectors x'' and x''' are called the *affine principal normal* and the *affine binormal*, respectively, of the curve C at the point considered. The vectors x', x'', x''' with the initial point at the point x of the curve C constitute the *affine fundamental trihedral* at x and they satisfy the following relation [1, p. 72; 3, p. 78]

$$(x', x'', x''') = 1. \tag{1.2}$$

The plane determined by the point x and the edges x', x'' of the affine fundamental trihedral is the *osculating plane* at x; the plane determined by x and the edges x'', x''' is the *affine normal plane* and the plane determined by x and the edges x', x''' is the *affine rectifying plane* of the curve C at the point x.

Sometimes it is convenient to use the vector $kx'/4+x'''$ which is called the *binormal of Winternitz* [1, p. 76]. The invariants k and t (functions of the affine arc length s) are called the *affine curvature* and the *affine torsion* respectively.

For the affine fundamental trihedral and for k and t some geometrical characterizations have been given by Blaschke [1, chap. 3], Salkowski [3, p. 76] and Haack [2]. The purpose of the present paper is to give a new geometrical construction for the mentioned elements, which we believe is simpler than those previously obtained.

2. **Geometrical elements associated to an ordinary point of a space curve.** Let us consider the space curve C represented by the vector equation $x=x(s)$ (s=affine arc length) in the neighborhood of the point $x_0=x(0)$. If we denote by $x_0^{(i)}$ the derivatives $d^{(i)}x/ds^i$ at the point $s=0$, since x_0', x_0'', x_0''' are not coplanar (by (1.2)), any point y of the space can be expressed in the form

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$y = x_0 + \xi_1 x_0' + \xi_2 x_0'' + \xi_3 x_0''' \tag{2.1}$$

where ξ_1, ξ_2, ξ_3 may be considered as the coordinates of the point y with respect to the affine fundamental trihedral.

For the points $x(s)$ of the curve C we have

$$x(s) = x_0 + s x_0' + \frac{1}{2} s^2 x_0'' + \frac{1}{3!} s^3 x_0'' + \frac{1}{4!} s^4 x_0''' + \cdots$$

and taking into account the relation (1.1) we obtain

$$\begin{aligned} \xi_1 &= s - \frac{1}{4!} t s^4 - \frac{1}{5!} t' s^5 + \cdots, \\ \xi_2 &= \frac{1}{2} s^2 - \frac{1}{4!} k s^4 - \frac{1}{5!} (k' + t) s^5 + \cdots, \\ \xi_3 &= \frac{1}{3!} s^3 - \frac{1}{5!} k s^5 + \cdots, \end{aligned} \tag{2.2}$$

where $k' = dk/ds$, $t' = dt/ds$.

Let us now consider the following geometrical elements associated to the curve C at the point x_0:

(a) *The quadric cone K.* By K we denote the quadric cone determined by the tangent of C at the point x_0 and the parallel lines through x_0 to the tangents of C at four neighboring points as each of these points independently approaches x_0 along C. This quadric cone K has been considered by Haack [2] and its equation in terms of the coordinates ξ_i is [2, p. 159]

$$\xi_2^2 + k\xi_3^2 - 2\xi_1\xi_3 = 0. \tag{2.3}$$

(b) *The osculating cubical parabola Q.* A twisted cubic which has the plane at infinity as osculating plane is called a cubical parabola. The parametric equations of a cubical parabola have the following form

$$\xi_i = a_i + b_i u + c_i u^2 + d_i u^3 \qquad (i = 1, 2, 3). \tag{2.4}$$

In order to find the cubical parabola which has contact of the highest order with the curve C at the point x_0, let us put

$$s = u + m_1 u^2 + m_2 u^3 + m_3 u^4 + \cdots \tag{2.5}$$

in the equations (2.2). We obtain

$$\xi_1 = u + m_1 u^2 + m_2 u^3 + \left(m_3 - \frac{1}{4!} t\right) u^4 + \cdots,$$

$$\xi_2 = \frac{1}{2}u^2 + m_1u^3 + \left(m_2 + \frac{1}{2}m_1^2 - \frac{1}{4!}k\right)u^4 + \cdots,$$

$$\xi_3 = \frac{1}{3!}u^3 + \frac{1}{2}m_1u^4 + \cdots.$$

If we take the values

$$m_1 = 0, \qquad m_2 = \frac{1}{4!}k, \qquad m_3 = \frac{1}{4!}t$$

we see that the cubical parabola

$$\xi_1 = u + \frac{1}{4!}ku^3, \qquad \xi_2 = \frac{1}{2}u^2, \qquad \xi_3 = \frac{1}{3!}u^3 \tag{2.6}$$

is the only cubical parabola which has a fourth order contact with the curve C at the point x_0. Consequently the equations (2.6) are the parametric equations of the osculating cubical parabola Q of C at x_0.

The quadric cone K and the twisted cubic Q are affinely connected to the curve C at the point x_0. We shall use also the following cone K^* which is projectively connected with C.

(c) *The osculating quadric cone* K^*. With K^* we denote the unique quadric cone with its vertex at the point x_0 which has seven-point contact with C at the point x_0. In order to find the equation of K^* let us substitute the expansions (2.2) in the general quadratic equation

$$A\xi_1^2 + B\xi_2^2 + C\xi_3^2 + D\xi_1\xi_2 + E\xi_1\xi_3 + F\xi_2\xi_3 = 0 \tag{2.7}$$

and then, demanding that this equation be satisfied identically in s as far as the terms in s^6, we find

$$A = D = F = 0, \qquad 3B + 2E = 0, \qquad 10C - 15kB - 3kE = 0.$$

Hence, the osculating quadric cone K^* is given by the equation

$$20\xi_2^2 + 21k\xi_3^2 - 30\xi_1\xi_3 = 0. \tag{2.8}$$

3. **Geometrical characterization of the affine fundamental trihedral of a space curve at an ordinary point.** From (2.6) we deduce that the straight line which connects the point x_0 of the curve C with the point at infinity of the osculating cubical parabola Q at x_0 has the direction of the vector

$$\frac{1}{4}kx_0' + x_0''',$$

that is, coincides with the binormal of Winternitz (§1).

The plane determined by this binormal and the tangent line of C at x_0 is the affine rectifying plane and the polar line of this affine rectifying plane with respect to the cone K turns out to be the affine principal normal line.

If we let $\xi_2=0$ in the equation (2.3) we find that the intersection of the affine rectifying plane with the quadric cone K is composed of the tangent line and of the straight line given by the equations $\xi_2=0$, $k\xi_3-2\xi_1=0$, that is, the line which has the direction of the vector $kx_0'/2+x_0'''$.

Since the vectors x_0', $kx_0'/4+x_0'''$ and $kx_0'/2+x_0'''$, x_0''' are harmonically separated it turns out that the affine binormal is the line harmonic conjugate to the intersection line (different from the tangent) of the cone K with the affine rectifying plane with respect to the tangent line and the binormal of Winternitz. The plane determined by the affine binormal and the affine principal normal is the affine normal plane. Thus the following theorem is established:

At an ordinary point x_0 of a space curve C let us consider the osculating cubical parabola Q and the cone K defined in §2. The direction of the point at infinity of Q gives the binormal of Winternitz of C at x_0. This binormal determines with the tangent line the affine rectifying plane, whose polar line with respect to the cone K is the affine principal normal of C at x_0. The harmonic conjugate of the intersection line (different from the tangent) of K with the affine rectifying plane with respect to the pair formed by the tangent line and the binormal of Winternitz is the affine binormal line of C at x_0.

Instead of the cone K and the cubical parabola Q in order to give a geometrical characterization of the affine fundamental trihedral connected with the space curve C at the point x_0, we can use only the cones K and K^* (§2). For this purpose let us seek the straight line which passes through the point x_0 and has the same polar plane with respect to both cones K and K^*. If we use the coordinates ξ_1, ξ_2, ξ_3 of (2.1) the polar plane with respect to K of the line which connects x_0 with the point ξ_1^0, ξ_2^0, ξ_3^0 is given by the equation

$$\overset{0}{\xi}_3\xi_1-\overset{0}{\xi}_2\xi_2-(k\overset{0}{\xi}_3-\overset{0}{\xi}_1)\xi_3=0 \tag{3.1}$$

and the polar plane of the same line with respect to the cone K^* is

$$15\overset{0}{\xi}_3\xi_1-20\overset{0}{\xi}_2\xi_2-(21k\overset{0}{\xi}_3-15\overset{0}{\xi}_1)\xi_3=0. \tag{3.2}$$

In order that the planes (3.1) and (3.2) coincide, we must have either $\xi_2^0=\xi_3^0=0$ or $\xi_1^0=\xi_3^0=0$. The first solution corresponds to the tangent line whose common polar plane with respect to the cones K and K^* is the osculating plane. The second solution corresponds to the affine principal normal whose polar plane with respect to the cones K and K^* is the affine rectifying plane. Then, upon excluding the tangent line, the affine principal normal is the only line which has the same polar plane with respect to the quadric cones K and K^*.

We have seen that the affine rectifying plane intersects the cone K, excluding the tangent line, in the line R of the vector $kx_0'/2+x_0'''$. Analogously we find that the affine rectifying plane intersects the cone K^* besides the tangent in the line R^* of the vector $7kx_0'+10x_0''$. Consequently the cross ratio of the tangent T, affine binormal B and the lines R and R^* has the value

$$\lambda = (TBRR^*) = \left(\infty, 0, \frac{1}{2}k, \frac{7}{10}k\right) = \frac{7}{5}\cdot$$

Thus we have obtained:

Let K and K^* be the quadric cones attached to the space curve C at the point x_0 defined in §2. *The affine principal normal of C at x_0 is the only line which has the same polar plane with respect to both cones K and K^*; its common polar plane is the affine rectifying plane. Let R and R^* be the intersection lines, the tangents excluded, of the rectifying plane with the cones K and K^* respectively; the affine binormal is the line B such that the cross ratio $(TBRR^*)$ has the value* 7/5.

4. **Geometrical interpretation of the affine curvature.** If we substitute the expressions (2.6) in (2.3) we find that the cubical parabola Q has with the cone K four coincident points in x_0 and two other intersection points corresponding to the values of u satisfying the equation

$$ku^2 = 6. \tag{4.1}$$

Let us put

$$u_1 = \left(\frac{6}{k}\right)^{1/2}, \qquad u_2 = -\left(\frac{6}{k}\right)^{1/2}. \tag{4.2}$$

The corresponding intersection points of Q with K will be the points

$$A_i\left(\xi_1 = u_i + \frac{1}{4!}ku_i^3,\ \xi_2 = \frac{1}{2}u_i^2,\ \xi_3 = \frac{1}{3!}u_i^3\right) \quad (i = 1, 2). \tag{4.3}$$

The straight line which connects A_1 and A_2 cuts the osculating plane in the point $\xi_1=\xi_3=0$, $\xi_2=3k^{-1}$. That gives the following new characterization of the affine principal normal: the affine principal normal is the line which connects the point x_0 of the curve with the intersection point of the osculating plane with the straight line A_1A_2.

The osculating plane of Q at A_i, taking into account (4.1), is

$$6u_i^2\xi_1 - 12u_i\xi_2 + 3\xi_3 - 2u_i^3 = 0 \tag{4.4}$$

and it cuts the edges of the affine fundamental trihedral in the following points

$$y_1 = x_0 + \frac{1}{3}u_i x_0', \qquad y_2 = x_0 - \frac{1}{6}u_i^2 x_0'', \qquad y_3 = x_0 + \frac{2}{3}u_i^3 x_0'''.$$

The absolute value of the volume of the tetrahedron whose vertices are the points x_0, y_1, y_2, y_3, taking into account (4.1) and (1.2), is

$$V = \frac{1}{6}(y_1 - x_0,\ y_2 - x_0,\ y_3 - x_0) = \frac{4}{3}k^{-3}. \tag{4.5}$$

Then we have the theorem:

Let us consider the points A_i ($i=1, 2$) in which the osculating cubical parabola Q intersects the cone K. The osculating plane of Q at any one of the points A_i determines with the affine fundamental trihedral a tetrahedron whose volume V is related to the affine curvature k by (4.5).

5. **Geometrical interpretation of the affine torsion.** We now give a geometrical interpretation of the affine torsion t. For this purpose let us seek the intersection line of two consecutive affine normal planes. The vector equation of the affine normal plane of C at x_0 is

$$y(\lambda, \mu) = x_0 + \lambda x_0'' + \mu x_0''' \tag{5.1}$$

and the analogous equation for the affine normal plane at the point $x(s)$, taking into account the relation (1.1), takes the form

$$\begin{aligned} \bar{y}(\bar{\lambda}, \bar{\mu}) &= x(s) + \bar{\lambda}x''(s) + \bar{\mu}x'''(s) \\ &= x_0 + \left[(1 - t\bar{\mu})s - \frac{1}{2}(t\bar{\lambda} + t'\bar{\mu})s^2 + \cdots\right]x_0' \\ &\quad + \left[\bar{\lambda} - k\bar{\mu}s + \frac{1}{2}(1 - k\bar{\lambda} - (k' + t)\bar{\mu})s^2 + \cdots\right]x_0'' \\ &\quad + \left[\bar{\mu} + \bar{\lambda}s - \frac{1}{2}k\bar{\mu}s^2 + \cdots\right]x_0'''. \end{aligned} \tag{5.2}$$

In order to find the intersection line of the planes (5.1) and (5.2) we must write

$$1 - t\bar{\mu} - \frac{1}{2}(t\bar{\lambda} + t'\bar{\mu})s + \cdots = 0,$$

$$\bar{\lambda} - k\bar{\mu}s + \frac{1}{2}(1 - k\bar{\lambda} - (k' + t)\bar{\mu})s^2 + \cdots = \lambda,$$

$$\bar{\mu} + \bar{\lambda}s - \frac{1}{2}k\bar{\mu}s^2 + \cdots = \mu$$

and by elimination of $\bar{\lambda}$ and $\bar{\mu}$ we find

$$(1 - \mu t) + \frac{1}{2}(\lambda t + \mu t')s + \cdots = 0.$$

For $s=0$ we have $\mu=1/t$; hence the intersection line of two consecutive affine normal planes is the line $y(\lambda)=x_0+\lambda x_0''+t^{-1}x_0'''$. Then, the intersection point of the affine binormal at the point x_0 with the consecutive affine normal plane or, what is the same, the intersection point of the affine binormal with the developable surface enveloped by the affine normal planes, is the point $B=x_0+t^{-1}x_0'''$.

Let us now trace through the point B the parallel line to the tangent of C at the point x_0; if we call E its intersection point with the quadric cone K we find that it is

$$E = x_0 + \frac{1}{2}kt^{-1}x_0' + t^{-1}x_0' .$$

Taking into account (1.2) for the volume of the tetrahedron whose vertices are the points x_0, B, A_1 (or A_2) and E we have the expression

$$V^* = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & t^{-1} & 1 \\ \xi_1 & \xi_2 & \xi_3 & 1 \\ \frac{1}{2}kt^{-1} & 0 & t^{-1} & 1 \end{vmatrix}$$

where the values of ξ_i $(i=1, 2, 3)$ are given by (4.3) and (4.2). Consequently we have

$$V^* = \frac{1}{4}t^{-2}. \tag{5.3}$$

Hence we have obtained the following geometrical interpretation for the affine torsion t of a space curve:

Through the intersection point B of the affine binormal at the point x_0 *of a space curve C with the developable surface enveloped by the affine normal planes, we trace the parallel line to the tangent of C at* x_0. *Let E be the point in which this parallel line intersects the quadric cone K. If* A_i *is any one of the points in which the osculating cubical parabola Q intersects the cone K, the volume of the tetrahedron whose vertices are the points* x_0, B, A_i, E *is related to the affine torsion of C at* x_0 *by the relation* (5.3).

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UNIVERSIDAD NACIONAL DEL LITORAL, ROSARIO

ALMOST PERIODICITY, EQUI-CONTINUITY AND TOTAL BOUNDEDNESS

W. H. GOTTSCHALK

Let X be a uniform space; that is to say, let X be a space provided with a system of indexed neighborhoods $U_\alpha(x)$ ($x \in X$, α = index), subject to the conditions (A. Weil): (1) If $x \in X$ and if α is an index, then $x \in U_\alpha(x)$; (2) If α and β are indices, then there exists an index γ such that $x \in X$ implies $U_\gamma(x) \subset U_\alpha(x) \cap U_\beta(x)$; (3) If α is an index, then there exists an index β such that $x, y, z \in X$ with $x, y \in U_\beta(z)$ implies $x \in U_\alpha(y)$. Let T be a topological group with identity σ and let f be a transformation of $X \times T$ into X. We agree to write $f^t(x)$ or $f_x(t)$ in place of $f(x, t)$ ($x \in X$, $t \in T$), whenever we wish. Furthermore, let f define a transformation group; that is to say, suppose $f^\sigma(x) = x$ and $f^s f^t(x) = f^{ts}(x)$ ($x \in X$; $t, s \in T$). We impose continuity conditions on f as the needs arise.

A subset E of T is said to be *relatively dense* provided there exists a compact set A in T such that each left translate of A intersects E. A point x of X is called *almost periodic* provided that if U is a neighborhood of x, then there exists a relatively dense set E in T for which $f(x, E) \subset U$. We observe that the notion of almost periodic point depends on the topology in T, the strongest type of almost periodicity occurring when T is provided with the discrete topology. It is easily proved that a set E in T is relatively dense if and only if there exists a compact set B in T such that $T = EB$. The set $f(x, T)$ is called the *orbit* of the point x.

THEOREM 1. *If the family $[f^t \mid t \in T]$ is equi-continuous at x, if f_x is continuous on T, and if x is almost periodic, then the orbit of x is totally bounded. Conversely, if the family $[f^t \mid t \in T]$ is equi-uniformly continuous and if the orbit of x is totally bounded, then x is almost periodic.*

PROOF. Suppose the hypotheses of the first statement hold. Let α be an index. There exists an index β such that the β-neighborhood of each compact set in X is contained in the union of finitely many α-neighborhoods. By hypothesis we can find an index γ such that $f^t(U_\gamma(x)) \subset U_\beta(f^t(x))$ ($t \in T$). There are sets E and A in T such that $T = EA$, A is compact, and $f(x, E) \subset U_\gamma(x)$. Hence, $f(x, T) \subset f(U_\gamma(x), A) \subset U_\beta(f(x, A))$. Since $f(x, A)$ is compact, $U_\beta(f(x, A))$ is contained in

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the union of finitely many α-neighborhoods. Thus the orbit of x is totally bounded.

Suppose the hypotheses of the second statement hold. Let α be an index. By hypothesis there exists an index β such that $f^t(U_\beta(y)) \subset U_\alpha(f^t(y))$ $(y \in X, t \in T)$. Choose finitely many elements $t_1, \cdots, t_n$ of T so that $f(x, T) \subset \bigcup_{i=1}^n V_i$, where $V_i = U_\beta(f(x, t_i))$. If $t \in T$, then for some i, $f(x, t) \in V_i$ whence $f(x, tt_i^{-1}) \in U_\alpha(x)$. Hence, x is almost periodic.

Corollary 1. *If the family* $[f^t | t \in T]$ *is equi-uniformly continuous, if* f_x *is continuous on* T, *and if* x *is almost periodic, then* x *is almost periodic with respect to the discrete topology in* T.

Corollary 2 (Bochner's Theorem). *If* $x(\tau)$, $-\infty < \tau < +\infty$, *is a complex-valued continuous function, then* $x(\tau)$ *is an almost periodic function in the sense of Bohr if and only if each sequence of translates of* $x(\tau)$ *contains a uniformly convergent subsequence.*

Proof. Define Y to be the collection of all complex-valued continuous functions $y(\tau)$, $-\infty < \tau < +\infty$, and define indexed neighborhoods in Y so that $y \in U_n(y_0)$ if and only if $|y(\tau) - y_0(\tau)| < 1/n$ $(-\infty < \tau < +\infty)$ where the index n is a positive integer. Construct a transformation group g in Y by translation of the functions in Y, taking T to be the additive group of reals with its natural topology. Now x is an almost periodic point if and only if $x(\tau)$ is an almost periodic function. Clearly, the family $[g^t | t \in T]$ is equi-uniformly continuous. Also if $x(\tau)$ is an almost periodic function, then $x(\tau)$ is uniformly continuous whence g_x is continuous on T. Hence, by Theorem 1, x is an almost periodic point if and only if the orbit of x is totally bounded. The conclusion follows.

We point out that A. Weil [2, pp. 130–133][1] has essentially taken Theorem 1 as the definition of an almost periodic point with respect to a transformation group.

If X is an arbitrary set, if Y is a uniform space, and if Φ is a nonvacuous collection of functions on X to Y, then we consider Φ to be a uniform space in the following manner: If α is an index belonging to Y and if $\phi \in \Phi$, then the α-neighborhood $U_\alpha(\phi)$ of ϕ is taken to be the set of all elements ψ of Φ such that $\psi(x) \in U_\alpha(\phi(x))$ for every element x of X.

The following lemma will be recognized as a generalization of Ascoli's theorem and its converse.

Lemma 1. *Let* X *and* Y *be uniform spaces and let* Φ *be a nonvacuous*

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

collection of functions on X to Y. If X and Y are totally bounded and if the family Φ is equi-uniformly continuous, then the space Φ is totally bounded. Conversely, if the individual functions in Φ are uniformly continuous and if the space Φ is totally bounded, then the family Φ is equi-uniformly continuous.

PROOF. Suppose X and Y are totally bounded and Φ is equi-uniformly continuous. Let α be an index belonging to Φ and, hence, to Y. Choose an index β belonging to Y so that $a, b, c \in Y$ with $a, b \in U_\beta(c)$ implies $a \in U_\alpha(b)$. We can find an index γ belonging to Y and finitely many points $y_1, \cdots, y_n$ of Y so that $y \in Y$ implies $U_\gamma(y) \subset U_\beta(y_j)$ for some integer j, $1 \leqq j \leqq n$. There exists an index δ belonging to X such that $\phi(U_\delta(x)) \subset U_\gamma(\phi(x))$, $(\phi \in \Phi,\ x \in X)$. Choose finitely many points $x_1, \cdots, x_m$ of X so that $X = \bigcup_{i=1}^m U_\delta(x_i)$. Hence if $\phi \in \Phi$, then to each integer i, $1 \leqq i \leqq m$, there corresponds at least one integer j, $1 \leqq j \leqq n$, such that $\phi(U_\delta(x_i)) \subset U_\beta(y_j)$. Letting I, J denote the first m, n positive integers, each element ϕ of Φ defines in the obvious manner a class $C(\phi)$ of transformations of I into J. Choose finitely many elements $\phi_1, \cdots, \phi_r$ of Φ so that the class $\bigcup_{k=1}^r C(\phi_k)$ is maximal. It follows that $\Phi = \bigcup_{k=1}^r U_\alpha(\phi_k)$. Hence, Φ is totally bounded.

Suppose now that the functions in Φ are uniformly continuous and Φ is totally bounded. Let α be an index belonging to Y. There exists an index γ belonging to Y so that $a \in U_\gamma(c)$, $b \in U_\gamma(d)$ and $c \in U_\gamma(d)$ implies $a \in U_\alpha(b)$. Choose finitely many elements $\phi_1, \cdots, \phi_n$ of Φ so that $\Phi = \bigcup_{i=1}^n U_\gamma(\phi_i)$. Select indices $\beta(i)$ $(i = 1, \cdots, n)$ belonging to X which have the property that $\phi_i(U_{\beta(i)}(x)) \subset U_\gamma(\phi_i(x))$ $(x \in X;\ i = 1, \cdots, n)$. There exists an index β belonging to X for which $U_\beta(x) \subset \bigcap_{i=1}^n U_{\beta(i)}(x)$ $(x \in X)$. We conclude that $\phi(U_\beta(x)) \subset U_\alpha(\phi(x))$ $(\phi \in \Phi,\ x \in X)$. Hence, Φ is equi-uniformly continuous and the proof is completed.

We say that the transformation group f is *almost periodic* provided that if α is an index, then there exists a relatively dense set E in T for which $x \in X$ implies $f(x, E) \subset U_\alpha(x)$. It may be verified that in order for f to be almost periodic it is both necessary and sufficient that if α is an index, then there exists a compact set A in T such that to each element t in T there corresponds an element s in A for which $f^t(x) \in U_\alpha(f^s(x))$ $(x \in X)$. If T has the discrete topology, this characterization reduces essentially to Montgomery's definition [1, p. 323] of an almost periodic transformation group.

In the following theorem we denote $[f^t \mid t \in T]$ by G and, somewhat improperly, call G the transformation group.

THEOREM 2. *If X is compact and if f is continuous on $X \times T$, then* *the following statements are pairwise equivalent*: (1) *The transformation group G is almost periodic*; (2) *The family G is equi-uniformly continuous*; (3) *The space G is totally bounded.*

PROOF. By Lemma 1, (2) is equivalent to (3).

Suppose (1) is satisfied. Let α be an index. There exists an index γ such that $a\in U_\gamma(c)$, $b\in U_\gamma(d)$ and $c\in U_\gamma(d)$ implies $a\in U_\alpha(b)$. It is possible to find a compact set A in T such that if $t\in T$, then $f^t(x)\in U_\gamma(f^s(x))$ $(x\in X)$, for some element s in A. Since f is uniformly continuous on $X\times A$, we can choose an index β for which $f^s(U_\beta(x))\subset U_\gamma(f^s(x))$ $(x\in X,\ s\in A)$. It follows that $f^t(U_\beta(x))\subset U_\alpha(f^t(x))$ $(x\in X,\ t\in T)$. Hence, (2) is satisfied.

Suppose (2) is satisfied. Let α be an index. There exists an index β such that $f^t(U_\beta(x))\subset U_\alpha(f^t(x))$ $(x\in X,\ t\in T)$. Since G is totally bounded, we can select finitely many elements $t_1, \cdots, t_n$ in T so that $G=\bigcup_{i=1}^{n} U_\beta(f^{t_i})$. Let t be an element of T. Then for some i, $f(x, t)\in U_\beta(f(x, t_i))$ $(x\in X)$, whence $f(x, tt_i^{-1})\in U_\alpha(x)$ $(x\in X)$. Thus (1) is satisfied and the proof is completed.

COROLLARY 3. *If X is compact, if f is continuous on $X\times T$, and if f is almost periodic, then f is almost periodic with respect to the discrete topology in T.*

COROLLARY 4 (SHARPENED DIRICHLET-KRONECKER THEOREM). *If $t, a_1, \cdots, a_k$ are nonzero real numbers and if ϵ is a positive number, then there exists a relatively dense set N of integers such that $n\in N$ implies the existence of integers $m_1, \cdots, m_k$ for which $|nt-m_ia_i|<\epsilon$ $(i=1, \cdots, k)$.*

PROOF. Let $X_1, \cdots, X_k$ denote pairwise disjoint circle boundaries in the plane with circumferences $|a_1|, \cdots, |a_k|$. Take $X=\bigcup_{i=1}^{k} X_i$ with the natural uniformity. Define the uniformity-preserving homeomorphism ϕ of X onto X by rotating each circle X_i through arc length t. The transformation group generated by the integral powers of ϕ satisfies (2) of Theorem 2 and, hence, is almost periodic. The conclusion follows from the definition of almost periodicity.

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RECURSIVE PROPERTIES OF TRANSFORMATION GROUPS

W. H. GOTTSCHALK AND GUSTAV A. HEDLUND

Introduction. If a continuous flow, the transformations of which are denoted by f^t, is defined in a topological space X, a point x of the space can be classified according to the behavior of the continuous orbit of the point. Among the types of points which have been considered are the periodic points, almost periodic points and recurrent points. If we fix the value of the parameter t, we obtain a transformation of the space X into itself. This transformation and its iterates determine a "discrete" flow and the "discrete" orbit of a point. Again a point of the space can be classified according to the behavior of the discrete orbit of the point. The question then arises as to whether a point belongs to the same class in the two cases.

Since the continuous orbit of a point contains the discrete orbit, many properties are retained when we pass from the discrete flow to the continuous flow. For example, if a point is periodic with respect to the discrete flow, it is clearly periodic with respect to the continuous flow. It is in the passage from the continuous to the discrete flow that the results are more difficult to predict.

In this note, the problem is generalized by replacing the parameter space of reals by an additive, abelian, locally compact, topological group. The action of such a transformation group T on a point and the action of certain subgroups G on the point are then related. It is shown that a general recursive property of T carries over to G (Theorem 1). It follows immediately that in the case of a continuous flow, if a point is either almost periodic, recurrent, or strongly recurrent with respect to the continuous flow, it retains this property with respect to the discrete flow determined by fixing the parameter t. It also follows that these properties carry over from a discrete flow to a sub-discrete flow. (For recurrence and almost periodicity, this was first proved by Gottschalk [2][1] and subsequently extended to discrete semi-flows by Erdős and Stone [1].)

Theorem on recursive points. Let X be a topological space and let T be an additive abelian locally compact topological group. Let f be a continuous transformation of $X \times T$ into X. We agree to write $f^t(x)$ in place of $f(x, t)$ $(x \in X, t \in T)$ whenever we wish. Furthermore, let f

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

define a transformation group; that is to say, suppose $f^0(x) = x$ and $f^s f^t(x) = f^{t+s}(x)$ $(x \in X;\ t,\ s \in T)$.

Let there be distinguished in T certain nonvacuous sets, called *admissible*, which satisfy this condition: If A is an admissible set and if B is a set in T such that for some compact set C in T we have $a \in A$ implies $B \cap (a + C) \neq \emptyset$, then B is an admissible set. If $x \in X$ and if S is a set in T, then x is said to be *recursive* with respect to S provided that if U is a neighborhood of x, then there exists an admissible set A such that $A \subset S$ and $f(x, A) \subset U$.

We suppose throughout that x is a fixed point of X and that G is a fixed topologically closed subgroup of T such that $T = F + G$ for some compact set F in T. We define H to be the set of all elements t of T such that $x \in \bar{f}(x,\ t + G)$, where $\bar{f}(x,\ t+G)$ denotes the closure of $f(x, t+G)$.

LEMMA 1. *H is a topologically closed semi-group ($H + H \subset H$) which contains G.*

PROOF. Suppose t, $s \in H$ and U is a neighborhood of x. There exists an element p in G such that $f(x,\ t+p) \in U$. Select a neighborhood V of x for which $f(V,\ t+p) \subset U$. For some element q in G, $f(x,\ s+q) \in V$. Hence, $f(x,\ t+s+p+q) \in U$. Thus H is a semi-group. Obviously, $G \subset H$.

Suppose $t \in \bar{H}$ and U is a neighborhood of x. Choose neighborhoods V of x and W of 0 for which $W = -W$ and $f(V,\ W) \subset U$. Let $s \in (t + W) \cap H$. For some element p in G, $f(x,\ s+p) \in V$. Now $f(x, t+p) = f^{t-s} f(x,\ s+p) \in f^{t-s}(V) \subset U$. Thus H is topologically closed and the proof is completed.

LEMMA 2. *There exists a compact set E in H such that $H = E + G$.*

PROOF. There exists a compact set F in T such that $T = F + G$. Define $E = F \cap H$. Using Lemma 1, we see that the set E has the required properties.

LEMMA 3. *If H is a group and if U is a neighborhood of x, then there exists a compact set C in H such that $f(x,\ C) \subset U$ and $t \in H$ implies $(t + C) \cap G \neq \emptyset$.*

PROOF. By Lemma 2, there exists a compact set E in H such that $H = E + G$. Define $D = -E$. Now $t \in H$ implies $(t + D) \cap G \neq \emptyset$. For each element d in D, there exists an element t_d in G so that $f(x,\ d + t_d) \in U$ and, hence, there exists a compact closed neighborhood W_d of d for which $f(x,\ W_d + t_d) \subset U$. Cover D by finitely many neighborhoods

$W_{d_1}, \cdots, W_{d_n}$. Define $C=H\cap\bigcup_{i=1}^{n}(W_{d_i}+t_{d_i})$. The set C has the required properties.

LEMMA 4. *If H is a group and if x is recursive with respect to H, then x is recursive with respect to G.*

PROOF. Let U be a neighborhood of x. By Lemma 3, there exists a compact set C in H such that $f(x, C)\subset U$ and $t\in H$ implies $(t+C)\cap G\neq\emptyset$. Choose a neighborhood V of x for which $f(V, C)\subset U$. There exists an admissible set A such that $A\subset H$ and $f(x, A)\subset V$. Hence, $f(x, A+C)\subset U$. Define $B=(A+C)\cap G$. The set B is admissible and contained in G. Also $f(x, B)\subset U$. The proof is completed.

LEMMA 5. *If M, N are subsets of T such that M is compact and $x\in\bar{f}(x, M+N)$, then there exists an element m in M for which $x\in\bar{f}(x, m+N)$.*

PROOF. Suppose the contrary. Then for each element m in M we can find a neighborhood U_m of x such that $f(x, m+N)\cap U_m=\emptyset$; selecting neighborhoods V_m of x and W_m of 0 so that $W_m=-W_m$ and $f(V_m, W_m)\subset U_m$, we have $f(x, m+W_m+N)\cap V_m=\emptyset$. Choose finitely many neighborhoods $m+W_m$ covering M and let V be a neighborhood of x contained in each of the corresponding V_m. It follows that $f(x, M+N)\cap V=\emptyset$. This is a contradiction.

LEMMA 6. *If H is a group and if $t\in T\sim H$ (t belongs to T but not to H), then $x\notin\bar{f}(x, t+H)$.*

PROOF. Suppose $x\in\bar{f}(x, t+H)$. By Lemma 2, there exists a compact set E in H such that $H=E+G$. By Lemma 5, there exists an element s in E for which $x\in\bar{f}(x, t+s+G)$. Hence, $t+s\in H$ and $t\in H$. This is a contradiction.

LEMMA 7. *If H is a group and if $t\in T\sim H$, then there exist neighborhoods V of x and W of t such that $f(x, W+H)\cap V=\emptyset$.*

PROOF. By Lemma 6, there exists a neighborhood U of x for which $f(x, t+H)\cap U=\emptyset$. Choose neighborhoods V of x and W_0 of 0 so that $W_0=-W_0$ and $f(V, W_0)\subset U$. Take $W=W_0+t$. The proof is completed.

LEMMA 8. *If H is a group and if x is recursive with respect to T, then x is recursive with respect to H.*

PROOF. Let U be a neighborhood of x. Choose neighborhoods V of x and W of 0 so that $W=-W$, $\overline{W}$ is compact and $f(V, W)\subset U$. Since we may suppose $H\neq T$, we may also suppose that $W+H\neq T$.

There exists a compact set F in T such that $T=F+H$. Define $E=F\cap[T\sim(W+H)]$. Now $T\sim(W+H)=E+H$. By Lemma 7, to each element e in E there correspond neighborhoods V_e of x and W_e of e so that $f(x, W_e+H)\cap V_e=\emptyset$. Hence, there exists a neighborhood V^* of x such that $f(x, E+H)\cap V^*=\emptyset$. We may suppose $V\subset V^*$, whence $f(x, E+H)\cap V=\emptyset$. There exists an admissible set A in T such that $f(x, A)\subset V$. Hence, $A\cap(E+H)=\emptyset$ and $A\subset W+H$. Define $B=H\cap(A+W)$. Now $f(x, A+W)\subset U$. Since B is an admissible set contained in H and $f(x, B)\subset U$, the proof is completed.

THEOREM 1. *If H is a group and if x is recursive with respect to T, then x is recursive with respect to G.*

PROOF. The conclusion follows immediately from Lemmas 4 and 8.

Applications. Now let T denote the additive group of either the integers or the reals with its natural topology. Then G is either T or the group of all multiples of a nonzero number. It is clear from Lemma 1 that in this case H is a group.

We consider the following six interpretations of admissibility. A set A in T is said to be admissible provided that:

(1) A contains a sequence marching to $+\infty$;

(2) A contains a sequence marching to $+\infty$ and a sequence marching to $-\infty$;

(3) There exists a positive number p such that each interval of non-negative numbers of length p contains a point of A;

(4) There exists a positive number p such that each interval of length p contains a point of A;

(5) For each positive number ϵ, $\lim\sup_{n\to+\infty} n^{-1}\mu\{U_\epsilon(A)\cap(0, n)\}>0$, where $U_\epsilon(A)$ is the ϵ-neighborhood of A in T, n is an integer, $(0, n)$ is the open interval with end points 0 and n, and μ denotes the usual measure of a set in T;

(6) For each positive number ϵ, $\lim\sup_{n\to+\infty} n^{-1}\mu\{U_\epsilon(A)\cap(0, n)\}>0$ and $\lim\sup_{n\to+\infty} n^{-1}\mu\{U_\epsilon(A)\cap(-n, 0)\}>0$.

With admissibility defined as 1, 2, 3, 4, 5 or 6, then the term "recursive" is replaced by "positively recurrent," "recurrent," "positively almost periodic," "almost periodic," "positively strongly recurrent" or "strongly recurrent," respectively.

Theorem 1 now yields immediately the following statement.

THEOREM 2. *If x is positively recurrent, recurrent, positively almost periodic, almost periodic, positively strongly recurrent or strongly recurrent with respect to T, then x has the same property with respect to G.*

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A REMARK ON DENSITY CHARACTERS

EDWIN HEWITT[1]

Let X be an arbitrary topological space satisfying the T_0-separation axiom [1, Chap. 1, §4, p. 58].[2] We recall the following definition [3, p. 329].

DEFINITION 1. *The least cardinal number of a dense subset of the space X is said to be the density character of X. It is denoted by the symbol $\Xi(X)$.*

We denote the cardinal number of a set A by $|A|$.

Pospíšil has pointed out [4] that if X is a Hausdorff space, then

$$|X| \leq 2^{2^{\Xi(X)}}. \tag{1}$$

This inequality is easily established. Let D be a dense subset of the Hausdorff space X such that $|D| = \Xi(X)$. For an arbitrary point $p \in X$ and an arbitrary complete neighborhood system $\mathcal{V}_p$ at p, let $\mathfrak{D}_p$ be the family of all sets $U \cap D$, where $U \in \mathcal{V}_p$. Thus to every point of X, a certain family of subsets of D is assigned. Since X is a Hausdorff space, $\mathfrak{D}_p \neq \mathfrak{D}_q$ whenever $p \neq q$, and the correspondence assigning each point p to the family $\mathfrak{D}_p$ is one-to-one. Since X is in one-to-one correspondence with a sub-hierarchy of the hierarchy of all families of subsets of D, the inequality (1) follows.

It may be remarked in passing that the inequality (1) does not obtain for all T_1-spaces. Let $\mathfrak{m}$ be a cardinal number greater than $2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\aleph_0}$. Let Z be a T_1-space of cardinal number $\mathfrak{m}$ and with the property that the only closed proper subsets of Z are finite or

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void. Then it is obvious that $\Xi(Z)=\aleph_0$, and that (1) does not obtain for the space Z.

For some Hausdorff spaces, the equality

$$|X| = 2^{2^{\Xi(X)}} \tag{2}$$

obtains. Pospíšil [4] has constructed a large family of such Hausdorff spaces, and has shown [5] that the Stone-Čech β for any discrete infinite space satisfies it as well. It is the purpose of this note to exhibit another class of Hausdorff spaces for which (2) holds.

THEOREM. *Let Λ be an index class such that $|\Lambda| = 2^{\mathfrak{m}}$ where $\mathfrak{m}$ is an infinite cardinal number. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of Hausdorff spaces such that $|X_\lambda|\geq 2$ and $\Xi(X_\lambda)\leq\mathfrak{m}$ for all $\lambda\in\Lambda$. Then $|\mathfrak{P}_{\lambda\in\Lambda}X_\lambda| = 2^{2^{\mathfrak{m}}}$ and $\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda)=\mathfrak{m}$.*

PROOF. We first consider the set Λ as a topological space itself. Clearly, it may be put into one-to-one correspondence with the Cartesian product $\mathfrak{P}_{\mu\in M}A_\mu$, where each A_μ is a Hausdorff space containing exactly two points and the index class M has cardinal number $\mathfrak{m}$. As is well known, this Cartesian product is a bicompact Hausdorff space with cardinal number $2^{\mathfrak{m}}$ and a basis of open sets with cardinal number $\mathfrak{m}$. We may consequently regard Λ as being a Hausdorff space with a basis $\mathcal{B}$ of open sets such that $|\mathcal{B}| = \mathfrak{m}$.

Let $y^0 = \{q_\lambda^0\}$ be a fixed point in the space $P_{\lambda\in\Lambda}X_\lambda$. Let D_λ be a dense subset in X_λ such that $|D_\lambda| = \Xi(X_\lambda)\leq\mathfrak{m}$. If α_0 is the least ordinal number with corresponding cardinal number $\mathfrak{m}$, then each set D_λ can be so well ordered that

$$D_\lambda = \{p_\lambda^1, p_\lambda^2, p_\lambda^3, \cdots, p_\lambda^\alpha, \cdots\}, \qquad \alpha < \alpha_0.$$

If $|D_\lambda| <\mathfrak{m}$, then the elements p_λ^α may be all taken identical from a certain point on. Of course, if $|D_\lambda| =\mathfrak{m}$, no repetitions need occur.

Let $\{\Lambda_1, \cdots, \Lambda_n\}$ be an arbitrary family of disjoint sets in $\mathcal{B}$, and let $\{\alpha_1, \cdots, \alpha_n\}$ be arbitrary ordinal numbers all less than α_0. Let $x(\Lambda_1, \cdots, \Lambda_n; \alpha_1, \cdots, \alpha_n) = \{r_\lambda\}$ be the point in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$ such that $r_\lambda = p_\lambda^{\alpha_i}$ for all $\lambda\in\Lambda_i$, $i=1, 2, 3, \cdots, n$, and $r_\lambda = q_\lambda^0$ for $\lambda\in\Lambda\cap(\sum_{i=1}^n\Lambda_i)'$. Let W be the set of all points $x(\Lambda_1, \cdots, \Lambda_n; \alpha_1, \cdots, \alpha_n)$ as $\{\Lambda_1, \cdots, \Lambda_n\}$ and $\{\alpha_1, \cdots, \alpha_n\}$ assume all possible values. It is clear that $|W| = \sum_{n=1}^\infty\mathfrak{m}^n\cdot\mathfrak{m}^n = \aleph_0\cdot\mathfrak{m} = \mathfrak{m}$. Furthermore, W is dense in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$. Let G be an arbitrary non-void open set in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$. By the definition of open sets and neighborhoods in a Cartesian product (see, for example, [2, pp. 829–830]), there exist a finite subset $\{\lambda_1, \cdots, \lambda_m\}$ of Λ and sets $U_{\lambda_1}, \cdots, U_{\lambda_m}$, where U_{λ_i} is an open set in X_{λ_i}, with the property that G contains all points $\{s_\lambda\}$

of $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$ such that $s_{\lambda_i}\in U_{\lambda_i}$ for $i=1, 2, 3, \cdots, m$. The sets D_λ being dense in the spaces X_λ, there is a point $p_{\lambda_i}^{\alpha_i}\in D_{\lambda_i}$ such that $p_{\lambda_i}^{\alpha_i}\in U_{\lambda_i}$ $(i=1, 2, 3, \cdots, m)$. Since Λ is a Hausdorff space under the topology defined by $\mathcal{B}$, there are sets $\Lambda_1, \cdots, \Lambda_m$ in $\mathcal{B}$ such that $\Lambda_i\cap\Lambda_j=0$ for $i\neq j$ and such that $\lambda_i\in\Lambda_i$ for all $i=1, 2, 3, \cdots, m$. It is obvious that the point $x(\Lambda_1, \cdots, \Lambda_m; \alpha_1, \cdots, \alpha_m)$ is in the set $W\cap G$. Having a nonvoid intersection with an arbitrary nonvoid open set in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$, W is dense in $\mathfrak{P}_{\lambda\in\Lambda}X_\lambda$.

It follows from the definition of $\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda)$ and the equality $|W|=\mathfrak{m}$ that $\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda)\leqq\mathfrak{m}$. On the other hand, we have

$$|\mathfrak{P}_{\lambda\in\Lambda}X_\lambda| \geqq \mathfrak{m}^{|\Lambda|} = 2^{2^{\mathfrak{m}}}. \tag{3}$$

Hence, by virtue of the inequality (1), it follows that

$$|\mathfrak{P}_{\lambda\in\Lambda}X_\lambda| = 2^{2^{\mathfrak{m}}} \tag{4}$$

and

$$\Xi(\mathfrak{P}_{\lambda\in\Lambda}X_\lambda) = \mathfrak{m}. \tag{5}$$

This completes the proof.

For a result similar to this, see [6].

The foregoing theorem, applied to various well known spaces, yields curious results.

COROLLARY 1. *The space of all real-valued functions of a real variable with the Cartesian product topology contains a countable dense subset.*

COROLLARY 2. *The space of all characteristic functions defined on a set of cardinal number* $2^{\aleph_0}$ *contains a countable dense subset under the Cartesian product topology.*

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THE KLINE SPHERE CHARACTERIZATION PROBLEM

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The object of this paper is to give a solution to the following problem proposed by J. R. Kline: Is a nondegenerate, locally connected, compact continuum which is separated by each of its simple closed curves but by no pair of its points homeomorphic with the surface of a sphere? The answer is in the affirmative.

A solution to the Kline problem gives a characterization of a simple closed surface. Partial solutions of this problem have been made by Hall [1, 2][1] and Jones [3]. Other characterizations of a simple closed surface have been given by Kuratowski [4], Zippin [5, 6], Wilder [7] and Claytor [8]. Previous to the giving of these characterizations, Moore gave [9] two sets of axioms, each set of which characterized a set topologically equivalent to a plane.

DEFINITION. We say that M disrupts X from Y in D if there is an arc from X to Y in D but each such arc contains a point of M.

We shall make use of the following lemma.

LEMMA. *Suppose that space is locally connected and cannot be separated by the omission of any pair of its points, that the boundary of the connected domain D is equal to the sum of the mutually exclusive sets M, N and E, each of which contains a point which is accessible from D, and that D' is a connected subdomain of D such that no point of D either disrupts D' from $E+M$ in $D+E+M$ or disrupts D' from $E+N$ in $D+E+N$. Then there is an open arc from M to N in D that does not disrupt D' from E in $D+E$.*

PROOF. Consider the arc AB in $D+B$ from a point A of D' to a point B of E. Let W_1 be the set of all points P of AB such that there is an open arc from P to E in D that does not intersect some open arc from M to N in D. Assume that the first point R of AB in the order from A to B on the closure of W_1 does not belong to D'.

If R disrupts D' from E in $D+E$, there are an arc from D' to M in $D+M-R$ and an arc from D' to N in $D+N-R$. In the sum of these two arcs plus D' there is an open arc from M to N in D which does not intersect RB. This is contrary to the definition of R. Hence, R does not disrupt D' from E in $D+E$.

Let $A'B'$ be an arc in $D+B'-R$ from a point A' of D' to a point

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[1] Numbers in brackets refer to the references cited at the end of the paper.

B' of E. Since D' is connected and does not contain R, we may suppose that A' is A. Let X be the last point of $A'B'$ in the order from A' to B' on AR and let Y be a point of AB between R and B such that RY contains no point of XB'.

Choose a metric for $XB'+XY$. Let P_1 be a point of $W_1 \cdot XY$ such that no point of W_1 at a distance of more than 1 from P_1 is between P_1 and X on XY. Denote by V_1 the set of all points P of XB' such that there are in D an open arc from P to E and an open arc from XP_1 to E such that the sum of these two open arcs does not intersect some open arc from M to N in D. Let Q_1 be a point of V_1 such that no point of V_1 at a distance of more than 1 from Q_1 is between Q_1 and X on XB'.

Denote by W_2 the set of all points P of XY such that there are in D an open arc from P to E and an open arc from XQ_1 to E such that the sum of these open arcs does not intersect some open arc from M to N in D. Let P_2 be a point of $W_2 \cdot XP_1$ such that no point of W_2 at a distance of more than $1/2$ from P_2 is between P_2 and X on XY.

In general, let W_n (or V_n) be the set of all points P of XY (or XB') such that there are an open arc from P to E in D and an open arc from XQ_{n-1} (or XP_n) to E in D such that the sum of these open arcs does not intersect some open arc from M to N in D. Let P_n (or Q_n) be a point of W_n (or V_n) on XP_{n-1} (or XQ_{n-1}) such that no point of W_n (or V_n) at a distance of more than $1/n$ from P_n (or Q_n) is between P_n (or Q_n) and X on XY (or XB').

Let P_0 and Q_0 be the limit points of $P_1, P_2, \cdots$ and $Q_1, Q_2, \cdots$ respectively. We note that for no integer n is P_n between X and P_0 on XY or is Q_n between X and Q_0 on XB'.

Since no pair of points separates space, there is an arc from A to E in space that contains neither P_0 nor Q_0. Let $A''B''$ be a subset of this arc irreducible from $AX+XP_0+XQ_0$ to E. We shall suppose that A'' is a point of $AX+XP_0$ since the argument to follow may be adjusted to take care of the case where it belongs to $AX+XQ_0$.

There is an integer i such that AA'' contains no point of W_i and $A''B''$ intersects neither P_0P_i nor Q_0Q_i. There is an arc $M'N'$ in $D+M'+N'$ from a point M' of M to a point N' of N such that there are open arcs $\langle P'B_1\rangle$ and $\langle Q'B_2\rangle$ in $D-\langle M'N'\rangle$, where P' and Q' are points of P_0P_i and Q_0Q_i respectively and B_j $(j=1, 2)$ is a point of E. Denote by P'' the first point of XP' in the order from X to P' on $P'B_1+Q'B_2$ and denote by Q'' the first point of XQ' in the order from X to Q' on $P'B_1+Q'B_2$. Let $P''B_3$ and $Q''B_4$ be arcs in $P'B_1+Q'B_2$ where B_3 and B_4 are points of B_1+B_2.

Denote the set $AX+XP''+XQ''$ by T. Since no point of AX is a point of W_i, $P''B_3+Q''B_4$ does not intersect AX. Then $P''B_3+Q''B_4$ intersects T only at $P''+Q''$.

Now $M'N'$ intersects T or else there is an open arc from A to E in $T+P''B_3$ and R is not as assumed. Also, $M'N'$ does not intersect T in only one point, for if it did, there would be an open arc from X to E in $T+P''B_3+Q''B_4$ which would not intersect $M'N'$. But X is not a point of W_i.

Let M'' and N'' be the first and last points of $M'N'$ in the order from M' to N' on T. Replace the part of $M'N'$ between M'' and N'' by an arc α from M'' to N'' in T. Now one of the points M'', N'' belongs to XP'', for if neither were a point of XP'', there would be an open arc from X to E in $XP''+P''B_3$ which would not intersect $M'M''+\alpha+N''N'$. This contradicts the fact that X is not a point of W_i. Also, one of M'', N'' is a point of XQ''. Since A'' is not a point of W_i, it is not between P'' and $M''+N''$ on XP''.

Let C be the first point of $A''B''$ in the order from A'' to B'' on $M+M'M''+N+N''N'+E+P''B_3+Q''B_4$. Now C is not a point of $E+P''B_3+Q''B_4$, for if it were, there would be an open arc from A'' to E in $A''C+P''B_3+Q''B_4$ that would not intersect $M'M''+\alpha+N''N'$. But A'' is not a point of W_i.

Assume that C is a point of $M+M'M''$. Consider an arc β from M to N in $A''C+M'M''+T+N''N'$ containing $A''C$. We note that β does not contain M''. Now A'' is not a point of XP'', for if it were, there would be an open arc from A'' to E in $P''B_3+Q''B_4+T$ which would not intersect β. Also, A'' is not a point of AX or else there would be an open arc from X to E in $P''B_3+Q''B_4+T$ which would not intersect β. But X is not a point of W_i. Hence, C is not a point of $M+M'M''$. Likewise, we find that C is not a point of $N+N''N'$.

This establishes the lemma. The assumption that R is not a point of D' leads to the contradiction that the point C of $M+M'M''+N+N''N'+E+P''B_3+Q''B_4$ belongs to neither $E+P''B_3+Q''B_4$, $M+M'M''$ nor $N+N''N'$.

THEOREM. *Suppose that S is a nondegenerate, compact continuous curve such that no pair of points but every simple closed curve of S separates it. Then S is topologically equivalent to the surface of a sphere.*

PROOF. Regard S as space. It is known [6, 10] that S is a simple closed surface if no arc separates it. We shall show that no arc separates S.

Before giving the details of the proof, we shall briefly outline what we intend to do. On the assumption that some arc separates space, we shall get a finite collection H_1 of connected domains such that their sum does not separate space and such that the sum of any two nonintersecting elements of H_1 separates the sum of the elements of H_1. Collections H_2, H_3, $\cdots$ are defined which satisfy corresponding conditions and which are such that the closure of each element of H_{n+1} is a subset of the sum of the elements of H_n. See the figure. There

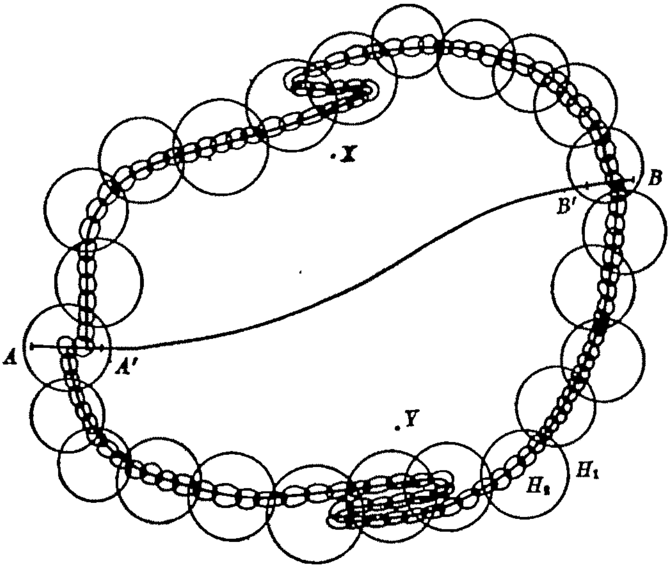

FIG. 1

are actually many more elements in H_1 and H_2 than are shown in the figure. The collections H_1, H_2, $\cdots$ are described in such a way that the common part of their sums is a simple closed curve not separating space. Hence, we shall show that the assumption that some arc separates space leads to the contradiction that some simple closed curve does not. We now consider the details of the proof.

Assume that an arc separates the point X from the point Y. Then there is an arc AB that separates X from Y such that no proper subarc of AB separates X from Y. Let D_X and D_Y be the complementary domains of AB containing X and Y respectively.

Description of collection H_1. Assume that a metric has been chosen for S and let ϵ_1 be a positive number less than one one-hundredth of the distance from A to B. We shall describe a collection H_1 of connected domains. The sum of the elements of H_1 will be denoted by H_1^*.

The collection H_1 of connected domains $h_{1,1}, h_{1,2}, \cdots, h_{1,t}$ $(t>100)$ will satisfy the following conditions:

(1) $h_{1,i}$ intersects $h_{1,j}$ only if i is equal to either $j-1$, j or $j+1$ $(h_{1,t+1}\equiv h_{1,1}$ and $h_{1,1-1}\equiv h_{1,t})$;

(2) $S-H_1^*$ is connected;

(3) some point of $S-H_1^*$ is accessible from $h_{1,i}$;

(4) the diameter of $h_{1,i}$ is less than ϵ_1;

(5) no connected subset of H_1^* that intersects $h_{1,i}$ and $h_{1,i+2}$ is of diameter less than $\epsilon_1/4$.

Denote by $D_1, D_2, \cdots, D_n$ the elements of a finite collection of connected domains covering S such that the diameter of each is less than $\epsilon_1/100$. Suppose that each of the domains $D_1, D_2, \cdots, D_j$ intersects the complement of D_X+D_Y, each of the domains $D_{j+1}, D_{j+2}, \cdots, D_k$ is a subset of D_X and each of the domains $D_{k+1}, D_{k+2}, \cdots, D_n$ is a subset of D_Y.

Let $\alpha_1, \alpha_2, \cdots, \alpha_j$ be a collection of arcs in the complement of D_X+D_Y+A+B such that α_i $(i=1, \cdots, j)$ intersects D_i and AB.

Let A' and B' be points of the arc AB such that AA' and $B'B$ are arcs that do not intersect $\alpha_1+\alpha_2+\cdots+\alpha_j$, each of the arcs AA' and $B'B$ is covered by an element of $D_1, D_2, \cdots, D_n$, some point of $A'B'$ is accessible from $D_Z (Z=X, Y)$ and if D_p is an element of $D_1, D_2, \cdots, D_n$ in D_Z, then no point of D_Z either disrupts D_p from AB' in D_Z+AB' or disrupts D_p from $A'B$ in $D_Z+A'B$.

Considering $D_X, D_{j+1}, \bar{D}_X\cdot(AA'-A'), \bar{D}_X\cdot(B'B-B')$ and $\bar{D}_X\cdot A'B'$ as D, D', M, N and E of the preceding lemma, we find that there is an arc α_{j+1} from $A'B'$ to D_{j+1} in $D_X+A'B'$ that does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$. Let D' be a component of $D_X-D_X\cdot\alpha_{j+1}$ that contains an open arc from a point of $AA'-A'$ to a point of $B'B-B'$. If D_{j+2} is not a subset of D', let α_{j+2} be an arc in $D_X-D'+A'B'$ from $A'B'$ to a point of D_{j+2}. If D_{j+2} is a subset of D', we shall apply the lemma to get an arc α_{j+2} from D_{j+2} to $A'B'$ in $D_X+A'B'$ such that $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$. The procedure is described in the following paragraph.

Let R be a point of D'. Since no point of D_X disrupts D_{j+2} from AB' in D_X+AB', there is an arc β from D_{j+2} to AB' in $D_X+AB'-R$. A subarc of β in $D'+AB'+\alpha_{j+1}-R$ intersects D_{j+2} and $AB'+\alpha_{j+1}$. Hence, R does not disrupt D_{j+2} from $AB'+\alpha_{j+1}$ in $D'+\bar{D}'\cdot(AB'+\alpha_{j+1})$. Also, R does not disrupt D_{j+2} from $A'B+\alpha_{j+1}$ in $D'+\bar{D}'\cdot(A'B+\alpha_{j+1})$. Applying the lemma, we find that there is an arc from D_{j+2} to $A'B'+\alpha_{j+1}$ in $D'+A'B'+\alpha_{j+1}$ that does not disrupt $AA'-A'$ from $B'B-B'$ in $D'+AB-A'B'$. Then there is an arc α_{j+2} in $D_X+A'B'$

from D_{j+2} to $A'B'$ such that $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$.

Likewise, we find that there is an arc α_{j+3} from D_{j+3} to $A'B'$ in $D_X+A'B'$ such that $\alpha_{j+1}+\alpha_{j+2}+\alpha_{j+3}$ does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$. A continuation of this process gives that there are arcs $\alpha_{j+1}, \alpha_{j+2}, \cdots, \alpha_k$ in $D_X+A'B'$ whose sum does not disrupt $AA'-A'$ from $B'B-B'$ in $D_X+AB-A'B'$ and such that $\alpha_p (p=j+1, \cdots, k)$ intersects D_p and $A'B'$. Also, there are arcs $\alpha_{k+1}, \alpha_{k+2}, \cdots, \alpha_n$ in $D_Y+A'B'$ whose sum does not disrupt $AA'-A'$ from $B'B-B'$ in $D_Y+AB-A'B'$ and such that α_p $(p=k+1, \cdots, n)$ intersects D_p and $A'B'$.

Let G be the collection of all domains g such that g is a component of the common part of the complement of $AB+\alpha_1+\alpha_2+\cdots+\alpha_n$ and some domain of $D_1, D_2, \cdots, D_n$. If P is a point of D_i, there is an arc in D_i from P to α_i. Hence, if g is an element of G, some point of $AB+\alpha_1+\alpha_2+\cdots+\alpha_n$ is accessible from g.

Let g_A and g_B be domains of diameters less than $\epsilon_1/100$ that cover $AA'-A'$ and $B'B-B'$ respectively but no point of $A'B'+\alpha_1+\alpha_2+\cdots+\alpha_n$. There exists a finite collection G_X of domains of G such that this collection but no collection of fewer elements of G satisfies the condition that the sum of the elements of G_X is a connected subset of D_X and intersects both g_A and g_B. Denote the elements of G_X by $g_2, g_3, \cdots, g_r$, where g_A intersects g_2, g_i $(i=2, \cdots, r-1)$ intersects g_{i+1} and g_r intersects g_B. Denote g_A by g_1 and g_B by g_{r+1}.

Also, there exists a finite collection G_Y of domains of G such that this collection but no collection of fewer elements of G satisfies the condition that the sum of the elements of G_Y is a connected subset of D_Y and intersects both g_A and g_B. Denote the elements of G_Y by $g_{r+2}, g_{r+3}, \cdots, g_s$ where g $(i=r+1, \cdots, s-1)$ intersects g_{i+1} and g_s intersects g_1.

Let E denote $g_1+g_2+\cdots+g_s$ plus all points that it separates from $A'B'$. Each component of the common part of E and an element of G intersects an element of $g_1, g_2, \cdots, g_s$. However, it is to be noted that no such component intersects two g_i's that do not belong to a consecutive set of three domains of $g_1, g_2, \cdots, g_s, g_1, g_2$. Denote by g_i' the sum of g_i and all such components that intersect g_i. We note that g_i' is a diameter less than $\epsilon_1/33$.

If three is a factor of s, denote the sum of the first three elements of $g_1', g_2', \cdots, g_s'$ by h_1, the sum of the next three elements by $h_2, \cdots$ and the sum of the last three elements by h_v. If three is a factor of $s+1$, then $h_1, h_2, \cdots$ and h_v are defined as before except that h_v is the sum of the last four elements of $g_1', g_2', \cdots, g_s'$ instead of the last three. If three is a factor of $s+2$, each of h_{v-1} and h_v is the sum of four elements of $g_1', g_2', \cdots, g_s'$. Since each h_i contains either g_A, g_B or an element g of G which does not intersect g_A+g_B and since a point of $AB+\alpha_1+\alpha_2+\cdots+\alpha_n$ is accessible from g, then a point of $A'B'+\alpha_1+\alpha_2+\cdots+\alpha_n$ is accessible from h_i. Now h_i is of diameter less than $\epsilon_1/8$ and the collection $h_1, h_2, \cdots, h_v$ satisfies conditions analogous to conditions (1), (2) and (3) to be satisfied by $h_{1,1}, h_{1,2}, \cdots, h_{1,t}$.

Let $h_{1,1}$ be the sum of $h_1, h_2, \cdots, h_n$ where some connected subset of $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects $h_1, h_2, \cdots$ and h_n but no such subset intersects both h_1 and h_{n+1}; let $h_{1,2}$ be the sum of $h_{n+1}, h_{n+2}, \cdots, h_m$ where some connected subset of $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects $h_{n+1}, h_{n+2}, \cdots$ and h_m but no such subset intersects both h_{n+1} and h_{m+1}; $\cdots$; and let $h_{1,t}$ be the sum of $h_{p+1}, h_{p+2}, \cdots, h_v$ where some connected subset of $h_1+h_2+\cdots+h_v$ of diameter less than $3\epsilon_1/4$ intersects h_{p+1}, $h_{p+2}, \cdots$ and h_v but no subset of $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects both h_{p+1} and h_1. We use $3\epsilon_1/4$ in the last case instead of $\epsilon_1/4$ in order to insure that no connected set in $h_1+h_2+\cdots+h_v$ of diameter less than $\epsilon_1/4$ intersects both h_{p+1} and h_1. The collection H_1 satisfies conditions (1), (2), (3), (4) and (5).

Description of collection H_2. Choose a positive number ϵ_2 less than one one-hundredth of the diameter of any connected set in H_1^* that intersects $h_{1,i}$ and $h_{1,i+2}$. We shall describe a collection H_2 of connected domains $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$ such that:

(1) $h_{2,i}$ intersects $h_{2,j}$ only if i is equal to either $j-1$, j or $j+1$ ($h_{2,s+1}\equiv h_{2,1}$ and $h_{2,1-1}\equiv h_{2,s}$);

(2) $S-H_2^*$ is connected;

(3) some point of $S-H_2^*$ is accessible from $h_{2,i}$;

(4) the diameter of $h_{2,i}$ is less than ϵ_2;

(5) no connected subset of H_2^* that intersects $h_{2,i}$ and $h_{2,i+2}$ is of diameter less than $\epsilon_2/4$;

(6) each $h_{1,i}$ contains 98 consecutive elements of $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$;

(7) if $H(n;\, i,\, j)$ denotes $h_{n,i-100}+\cdots+h_{n,i}+\cdots+h_{n,j}+\cdots+h_{n,j+100}$ and $h_{1,i}$ and $h_{1,j}$ intersect $h_{2,m}$ and $h_{2,k}$ respectively, then either $H(1;\, i,\, j)$ covers the closure of $H(2;\, m,\, k)$ and $H(1;\, j,\, i)$ covers the closure of $H(2;\, k,\, m)$ or $H(1;\, i,\, j)$ covers the closure of $H(2;\, k,\, m)$ and $H(1;\, j,\, i)$ covers the closure of $H(2;\, m,\, k)$.

Denote by C the component that contains $h_{1,5}+h_{1,6}$ of the common part of $h_{1,2}+h_{1,3}+\cdots+h_{1,9}$ and the complement of the closure of $h_{1,1}+h_{1,10}$. We shall show that if P is a point of $h_{1,5}+h_{1,6}$ and R is a point of $C-P$, then R does not disrupt P from $S-H_1^*$ in $S-H_1^*+C$. Let PQ be an arc in $S-R$ from P to a point Q of $S-H_1^*$. Let Q' be the first point of PQ in the order from P to Q on $S-C$. If PQ' intersects $h_{1,3}$, then there is an arc from $PQ'-Q'$ to $S-H_1^*$ in $S-H_1^*+h_{1,3}$ because a point of $S-H_1^*$ is accessible from $h_{1,3}$. Also, if PQ' intersects $h_{1,8}$, R does not disrupt P from $S-H_1^*$ in $S-H_1^*+C$. If PQ' intersects neither $h_{1,3}$ nor $h_{1,8}$, then Q' is a point of $S-H_1^*$. This demonstrates that R does not disrupt P from $S-H_1^*$ in $S-H_1^*+C$.

Let G be a finite collection of connected domains covering $h_{1,5}+h_{1,6}$ such that each intersects $h_{1,5}+h_{1,6}$ and is of diameter less than $\epsilon_2/1200$. No point of C disrupts an element of G from $S-H_1^*$ in $S-H_1^*+C$. Repeated applications of the preceding lemma give that there is a continuum K in $S-H_1^*+C$ that intersects $A'B'$ and each element of G but does not disrupt $h_{1,1}$ from $h_{1,10}$ in $h_{1,1}+h_{1,2}+\cdots+h_{1,10}$. Let G' be the set of all domains g' such that g' is either the common part of $h_{1,2}+h_{1,3}+\cdots+h_{1,9}$, the complement of K and an element of G, or the common part of the complement of K and h_i for i equal to 2, 3, 4, 7, 8 or 9.

There exists a finite collection G'' of elements of G' such that the sum of the elements of G'' is a connected domain intersecting $h_{1,1}$ and $h_{1,10}$ but the sum of no subcollection of G' having fewer elements than G'' is a connected domain intersecting $h_{1,1}$ and $h_{1,10}$. Assume that g_1 of G'' intersects $h_{1,1}$, g_i $(i=1,\cdots,r-1)$ intersects g_{i+1} and g_r intersects $h_{1,10}$.

There exists a collection $g_1', g_2', \cdots, g_r'$ of connected domains such that g_1' intersects $h_{1,1}$, g_i' intersects g_{i+1}', g_r' intersects $h_{1,10}$ and the closure of g_k' $(k=1,\cdots,r)$ is a subset of g_k.

Let E denote $h_{1,1}+g_1'+\cdots+g_r'+h_{1,10}+\cdots+h_{1,t}$ plus all points that it separates from $A'B'$. Each component of the common part of E and an element of G' intersects one of the domains $h_{1,1}$, $g_1', \cdots, g_r'$, $h_{1,10}$ but no such component intersects two of these domains that do not belong to a consecutive set of three of these domains. Add such components to the ones of $h_{1,1}$, g_i', $h_{1,10}$ that they intersect to form the sets h_1, g_i'', h_{10}. We note that the diameter of each g_i'' not intersecting $h_{1,2}+h_{1,3}+h_{1,4}+h_{1,7}+h_{1,8}+h_{1,9}$ is less than $\epsilon_2/400$.

Consecutive elements of $g_1'', g_2'', \cdots, g_r''$ may be combined by threes and fours in a manner previously described so as to get a collection $g_{1,1}, g_{1,2}, \cdots, g_{1,u}$ such that the collection $h_1, g_{1,1}, \cdots, g_{1,u}, h_{10}, h_{1,11}, \cdots, h_{1,t}$ satisfies conditions analogous to conditions (1), (2) and (3) to be satisfied by $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$. We note that the closure of $g_{1,i}$ $(2\leq i<u)$ is a subset of $h_{1,1}+h_{1,2}+\cdots+h_{1,10}$.

In a manner similar to that in which $h_{1,1}+h_{1,2}+\cdots+h_{1,10}$ was replaced by $h_1+g_{1,1}+\cdots+g_{1,n}+h_{10}$, we replace $h_{1,11}+\cdots+h_{1,20}$ by $h_{11}+g_{11,1}+\cdots+g_{11,n}+h_{20}, \cdots$ and $h_{1,t-m}+\cdots+h_t$ $(9\leqq m\leqq 18)$ by $h_{t-m}+g_{t-m,1}+\cdots+g_{t-m,n}+h_t$.

Let $g_{i,0}$ be the fourth element of $g_{i,1}, g_{i,2}, \cdots, g_{i,n}$ which follows all of these elements that intersect $h_{1,i+3}$. We note that $g_{i,0}$, the three domains immediately preceding $g_{i,0}$, and the three domains immediately following $g_{i,0}$ are each a subset of $h_{1,i+4}$ of diameter less than $\epsilon_2/100$.

In the manner described above, replace $g_{1,0}+\cdots+h_{10}+h_{11}+\cdots+g_{11,0}$ by $g'_{1,0}+h'_{2,2}+\cdots+h'_{2,r}+g'_{11,0}$; replace $g'_{11,0}+\cdots+g_{21,0}$ by $h'_{2,r+1}+\cdots+h'_{2,v}+g'_{21,0}; \cdots$ and replace $g'_{t-m,0}+\cdots+g'_{1,0}$ by $h'_{2,s}+\cdots+h'_{2,u}+h'_{2,1}$. The closure of $h'_{2,2}+\cdots+h'_{2,r}$ is a subset of $h_{1,4}+\cdots+h_{1,20}, \cdots$ and the closure of $h'_{2,s}+\cdots+h'_{2,u}+h'_{2,1}$ is a subset of $h_{1,t-m+3}+\cdots+h_{1,10}$. Consecutive elements of $h'_{2,1}, h'_{2,2}, \cdots, h'_{2,u}$ may be combined in a manner previously described so as to form a collection H_2 of connected domains $h_{2,1}, h_{2,2}, \cdots, h_{2,s}$ satisfying conditions (1), (2), (3), (4), (5), (6) and (7).

Description of simple closed curve J. For each positive integer i greater than one, we define a collection H_i of connected domains $h_{i,1}, h_{i,2}, \cdots, h_{i,n}$ satisfying conditions analogous to those satisfied by H_2 where ϵ_i is a positive number less than one one-hundredth the diameter of any connected set in H_{i-1}^* intersecting $h_{i-1,j}$ and $h_{i-1,j+2}$. We shall show that the common part J of $H_1^*, H_2^*, \cdots$ is a simple closed curve that does not separate S.

As the closure of H_{i+1}^* is a connected subset of H_i^* (condition 7) and as each $h_{i,j}$ contains an element of H_{i+1} (condition 6), then J is a nondegenerate continuum. This continuum does not separate space because the complement of each H_i^* is connected.

To show that J is a simple closed curve, we shall show that any pair of points P, Q of J separates it. Suppose that h_{i,P_i} and h_{i,Q_i} are elements of $h_{i,1}, \cdots, h_{i,n}$ that contain P and Q respectively. For convenience in notation, we shall assume that it is $H(i;\, P_i, Q_i)$ that covers the closure of $H(i+1;\, P_{i+1}, Q_{i+1})$ and that it is $H(i;\, Q_i, P_i)$ that covers the closure of $H(i+1;\, Q_{i+1}, P_{i+1})$. If J_{PQ} is the common part of $H(1;\, P_1, Q_1), H(2;\, P_2, Q_2), \cdots$, we find that J is the sum of two continua J_{PQ} and J_{QP} which have only P and Q in common.

Hence, the assumption that an arc separates S leads to the conclusion that some simple closed curve does not.

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ON THE EXTENSION OF HOMEOMORPHISMS ON THE INTERIOR OF A TWO CELL

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The subject under discussion in this paper is the study of the existence and properties of extensions of homeomorphisms of the interior I of a two cell with boundary C onto a plane bounded region. Particular emphasis will be placed on the action of the extension on C. Application of the topological results will then be made to conformal maps on the interior of the unit circle.

The hypothesis that $f(I)=R$ is a homeomorphism of the interior I of a two cell with boundary C onto a plane bounded region R with boundary $F(R)$ will be assumed throughout the paper. The usual terminology of transformation theory will be used: the transformation $g(A)=B$ is said to be light if each $f^{-1}(x)$, $x\in B$, is totally disconnected, and non-alternating if for each x, $y\in B$, $f^{-1}(x)$ does not separate $f^{-1}(y)$.[1]

1. Action of extensions on the boundary.

THEOREM 1. *Suppose f is uniformly continuous. Then there exists a continuous extension g of f such that $g(\bar{I})=\bar{R}$ and $g=f$ on I. Moreover $g(C)=F(R)$ is a non-alternating transformation.*

PROOF. The existence of the extension is well known, since f is uniformly continuous. Moreover $g(C)=F(R)$. To prove this, we notice that $g(\bar{I})$ is compact and must contain $\bar{R}$. Since $g(I)=R$, then $g(C)\supset F(R)$. Suppose $g(C)\neq F(R)$; then there is a point $x\in C$ such that $g(x)\in R$. Let $(x_i)\to x$, $x_i\in I$; then $(f(x_i))\to g(x)$. Since $g(x)\in R$, then $(x_i)\to f^{-1}g(x)\in I$. This is a contradiction and $g(C)=F(R)$.

Suppose $g(C)=F(R)$ is not non-alternating; then there exist points x_1, x_2, y_1, $y_2\in C$ such that $g(x_1)=g(x_2)$, $g(y_1)=g(y_2)$, $g(x_1)\neq g(y_1)$, and x_1+x_2 separates y_1+y_2 on C. Let A_1 and A_2 be interiors of arcs x_1x_2 and y_1y_2 respectively, where $x_1x_2\subset I+x_1+x_2$, $y_1y_2\subset I+y_1+y_2$, $x_1x_2\cdot y_1y_2=p=A_1A_2$. Both $g(x_1x_2)$ and $g(y_1y_2)$ are simple closed curves and $g(x_1x_2)\cdot g(y_1y_2)=f(p)$. Moreover points of $g(y_1y_2)$ are contained both in the interior and exterior of $g(x_1x_2)$. For A_1 separates A_2 into two parts, one in each component of $I-A_1$; then $f(A_1)$ separates $f(A_2)$ into two parts, one in each component of $R-f(A_1)$. But one compo-

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942, pp. 127–129, 138–140, 165–170 for properties of non-alternating maps.

nent of $R-f(A_1)$ is contained in the interior of $f(A_1)+g(x)$, the other in the exterior. This furnishes a contradiction, since $f(p)$ would then be a cut point of $g(y_1y_2)$, and the theorem is proved.

DEFINITION. Let R be a plane region with boundary $F(R)$; we say that A is a *cut into* R if $A\subset R$ and if there exists a point $x\in F(R)$ such that $A+x$ is an arc. We shall call x the *end* of the cut A.

DEFINITION. We say that f possesses *property* P if

(1) A is a cut into R implies $f^{-1}(A)$ is a cut into I, and if

(2) A_1 and A_2 being cuts into R with distinct ends implies $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I with distinct ends.

LEMMA 1. *A necessary and sufficient condition that the boundary* $F(R)$ *of a bounded simply connected plane region* R *be locally connected is that for each sequence* (x_i), $x_i\in R$, *such that* (x_i) *converges to* x, $x\in F(R)$, *there exists an infinite subsequence* (y_j) *of* (x_i) *and an arc* a_1x, $a_1x\subset R+x$, *such that* $a_1x\supset(y_j)$.

PROOF. Let R be a bounded simply connected plane region with locally connected boundary $F(R)$. To show the necessity portion of the lemma, we proceed to show two preliminary statements.

(1) There exists a sequence (R_i) of regions contained in R such that $R_{i+1}\subset R_i$, each R_i possesses property S,[2] each R_i contains an infinite number of points of (x_i), and the diameter of R_i goes to 0 with $1/i$.

To prove (1), we note first that R possesses property S, since $F(R)$ is locally connected.[3] Then R may be decomposed into a finite number of regions, each with property S and each of diameter less than 1/2.[4] One of these, say R_1, contains an infinite number of points of (x_i). Then R_1 possesses property S and hence may be decomposed into a finite number of regions possessing property S and of diameter less than 1/4. One of these, say R_2, contains an infinite number of points of (x_i). By continuing the process, we obtain the required (R_i).

(2) Let $A=pq-p$ be a cut into a region R_i of (1), $p\in R\cdot F(R_i)$, $q\in R_i$. Let $R_j\subset R_i$ be such that $\overline{R}_j\cdot A=0$, and let $r\in R_j$. Then there exists an arc qr, $qr\subset(R_i-A)+q$, such that $qr\cdot F(R_j)$ consists of a single point, where $F(R_n)$ is used to denote the boundary of R_n.

To prove (2) we note first that R_i-A is connected, since a cut into a region does not disconnect it. Furthermore q is accessible from R_i-A since a sufficiently small neighborhood of q is contained in R_i-A except for points of the arc pq. Let qs be an arc contained in

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[2] R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fund. Math. vol. 3 (1922) pp. 232–237.

[3] R. L. Moore, loc. cit. p. 235.

[4] G. T. Whyburn, loc. cit. p. 21.

$(R_i - A) + q$, where $z \in R_j$. Let y be the first point of $qz \cdot F(R_j)$ in the order q, z. Then $qy \subset (R_i - R_i \cdot \overline{R}_j) + y$, $y \in F(R_j)$. Since R_j possesses property S, each boundary point is accessible from R.[5] Then let yr be an arc, $yr \subset R_j + y$. Then $qr = qy + yr$ is the desired arc, and (2) is justified.

We now proceed to show the necessity portion of the lemma. Let $y_1 \in R_1 \cdot (x_i)$. Let R_2' be such that R_2' is a member of the sequence (R_i) and $y_1 \notin \overline{R}_2'$. Such an R_2' exists since R_i is arbitrarily close to $x \in F(R)$ for i sufficiently large. Let $y_2 \in R_2' \cdot (x_i)$. Let $y_1y_2 = A$ be an arc contained in R and such that $A \cdot F(R_2')$ consists of a single point. Such an arc may be constructed by an argument similar to the one used in (2). Let R_3' be such that R_3' is a member of (R_i) and $y_1y_2 \cdot \overline{R}_3' = 0$; let $y_3 \in R_3' \cdot (x_i)$. Since $R_2' \cdot y_1y_2$ is a cut into R_2' we may use (2) to obtain an arc y_2y_3 contained in $(R_2' - R_2' \cdot y_1y_2) + y_2$ and intersecting $F(R_3')$ in a single point. Let R_4' be a member of (R_i) such that $\overline{R}_4' \cdot (y_1y_2 + y_2y_3) = 0$, let $y_4 \in R_4' \cdot (x_i)$, and continue the process indefinitely. We thus obtain a sequence $(y_{i-1}y_i)$ of arcs, $y_{i-1}y_i \subset R$, $y_i \in (x_i)$, $y_{i-1}y_i$ intersects $y_1y_2 + y_2y_3 + \cdots + y_{i-2}y_{i-1}$ in the single point y_{i-1}, and $y_{i-1}y_i$ approaches x as a limit. Let $A = \sum_{i=2}^{\infty} y_{i-1}y_i + x$. Then every point of A except y_1 and x are cut points. Moreover, A is compact, since the arcs $y_{i-1}y_i$ approach x as limit. Then A is an arc satisfying all the conditions, and the necessity is shown.

Suppose for each sequence the condition of the lemma is satisfied and suppose $F(R)$ is not locally connected. Then there is a point $x \in F(R)$ such that x is not regularly accessible from R.[6] That is, for some $e > 0$ there exists a sequence (x_i), $x_i \in R$ and $(x_i) \to x$, such that x_i may not be joined to x by an arc of diameter less than e in $R + x$. By hypothesis, there exists an arc $a_1x \subset R + x$ such that a_1x contains an infinite subsequence of (x_i). This arc may be supposed to be of diameter less than e. This is a contradiction and the theorem is proved.

LEMMA 2. *If* $F(R)$ *is locally connected and* f *possesses property* P, *then* f *is uniformly continuous.*

PROOF. Suppose f is not uniformly continuous. Then for some $e > 0$ and each d_i of a sequence $(d_i) \to 0$, there exist x_i, $y_i \in I$ such that $\rho(x_i, y_i) < d_i$ and $\rho(f(x_i), f(y_i)) \geq e$. We may suppose that $(x_i) \to x$; then $(y_i) \to x$ and $x \in C$. We may also suppose that $(f(x_i)) \to z_1$ and $(f(y_i)) \to z_2$. Then $z_1 \neq z_2$. Moreover $z_1 \in F(R)$ and $z_2 \in F(R)$. For suppose $z_1 \in R$; then $(x_i) \to f^{-1}(z_1) \in I$. But this is impossible; thus z_1, $z_2 \in F(R)$. By Lemma 1 there exist cuts A_1 and A_2 which contain infinite subsequences of $(f(x_i))$ and $(f(y_i))$ respectively and which have z_1 and z_2 as ends. By property P, $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I with different ends. Since each of them must have x as an end, this is a contradiction and the lemma is established.

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[5] G. T. Whyburn, loc. cit. p. 111.

[6] G. T. Whyburn, loc. cit. p. 112.

THEOREM 2. *Let $F(R)$ be locally connected. Then a necessary and sufficient condition that f have property* P *is that there exist an extension g of f, where $g(\bar{I})=\bar{R}$, $g=f$ on I, and $g(C)=F(R)$ is light and non-alternating.*

PROOF. Suppose f has property P. Then by Lemma 2 and Theorem 1, g exists and $g(C)=F(R)$ is non-alternating. Suppose $g(C)=F(R)$ is not light; let $x\in F(R)$ be such that $g^{-1}(x)$ is not totally disconnected. Let M be a nondegenerate component of $g^{-1}(x)$. Since $I+C$ is a two cell, there exists a set $N\subset I$ which is homeomorphic to the graph of the equation $y=\sin 1/x$, $0<x\leqq 1$, and such that $\bar{N}=N+M$. Then $f(N)$ is a cut into R with end x. Since $f^{-1}f(N)=N$ is not a cut into I, then f does not possess property P, and we have a contradiction. This shows the necessity of the theorem.

Suppose f may be extended to g, where $g(\bar{I})=\bar{R}$, $g=f$ on I, and $g(C)=F(R)$ is light and non-alternating. Let A be a cut into R with end x. Suppose $A=a_1x-x$. Let $(x_i)\to x$, where $x_i\in A$ and where x_i precedes x_{i+1} on the arc a_1x ordered from a_1 to x. We may also suppose that $(f^{-1}(x_i))$ converges to a point y. Let $A_i=x_ix_{i+1}$ be the subarc of A joining x_i to x_{i+1}. Then $\lim\sup f^{-1}(A_i)\subset g^{-1}(x)$ by the continuity of g. Since $\lim\inf f^{-1}(A_i)\supset y$, then $\lim\inf f^{-1}(A_i)\neq 0$ and $\lim\sup f^{-1}(A_i)$ is connected.[7] Let $B=f^{-1}(A)$. Then $\lim\sup f^{-1}(A_i)=\bar{B}-B$. Then $\bar{B}-B$ is both connected and totally disconnected, and hence is a single point. Then $B=f^{-1}(A)$ is a cut into I.

Suppose A_1 and A_2 are cuts into R with ends x_1 and x_2 respectively, $x_1\neq x_2$, while $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I, both with end y. Then $g(f^{-1}(A_1))=A_1$ and $g(f^{-1}(A_2))=A_2$ are cuts into I, both with end $g(y)$. This is a contradiction and the theorem is established.

2. Application to conformal maps.

THEOREM 3. *Let I be the interior of the unit circle, and let f be a one-to-one conformal map. Then f possesses property* P.[8]

COROLLARY 3.1. *Let $f(I)=R$ be a one-to-one conformal map of the interior I of the unit circle C onto a bounded plane region R with bound-*

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[7] G. T. Whyburn, loc. cit. p. 14.

[8] For proof, see C. Carathéodory, *Conformal representation*, London, 1932, pp. 82–85.

ary $F(R)$. Then a necessary and sufficient condition that f be extensible to g on $\bar{I}$ is that $F(R)$ be locally connected. In case $F(R)$ is locally connected, and if g denotes the extension, then the mapping $g(C)=F(R)$ is light and non-alternating.

PROOF. The sufficiency follows from Theorem 3 and Theorem 2. If f is extensible to g on $\bar{I}$, then $g(C)=F(R)$ gives a continuous map of the locally connected continuum C onto $F(R)$. Then $F(R)$ itself must be locally connected.

COROLLARY 3.2. *Let $f(I)=R$ be a one-to-one conformal map of the interior I of the unit circle onto a bounded plane region R with boundary $F(R)$. Then a necessary and sufficient condition that f be uniformly continuous is that $F(R)$ be locally connected.*

PROOF. If f is uniformly continuous, then f is extensible to g on $\bar{I}$ and $F(R)$ is locally connected by 3.1. If $F(R)$ is locally connected, then f is extensible to g on $\bar{I}$ by 3.1. Then g is uniformly continuous and so is f.

COROLLARY 3.3. THE OSGOOD-CARATHÉODORY THEOREM. *Let $f(I)=R$ be a one-to-one conformal map of the interior I of the unit circle C onto the interior R of a simple closed curve J. Then f may be extended to g on $\bar{I}$ such that $g(\bar{I})=\bar{R}$ is a homeomorphism.*

PROOF. The mapping $g(C)=J$ is light and non-alternating; the only light and non-alternating transformation of one simple closed curve onto another is a homeomorphism.[9] Hence $g(\bar{I})=\bar{R}$ is a homeomorphism.

Note. Another example of a map with property P is the inverse of the relative distance transformation[10] on a plane bounded region with property S. The following topological theorem is usually proven by means of this transformation:[11] *if B is any boundary curve, there exists a light and non-alternating transformation $g(C)=B$, where C is a simple closed curve.* If B is restricted to a boundary curve which contains at least one simple closed curve, we may prove the theorem by means of conformal maps as follows. We may suppose that B is the boundary of a bounded plane region R. Then there exists a 1-1 conformal map f of the interior of the unit circle onto R, since R is simply connected. Then f is extensible to g on $\bar{I}$ and $g(C)=B$ is light and non-alternating.

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[9] G. T. Whyburn, loc. cit. p. 165.

[10] G. T. Whyburn, loc. cit. pp. 155–162.

[11] G. T. Whyburn, loc. cit. p. 166.

NOTE ON THE KUROSCH-ORE THEOREM

R. P. DILWORTH

1. Introduction. The Kurosch-Ore theorem[1] asserts that if an element of a modular lattice has two decompositions into irreducibles, then each irreducible of one decomposition may be replaced by a suitably chosen irreducible from the other decomposition. It follows that the number of irreducibles in the two decompositions is the same.

The purpose of the present note is to study the manner in which the irreducibles of two decompositions can replace one another. Now from the Kurosch-Ore theorem it is not even clear that each irreducible of one decomposition is suitable for replacing some irreducible of the other decomposition. However, this follows from the following precise theorem:

THEOREM 1. *Let a be an element of a modular lattice and let $a=q_1\cap\cdots\cap q_n=q_1'\cap\cdots\cap q_n'$ be two reduced decompositions into irreducibles. Then the q's may be renumbered in such a way that*

$$a = q_1\cap\cdots\cap q_{i-1}\cap q_i'\cap q_{i+1}\cap\cdots\cap q_n, \qquad i = 1, \cdots, n.$$

Along the same line of ideas, the following theorem on simultaneous replacement is also proved.

THEOREM 2. *Let a be an element of a modular lattice and let $a=q_1\cap\cdots\cap q_n=q_1'\cap\cdots\cap q_n'$ be two reduced decompositions into irreducibles. Then for each q_i, there exists q_j' such that q_j' can replace q_i in the first decomposition and q_i can replace q_j' in the second decomposition.*

On the other hand, an example is given which shows that, in general, it is impossible to renumber the q's in such a way that simultaneously q_i may replace q_i' and q_i' replace q_i.

As the principal tool in the investigation we introduce the concept of a *superdivisor* r of an element a. r has the fundamental property that its crosscut with any proper divisor of a is never equal to a. The superdivisors of a are closed under crosscut and indeed form a dual-ideal $\mathfrak{r}_a$ which properly divides a.

A surprising by-product of the investigation is the fact that in a

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[1] A simple proof is given in Birkhoff [1, p. 54]. Numbers in brackets refer to the references cited at the end of the paper.

modular lattice satisfying the ascending chain condition, $\mathfrak{r}_a$ can be used to prove the existence of covering ideals. Thus, in this case, the customary use of transfinite induction can be avoided.

2. **Properties of superdivisors.** Let M denote a modular lattice of elements $a, b, c, \cdots$. $a \supseteq b$ will denote ordinary lattice inclusion while $a \supset b$ will denote proper inclusion. We recall that an element q of M is (crosscut) irreducible if $q = x \cap y$ implies either $q = x$ or $q = y$.

DEFINITION 1. A divisor r of a is a *superdivisor* of a if $r \cap x = a$ implies $x = a$ for all x in M.

The following lemmas give the basic properties of superdivisors.[2]

LEMMA 1. *If r is a superdivisor of a and $s \supseteq r$, then s is a superdivisor of a.*

For $s \cap x = a$ implies $r \cap x = a$ implies $x = a$.

LEMMA 2. *If r and s are superdivisors of a, then $r \cap s$ is a superdivisor of a.*

For $(r \cap s) \cap x = a$ implies $r \cap (s \cap x) = a$ implies $s \cap x = a$ implies $x = a$.

COROLLARY. *The superdivisors of a form a dual-ideal $\mathfrak{r}_a$ of M.*

LEMMA 3. *If q is an irreducible divisor of a and $x \supset q$, then x is a superdivisor of a.*

For if $x \cap y = a$, then $q = q \cup a = q \cup (x \cap y) = x \cap (q \cup y)$ by the modular law. Since q is irreducible and $q \neq x$, it follows that $q = q \cup y$. Hence $y = q \cap y = q \cap x \cap y = q \cap a = a$. Thus x is a superdivisor of a.

Now if $a = q_1 \cap \cdots \cap q_n$ is a reduced decomposition of a into irreducibles, we shall set $Q_i = q_1 \cap \cdots \cap q_{i-1} \cap q_{i+1} \cap \cdots \cap q_n$. Clearly $a = q_i \cap Q_i$ and $Q_i \neq a$.

LEMMA 4. *Let $a = q_1 \cap \cdots \cap q_n$ be a reduced decomposition into irreducibles. Then if r is a superdivisor of a, $q_i \cup (r \cap Q_i)$ is also a superdivisor of a.*

By Lemma 3 if $q_i \cup (r \cap Q_i)$ is not a superdivisor of a, then $q_i \supseteq r \cap Q_i$. But then $r \cap Q_i = r \cap q_i \cap Q_i = a$. Hence $Q_i = a$ which contradicts $Q_i \neq a$.

LEMMA 5. *Let $a = q_1 \cap \cdots \cap q_n$ be a reduced decomposition into ir-*

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[2] If the descending chain condition holds, it is easy to show that r is a superdivisor if and only if $r \supseteq u_a$ where u_a is the union of the elements covering a. Cf. Dilworth [**2**, p. 288].

reducibles and let $x\supseteq r_1\cap Q_1,\ r_2\cap Q_2,\ \cdots,\ r_i\cap Q_i$ *where* $r_1, \cdots, r_i$ *are superdivisors of* a. *Then* $x\supseteq r\cap q_{i+1}\cap\cdots\cap q_n$ *where* r *is a superdivisor of* a.

Now clearly $x\supseteq r\cap q_1\cap\cdots\cap q_n$ for any superdivisor r of a. Let k be maximal such that $x\supseteq r\cap q_k\cap\cdots\cap q_n$ for some superdivisor r. Suppose $k\leqq i$. Then by the hypothesis of the lemma $x\supseteq r_k\cap Q_k$. Let $r'=r\cap r_k$. Then $x\supseteq(r'\cap q_k\cap\cdots\cap q_n)\cup(r'\cap Q_k)=r'\cap q_{k+1}\cap\cdots\cap q_n\cap(q_k\cup(r'\cap Q_k))$. But $q_k\cup(r'\cap Q_k)$ is a superdivisor of a by Lemma 4. Hence $r'\cap(q_k\cup(r'\cap Q_k))=r''$ is a superdivisor of a by Lemma 2. But then $x\supseteq r''\cap q_{k+1}\cap\cdots\cap q_n$ contrary to the maximal property of k. Thus $k>i$ and the conclusion of the lemma follows.

3. **Decomposition theory.** The application of superdivisors to decomposition problems rests on the following lemma:

LEMMA 6. *Let* $a=q_1\cap\cdots\cap q_n$ *be a reduced decomposition of* a. *Then* q_i *may be replaced by an irreducible divisor* q *of* a *if and only if* $q\supseteq r\cap Q_i$ *is false for every superdivisor* r *of* a.

Let us suppose that q can replace q_i. Then $a=q\cap Q_i$. Hence if $q\supseteq r\cap Q_i$ for some superdivisor r, then $r\cap Q_i=r\cap q\cap Q_i=r\cap a=a$ and $Q_i=a$ which is impossible. Thus $q\supseteq r\cap Q_i$ fails for every superdivisor r. Conversely suppose $q\supseteq r\cap Q_i$ holds for no superdivisors r. Then

$$q\supseteq q\cap Q_i=(q\cap Q_i)\cup(q_i\cap Q_i)=[q_i\cup(q\cap Q_i)]\cap Q_i.$$

Hence $q_i\cup(q\cap Q_i)$ is not a superdivisor of a and by Lemma 3 we have $q_i\supseteq q\cap Q_i$. Thus $q\cap Q_i=q\cap q_i\cap Q_i=a$ and q can replace q_i in the decomposition.

The theorems stated in the introduction can now be proved.

PROOF OF THEOREM 1. Let S_i' denote the set of irreducibles of the second decomposition which can replace q_i in the first decomposition. Now suppose that there are k of the sets S_i' which together contain less than k irreducibles. Renumbering if necessary, we can suppose that $S_1', \cdots, S_k'$ are composed of the irreducibles $q_1', \cdots, q_l'$ where $l<k$. It follows that q_j' cannot replace q_i if $j>l$ and $i\leqq k$. Hence by Lemma 6, $q_j'\supseteq r_{ji}\cap Q_i$ for some superdivisor r_{ji} of a if $j>l$ and $i\leqq k$. From Lemma 5 we conclude that $q_j'\supseteq r_j\cap q_{k+1}\cap\cdots\cap q_n$ for some superdivisor r_j of a if $j>l$. Thus $q_{l+1}'\cap\cdots\cap q_n'\supseteq r\cap q_{k+1}\cap\cdots\cap q_n$ where $r=r_{l+1}\cap\cdots\cap r_n$ is a superdivisor of a. But then

$$a=q_1'\cap\cdots\cap q_l'\cap q_{l+1}'\cap\cdots\cap q_n'\supseteq r\cap q_1'\cap\cdots\cap q_l'\cap q_{k+1}\cap\cdots\cap q_n\supseteq a.$$

Hence $a=r\cap q_1'\cap\cdots\cap q_l'\cap q_{k+1}\cap\cdots\cap q_n$. Since r is a superdivisor of a, we have

$$a = q_1' \cap \cdots \cap q_l' \cap q_{k+1} \cap \cdots \cap q_n.$$

Since $l<k$, the number of components in this decomposition is less than n, contrary to the Kurosch-Ore theorem. Thus every k of the sets S_i' contain at least k irreducibles. It follows from the Radó-Hall theorem on representatives of sets that there exists a distinct set of representatives for the sets $S_1', \cdots, S_k'$. Renumbering if necessary, we may suppose that these representatives are $q_1', \cdots, q_k'$. But then q_i' can replace q_i and the theorem is proved.

Proof of Theorem 2. Renumbering if necessary, we may suppose that $q_1', \cdots, q_l'$ can replace q_i while the others cannot. According to Lemma 6, $q_j' \supseteq r_j \cap Q_i$, $j=l+1, \cdots, n$, where r_j is a superdivisor of a. Now suppose that q_i can replace none of the irreducibles $q_1', \cdots, q_l'$ in the second decomposition. Again by Lemma 6 we have $q_i \supseteq r_j' \cap Q_j'$, $j=1, \cdots, l$. From Lemma 5 we conclude that $q_i \supseteq r' \cap q_{l+1}' \cap \cdots \cap q_n'$ for some superdivisor r'. Now $q_{l+1}' \cap \cdots \cap q_n' \supseteq r_{l+1} \cap \cdots \cap r_n \cap Q_i = r \cap Q_i$ where r is a superdivisor of a. Hence $q_i \supseteq r' \cap r \cap Q_i$ where $r' \cap r$ is a superdivisor of a. But then $r' \cap r \cap Q_i = r' \cap r \cap Q_i \cap q_i = a$ and $Q_i = a$ contrary to hypothesis. Hence q_j' can be replaced by q_i for some $j \leqq l$. Thus q_i and q_j' can replace one another.

In order to see that a sharper theorem on simultaneous replacement cannot be proved in general, consider the lattice of subspaces of the seven-point projective plane. If $1, \cdots, 7$ denote the points, let the lines (and the points they contain) be denoted by $l_1(124)$, $l_2(235)$, $l_3(346)$, $l_4(457)$, $l_5(156)$, $l_6(267)$, $l_7(137)$. $l_1, \cdots, l_7$ are the irreducibles of the lattice. Let us consider the decompositions of the null space z. We have

$$z = l_1 \cap l_2 \cap l_3 = l_5 \cap l_6 \cap l_7.$$

Now the possible sets of replacements of l_1, l_2, l_3 respectively are (l_5, l_6, l_7), (l_6, l_5, l_7), and (l_6, l_7, l_5). But l_1, l_2, l_3 is a possible set of replacements only for (l_5, l_7, l_6), (l_7, l_5, l_6), and (l_7, l_6, l_5). Hence it is not possible in this case to renumber the irreducibles in such a way that corresponding irreducibles can replace one another.

4. Existence of covering ideals.

It is well known that the lattice of dual-ideals of a lattice M contains M as the sublattice of principal ideals.[3] Hence if $\mathfrak{a}$ is a dual-ideal, by $\mathfrak{a} \supseteq a$ we shall mean $\mathfrak{a} \supseteq (a)$ where

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[3] For the general properties of dual-ideals used in this paper see Dilworth [3, pp. 329–331].

(a) is the principal ideal generated by a. Also $\mathfrak{a} > a$ ($\mathfrak{a}$ "covers" a) means $\mathfrak{a} \supset a$ and no ideal exists which properly contains $\mathfrak{a}$ and is properly contained in (a). We give a proof of the existence of covering ideals which does not require transfinite induction.

THEOREM 3. *Let an element a of a modular lattice M have a decomposition into irreducibles. Then if $\mathfrak{a} \supseteq a$, there exists a dual-ideal $\mathfrak{p}$ such that $\mathfrak{a} \supseteq \mathfrak{p} > a$.*

PROOF. Let $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{r}_a$ where $\mathfrak{r}_a$ is the dual-ideal of all superdivisors of a. Then $\mathfrak{a}' \neq (a)$. For if $\mathfrak{a}' = (a)$, then $x \cap r = a$ where $x \in \mathfrak{a}$ and r is a superdivisor of a. But then $x = a$ and $\mathfrak{a} = (a)$ contrary to $\mathfrak{a} \supset a$. Now let $a = q_1 \cap \cdots \cap q_n$ be a reduced decomposition of a into irreducibles. Clearly $\mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} \supseteq \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_i$. Suppose $\mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} = \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_i$. Then $\mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} \cap q_{i+1} \cap \cdots \cap q_n = a$ and hence $r \cap Q_i = a$ where r is a superdivisor of a. Thus $Q_i = a$ contrary to assumption. Next suppose that $\mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} \supset \mathfrak{b} \supset \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_i$. Then since the lattice of dual-ideals is modular we have $\mathfrak{b} = \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} \cap (\mathfrak{b} \cup q_i)$. Let $b \in \mathfrak{b}$. If $q_i \supseteq b$, then $q_i \supseteq \mathfrak{b}$ and $\mathfrak{b} = \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} \cap q_i$ contrary to hypothesis. Hence $q_i \cup b$ is a proper divisor of q_i for every $b \in \mathfrak{b}$. By Lemma 3, $q_i \cup b$ is a superdivisor of a for every $b \in \mathfrak{b}$. Hence $q_i \cup b \in \mathfrak{r}_a$ for every $b \in \mathfrak{b}$. Thus $q_i \cup \mathfrak{b} \supseteq r_a$. But then $\mathfrak{b} = \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1}$ contrary to assumption. Thus $\mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{i-1} > \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_i$. But then $\mathfrak{r}_a > \mathfrak{r}_a \cap q_1 > \cdots > \mathfrak{r}_a \cap q_1 \cap \cdots \cap q_{n-1} > a$ is a finite complete chain joining $\mathfrak{r}_a$ to a. By the general theory of modular lattices (Birkhoff [1]) it follows that the quotient lattice $\mathfrak{r}_a/a$ is of finite dimension. Since $\mathfrak{a}' \in \mathfrak{r}_a/a$ we have $\mathfrak{a}' \supseteq \mathfrak{p} > a$ for some dual-ideal $\mathfrak{p}$ and the theorem is proved.

Now if the ascending chain condition holds in a modular lattice M, then every element has a decomposition into irreducibles and hence, by Theorem 3, there exist dual-ideals covering a for every a not the unit of M.

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DERIVATIVES OF COMPOSITE FUNCTIONS

JOHN RIORDAN

1. **Introduction.** The object of this note is to show the relation of the Y polynomials of E. T. Bell [1],[1] first to the formula of diBruno for the nth derivative of a function of a function, then to the more general case of a function of many functions. The subject belongs to the algebra of analysis in the sense of Menger [4]; all that is asked is the relation of the derivative of the composite function to the derivatives of its component functions when they exist and no questions of analysis are examined.

2. **Function of a single function.** Following Dresden [3], take the composite function in the form:

$$F(x) = f[g(x)]; \tag{1}$$

and for convenience write:

$$D_x^s F(x) \equiv F_s, \qquad [D_u^s f(u)]_{u=g(x)} \equiv f_s, \qquad D_x^s g(x) \equiv g_s,$$

with $D_x = d/dx$.

Then, the first few derivatives of $F(x)$ are as follows:

$$F_1 = f_1 g_1, \qquad F_2 = f_1 g_2 + f_2 g_1^2, \qquad F_3 = f_1 g_3 + 3 f_2 g_2 g_1 + f_3 g_1^3.$$

If these are generalized to

$$F_n = \sum_{i=1}^{n} F_{n,i} f_i \tag{2}$$

the coefficients $F_{n,i}$ are dependent only on the derivatives g_1 to g_i, and hence may be determined by specialization of f. A convenient choice used by Schlömilch [6] is $f(g) = \exp(ag)$, so that

$$f_i = a^i \exp ag$$

and

$$e^{-ag} F_n = e^{-ag} D_x^n e^{ag} = \sum_{i=1}^{n} F_{n,i}(g_1, \cdots, g_i) a^i, \tag{3}$$

a generating identity for the $F_{n,i}$, closely related to the definition equation of the Y polynomials (Bell, loc. cit. p. 269), namely

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$e^{-y} D_x^n e^y = Y_n(y_1, \cdots, y_n), \qquad y_s = D_x^s y.$$

Indeed

$$Y_n(ag_1, \cdots, ag_n) = \sum_{i=1}^{n} F_{n,i}(g_1, \cdots, g_i)a^i \tag{4}$$

and, by combination of this with (2),

$$F_n = Y_n(ag_1, \cdots, ag_n), \qquad (a^i \equiv f_i). \tag{5}$$

For concreteness, the first few Y's are listed as follows:

$$Y_0 = 1, \qquad Y_1 = y_1 = ag_1, \qquad Y_2 = y_2 + y_1^2 = ag_2 + a^2 g_1^2,$$
$$Y_3 = y_3 + 3y_2y_1 + y_1^3 = ag_3 + 3a^2 g_2g_1 + a^3 g_1^3.$$

Recurrence and other relations for the F_n—before the identification $a^i = f_i$—may then be taken from Bell with slight changes for the change in arguments. The more important ones are as follows:

$$\begin{aligned} F_{n+1} &= (ag_1 + D_x)F_n \\ &= ag(F + g)^n = \sum_{i=0}^{n} \binom{n}{i} ag_{i+1}F_{n-i} \\ &= \left(ag_1 + \sum_{i=1}^{n} g_{i+1}\frac{\partial}{\partial g_i}\right)F_n, \end{aligned} \tag{6}$$

$$\exp tF = \exp a[\exp tg - g_0], \tag{7}$$

where the last is symbolic and equivalent to

$$F_0 + tF_1 + t^2F_2/2! + \cdots = \exp a[tg_1 + t^2g_2/2! + \cdots],$$

and to diBruno's formula

$$F_n = \sum \frac{n!a^\sigma}{s_1! \cdots s_n!}\left(\frac{g_1}{1!}\right)^{s_1} \cdots \left(\frac{g_n}{n!}\right)^{s_n} \tag{8}$$

with summation over all non-negative integral solutions of $n = \sum is_i$ and $\sigma = \sum s_i$.

Instances of (7) with special choices of g may be used for verification of the numerical coefficients of F_n. Thus, with $g_i = 1$, all i,

$$\exp tF = \exp a(e^t - 1),$$

and

$$F_n = \sum S_{i,n}a^i, \qquad g_i = 1, \tag{9}$$

with $S_{i,n}=\Delta^i 0^n/i!$ the Stirling number of the second kind, showing that the sum of numerical coefficients of $F_{n,i}$ is $S_{i,n}$. Again with $g_1=0$, $g_i=1$, $i>1$,

$$\exp tF = \exp a(e^t - 1 - t),$$

and

$$F_n = \sum C_{i,n}a^i, \qquad g_1 = 0,\ g_i = 1,\ i > 1, \tag{10}$$

with $C_{i,n}$ an associated Stirling number with recurrence

$$C_{i,n+1} = iC_{i,n} + nC_{i-1,n-1}.$$

These numbers have a combinatorial interpretation similar to that for $S_{i,n}$, namely, $C_{i,n}$ is the number of rhyme schemes of n verses with i rhymes, such that each rhyme appears at least twice. This kind of verification of course may be extended at will. These results are in agreement with results of Wall [8] and Opatowski [5].

3. **Function of many functions.** The essentials of the general case are in the two function case

$$F(x) = f[g(x), h(x)]. \tag{11}$$

If we write

$$f_{ij} = \left[\frac{\partial^i}{\partial u^i}\frac{\partial^j}{\partial v^j} f(u, v)\right]_{u=g(x),\, v=h(x)}$$

the first two derivatives are

$$F_1 = f_{01}g_1 + f_{10}h_1,$$
$$F_2 = f_{20}g_1^2 + 2f_{11}g_1h_1 + f_{02}h_1^2 + f_{10}g_2 + f_{01}h_2,$$

and in general

$$F_n = \sum_i \sum_j f_{ij}F_{n,ij}(g_1 \cdots g_i;\ h_1 \cdots h_j).$$

Then as before the generating function for coefficients $F_{n,ij}$ is $\exp(-ag+bh)D_x^n \exp(ag+bh)$ and

$$F_n = Y_n(ag_1 + bh_1, \cdots, ag_n + bh_n), \qquad a^ib^j = f_{ij}, \tag{12}$$

$$\exp tF = \exp\left[a(\exp tg - g_0) + b(\exp th - h_0)\right]. \tag{13}$$

The last shows that (cf. Bell [1, equations 4.9 and 7.5])

$$F_n = (G + H)^n = \sum \binom{n}{i} G_{n-i}H_i \tag{14}$$

with

$$G_n = Y_n(ag_1, \cdots, ag_n), \qquad H_n = Y_n(bh_1, \cdots, bh_n).$$

The numerical coefficients associated with Y_n then reappear here and in the general case, as noted by Opatowski [5], who used the generalization of diBruno's formula given by Teixeira [7]; compare also Dederick [2].

The extension of (12) and (14) to the general case is purely a matter of notation, which should be sufficiently evident.

4. **Function of a function of a function.** For completeness, we note briefly another extension to a function of a function of a function, which contains the essence of a general extension. Take

$$F = f[g(h(x))]. \tag{15}$$

Then

$$\begin{aligned} F_n &= Y_n(fG_1, \cdots, fG_n), \qquad && f^i \equiv f_i, \\ &= Y_n(\Gamma h_1, \cdots, \Gamma h_n), && \Gamma^i \equiv \Gamma_i, \end{aligned} \tag{16}$$

with

$$\begin{aligned} g_n &= Y_n(gh_1, \cdots, gh_n), \qquad && g^i \equiv g_i, \\ \Gamma_n &= Y_n(fg_1, \cdots, fg_n), && f^i \equiv f_i. \end{aligned}$$

The last two expressions exemplify a kind of associative equality which permits writing the general result in a variety of ways.

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BELL TELEPHONE LABORATORIES

A SECOND NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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In a recent paper[1] the author proved that if Ω is any compact metric space containing non-denumerably many points and $C(\Omega)$ is the Banach space of all continuous functionals over Ω, then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure. The purpose of this note is to obtain the same result, assuming only that Ω contains infinitely many points. This leads to a certain necessary and sufficient condition for weak differentiability of Pettis integrals.

The author's previous paper (cited above) will be referred to hereafter as Note I. It will be assumed that the reader is familiar with the notation, terminology, and results of that paper.

Let B be the non-dense perfect set described in Note I, and let $\bar{B}$ be its complement. Let the intervals of $\bar{B}$ be arranged in a sequence $I_1, I_2, I_3, \cdots$ in such a way that if the order of I_j is greater than the order of I_k, then $j>k$. For each positive integer k, let $n(k)$ be the order of I_k. As in Note I, we now define a function $\phi(x, t)$ over the unit square so that for each x, $\phi(x, t)$ is continuous in t. This will serve to define a function from the unit interval to the space C.

For $x\in B$, let $\phi(x, t)=0$; for $x\in I_1$, let $\phi(x, t)=0$; for $x\in I_k$ and $t=1/k$ $(k=2, 3, 4, \cdots)$, let $\phi(x, t)=2^{2n(k)}/n(k)$; for $x\in I_k$ and $t=1/(k\pm 1/2)$ $(k=2, 3, 4, \cdots)$, let $\phi(x, t)=0$; for $t=0$ or 1, let $\phi(x, t)=0$. Now for each x, let $\phi(x, t)$ be extended linearly as a function of t between successive points already determined. We now denote by $\phi(x)$ the function whose values are the elements of C determined by $\phi(x, t)$.

THEOREM 1. *$\phi(x)$ is Pettis integrable. Its integral is the element of C*

$$\Phi_E(t) = \int_E \phi(x, t)dx.$$

That $\Phi_E(t)$ is defined and continuous in t is obvious. For $t=1/k$, $\Phi_E(t)=|E\cdot I_k|2^{2n(k)}/n(k)=|E\cdot I_k|/|I_k|n(k)\leq 1/n(k)$; for

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[1] M. E. Munroe, *A note on weak differentiability of Pettis integrals*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 167–174.

$t=1/(k\pm 1/2)$, $\Phi_B(t)=0$. $\Phi_B(t)$ is linear between successive points in this sequence, and $\Phi_B(0)=0$.

To show that $\Phi_B(t)$ is the integral of $\phi(x)$, we introduce the functions

$$\phi^{(k)}(x,t)=\begin{cases}\phi(x,t) & \text{for } t\geqq 1/(k+1/2),\\ 0 & \text{otherwise.}\end{cases}$$

For each x, $\phi^{(k)}(x,t)\to\phi(x,t)$ uniformly in t. Furthermore, the functions $\phi^{(k)}(x)$ determined by $\phi^{(k)}(x,t)$ are step functions, hence Pettis integrable. Therefore,[2] their integrals are the elements of C

$$\Phi_B^{(k)}(t)=\int_B\phi^{(k)}(x,t)dx.$$

Now

$$\begin{aligned}\left|\Phi_B(t)-\Phi_B^{(k)}(t)\right| &= \int_B\left[\phi(x,t)-\phi^{(k)}(x,t)\right]dx\\ &= \begin{cases}\Phi_B(t) & \text{for } t\leqq 1/(k+1/2)\\ 0 & \text{for } t>1/(k+1/2)\end{cases}\\ &\leqq 1/n(k)\quad \text{for all } t.\end{aligned}$$

Thus $\Phi_B^{(k)}(t)\to\Phi_B(t)$ uniformly in t, and Theorem 1 follows from a theorem of Pettis.[3]

THEOREM 2. *The function*

$$\Phi(E)=\int_E\phi(x)dx$$

is not weakly differentiable at any point of B.

Let $x_0\in B$, and let J_n be an interval having x_0 as its center and containing an interval of $\bar{B}$ of order n. This may be done with

$$|J_n|\leqq 2|G_n|=2(1/2^n+1/2^{2n-1})<1/2^{n-2}$$

where G_n is a gap of order n (see Note I). Let I_k be the interval of $\bar{B}$ of order n contained in J_n; then

$$\frac{\|\Phi(J_n)\|}{|J_n|}\geqq\frac{|\Phi_{J_n}(1/k)|}{|J_n|}\geqq\frac{1/n}{|J_n|}>\frac{2^{n-2}}{n}.$$

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[2] See Note I, footnote 5.

[3] B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. vol. 44 (1938) pp. 277–304. See Theorem 4.1.

Thus the norm of the difference quotient is unbounded for each $x_0 \in B$, and the failure of weak differentiability follows from a theorem of Banach.[4]

THEOREM 3. *If Ω is a compact metric space containing infinitely many points and if $C(\Omega)$ is the space of all continuous functionals defined over Ω, then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure.*

In this case Ω contains a homeomorph of the sequence $\{1/k\}$. The argument used in Theorem 4.1 of Note I now applies to the present theorem. We shall not bother to repeat it here.

THEOREM 4. *If M is an abstract M-space with unity,[5] a necessary and sufficient condition that every Pettis integral defined to M be almost everywhere weakly differentiable is that M be finite-dimensional.*

Obviously finite-dimensionality is sufficient. If M is infinite-dimensional, it contains an infinite-dimensional separable subspace which, by a theorem of Kakutani,[6] is equivalent to the space $C(\Omega)$ of Theorem 3. Hence, finite-dimensionality is also necessary.

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[4] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, vol. 1, Warsaw, 1932. See p. 224, Theorem 8.

[5] For a definition of this, see S. Kakutani, *Concrete representations of abstract (M)-spaces*, Ann. of Math. vol. 42 (1941) pp. 994–1024.

[6] Loc. cit. Theorem 2, p. 998.

A THEOREM ON ARBITRARY J-FRACTIONS

H. S. WALL

1. **Introduction.** We consider a J-fraction

$$(1.1)\qquad \cfrac{1}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{a_2^2}{b_3 + z - \cdots}}} \qquad (a_p \neq 0),$$

in which the coefficients a_p and b_p are any complex numbers, the a_p being different from zero, and z is a complex parameter. The system of linear equations

$$(1.2)\qquad -a_{p-1}x_{p-1} + (b_p + z)x_p - a_p x_{p+1} = 0, \qquad p = 1, 2, 3, \cdots;\ a_0 = 1,$$

can be solved for x_2, x_3, x_4, $\cdots$ uniquely in terms of arbitrarily chosen initial values x_0 and x_1. We denote by $X_p(z)$ and $Y_p(z)$ the solutions corresponding to $x_0 = -1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively: $X_0(z) = -1$, $X_1(z) = 0$, $Y_0(z) = 0$, $Y_1(z) = 1$. Then $X_{p+1}(z)/Y_{p+1}(z)$ is the pth approximant of the J-fraction, and we have the determinant formula

$$(1.3)\qquad X_{p+1}(z)Y_p(z) - X_p(z)Y_{p+1}(z) = 1/a_p, \qquad p = 0, 1, 2, \cdots.$$

The following theorem holds.

THEOREM OF INVARIABILITY. *If the series*

$$(1.4)\qquad \sum_{p=1}^{\infty} |X_p(z)|^2, \qquad \sum_{p=1}^{\infty} |Y_p(z)|^2$$

converge for a single value of the parameter z, then these series converge uniformly over every bounded domain of z.

This theorem was proved by Hellinger and Wall [3].[1] The uniformity of the convergence was not explicitly mentioned, but is contained

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

in the fact that the sums $\sum_{p=1}^{\infty}|\zeta_p|^2$ of [3, p. 121] are uniformly bounded over every finite domain of z^*.

In the present note we have derived some consequences of the above theorem. It will be convenient to make the following definition.

Definition. The *determinate case* or the *indeterminate case* is said to hold for the J-fraction (1.1) according as at least one of the series

$$\sum_{p=1}^{\infty}|X_p(0)|^2, \qquad \sum_{p=1}^{\infty}|Y_p(0)|^2 \tag{1.5}$$

diverges, or both of these series converge, respectively.

We shall prove that if the indeterminate case holds for the J-fraction, and if the J-fraction converges for a single value of z, then it represents a meromorphic function of z and converges except at the poles of this function. Hamburger [2] proved this theorem for J-fractions with real coefficients, and closely related theorems were proved by Hellinger and Wall [3] for J-fractions in which the a_p are real and $I(b_p) \geqq 0$, and by Dennis and Wall [1] for positive definite J-fractions. If, in particular, $b_p = 0$, $p = 1, 2, 3, \cdots$, then, if we drop the factor z, the J-fraction can be thrown into the form

$$\cfrac{1}{k_1 w + \cfrac{1}{k_2 + \cfrac{1}{k_3 w + \cfrac{1}{k_4 + \cdots}}}}, \qquad (w = z^2). \tag{1.6}$$

We show that if the series

$$\sum k_{2p+1}, \quad \sum k_{2p+1}(k_2 + k_4 + \cdots + k_{2p})^2 \tag{1.7}$$

are absolutely convergent and

$$\lim_{p\to\infty} |k_2 + k_4 + \cdots + k_{2p}| = \infty, \tag{1.8}$$

then the continued fraction (1.6) converges to a meromorphic function of w or else diverges to ∞ for every w. If the series (1.7) are absolutely convergent and (1.8) fails to hold, then the continued fraction diverges by oscillation for every w. This theorem was proved by Hamburger [2] for the case where the k_p are real, $k_{2p} \neq 0$, and $k_{2p+1} > 0$.

2. **Four entire functions.** We define four polynomials $U_n(z)$, $V_n(z)$

$P_n(z)$, $Q_n(z)$ by means of the following formulas:

$$
\begin{aligned}
U_n(z) &= a_n[Y_n(0)X_{n+1}(z) - Y_{n+1}(0)X_n(z)],\\
V_n(z) &= a_n[Y_n(0)Y_{n+1}(z) - Y_{n+1}(0)Y_n(z)],\\
P_n(z) &= a_n[X_n(0)X_{n+1}(z) - X_{n+1}(0)X_n(z)],\\
Q_n(z) &= a_n[X_n(0)Y_{n+1}(z) - X_{n+1}(0)Y_n(z)],
\end{aligned}
\qquad n = 1, 2, 3, \cdots . \tag{2.1}
$$

We find with the aid of the determinant formula (1.3) that these polynomials satisfy the identity

$$P_n(z)V_n(z) - Q_n(z)U_n(z) \equiv 1. \tag{2.2}$$

We now put $x_k = X_k(z)$, $p = n+1$, in (1.2), and get

$$a_{n+1}X_{n+2}(z) = (b_{n+1} + z)X_{n+1}(z) - a_nX_n(z).$$

We multiply both members of this identity by $X_{n+1}(0)$, subtract $a_{n+1}X_{n+2}(0)X_{n+1}(z)$ from both members, and obtain

$$
\begin{aligned}
a_{n+1}[X_{n+1}(0)X_{n+2}(z) - X_{n+2}(0)X_{n+1}(z)]&\\
= [\{b_{n+1}X_{n+1}(0) - a_{n+1}X_{n+2}(0)\} &+ zX_{n+1}(0)]X_{n+1}(z)\\
&- a_nX_{n+1}(0)X_n(z)\\
= a_n[X_n(0)X_{n+1}(z) - X_{n+1}(0)X_n(z)] &+ zX_{n+1}(0)X_{n+1}(z).
\end{aligned}
$$

Hence, by (2.1), we have the first of the following relations:

$$
\begin{aligned}
P_{n+1}(z) &= P_n(z) + zX_{n+1}(0)X_{n+1}(z),\\
Q_{n+1}(z) &= Q_n(z) + zX_{n+1}(0)Y_{n+1}(z),\\
U_{n+1}(z) &= U_n(z) + zY_{n+1}(0)X_{n+1}(z),\\
V_{n+1}(z) &= V_n(z) + zY_{n+1}(0)Y_{n+1}(z).
\end{aligned}
$$

The others may be obtained in a similar way. From these relations we now obtain immediately the following formulas.

$$
\begin{aligned}
P_{n+1}(z) &= \quad + z\sum_{p=1}^{n+1} X_p(0)X_p(z),\\
Q_{n+1}(z) &= -1 + z\sum_{p=1}^{n+1} X_p(0)Y_p(z),\\
U_{n+1}(z) &= 1 + z\sum_{p=1}^{n+1} Y_p(0)X_p(z),\\
V_{n+1}(z) &= z + z\sum_{p=1}^{n+1} Y_p(0)Y_p(z).
\end{aligned} \tag{2.3}
$$

By the theorem of invariability (§1) and formulas (2.2), (2.3) we find at once by Schwarz's inequality that the following theorem is true.

THEOREM 2.1. *Let the indeterminate case hold for the J-fraction* (1.1). *Then there exist four entire functions* $u(z)$, $v(z)$, $p(z)$, $q(z)$ *such that*

$$p(z)v(z) - q(z)u(z) = 1, \tag{2.4}$$

and such that

$$\begin{aligned} &\lim_{n\to\infty} U_n(z) = u(z), &&\lim_{n\to\infty} V_n(z) = v(z),\\ &\lim_{n\to\infty} P_n(z) = p(z), &&\lim_{n\to\infty} Q_n(z) = q(z), \end{aligned} \tag{2.5}$$

uniformly over every bounded region of the z-plane.

3. **Convergence theorem for J-fractions.** By means of (2.1) and (1.3) we find that

$$\begin{aligned} X_{n+1}(z) &= X_{n+1}(0)U_n(z) - Y_{n+1}(0)P_n(z),\\ Y_{n+1}(z) &= X_{n+1}(0)V_n(z) - Y_{n+1}(0)Q_n(z). \end{aligned} \tag{3.1}$$

Let $s_n = X_{n+1}(0)/Y_{n+1}(0)$. If $\lim_{n\to\infty} s_n = s$, a finite number, and if the indeterminate case holds for the J-fraction, it then follows from Theorem 2.1 and the relations (3.1) that

$$\lim_{n\to\infty}\frac{X_{n+1}(z)}{Y_{n+1}(0)} = su(z) - p(z), \qquad \lim_{n\to\infty}\frac{Y_{n+1}(z)}{Y_{n+1}(0)} = sv(z) - q(z), \tag{3.2}$$

uniformly over every finite region. Since, by (2.4),

$$[sv(z) - q(z)]u(z) - [su(z) - p(z)]v(z) = 1,$$

it follows that the limits (3.2) cannot vanish for one and the same value of z. Therefore, for every value of z,

$$\lim_{n\to\infty}\frac{X_{n+1}(z)}{Y_{n+1}(z)} = \frac{su(z) - p(z)}{sv(z) - q(z)},$$

which is a meromorphic function of z, or else is ∞ for every z. If $\lim_{n\to\infty} s_n = \infty$, then, for every value of z,

$$\lim_{n\to\infty}\frac{X_{n+1}(z)}{Y_{n+1}(z)} = \frac{u(z)}{v(z)},$$

which again is a meromorphic function of z or else is ∞ for every z. If the sequence $\{s_n\}$ has more than one limit-point, let $\lim_{n'\to\infty} s_{n'} = s'$,

$\lim_{n''\to\infty} s_{n''} = s''$, $s' \neq s''$. Then, if s' and s'' are both finite,

$$\lim_{n'\to\infty} \frac{X_{n'+1}(z)}{Y_{n'+1}(z)} = \frac{s'u(z) - p(z)}{s'v(z) - q(z)},$$

$$\lim_{n''\to\infty} \frac{X_{n''+1}(z)}{Y_{n''+1}(z)} = \frac{s''u(z) - p(z)}{s''v(z) - q(z)}.$$

These are unequal for every z inasmuch as

$$(s'u - p)(s''v - q) - (s''u - p)(s'v - q) = s' - s'' \neq 0,$$

and therefore the *J*-fraction diverges by oscillation for every z. The same evidently holds if one of the limits s', s'' is ∞. From these considerations we conclude that the following theorem is true.

THEOREM 3.1. *Let the indeterminate case hold for the J-fraction* (1.1). *If the J-fraction converges for a single value of* z, *then it represents a meromorphic function of* z *and converges except at the poles of that function. In terms of the entire functions of* §2,

$$\lim_{n\to\infty} \frac{X_{n+1}(z)}{Y_{n+1}(z)} = \frac{su(z) - p(z)}{sv(z) - q(z)} \quad \textit{or} \quad \frac{u(z)}{v(z)}$$

according as

$$\lim_{n\to\infty} \frac{X_{n+1}(0)}{Y_{n+1}(0)} = s \ (\textit{finite}) \ \textit{or} \ \infty,$$

respectively. If there is a single value of z *for which neither the J-fraction nor its reciprocal converges, then the J-fraction diverges by oscillation for every value of* z.

4. **Convergence theorem for *S*-fractions.** A *J*-fraction (1.1) in which the coefficients b_p are all equal to zero is called a *Stieltjes fraction* or *S-fraction*:

$$(4.1) \qquad \cfrac{1}{z - \cfrac{a_1^2}{z - \cfrac{a_2^2}{z - \cdots}}}$$

This has the property that its even part is a *J*-fraction in the variable z^2, multiplied by the factor z, namely,

$$\text{(4.2)}\qquad \cfrac{z}{z^2 - a_1^2 - \cfrac{(a_1a_2)^2}{z^2 - (a_2^2 + a_3^2) - \cfrac{(a_3a_4)^2}{z^2 - (a_4^2 + a_5^2) - \ddots}}}$$

If we divide both (4.1) and (4.2) by z, and then make the change of variable $z^2 = w$, these take the form

$$\text{(4.3)}\qquad \cfrac{1}{w - \cfrac{a_1^2}{1 - \cfrac{a_2^2}{w - \ddots}}},$$

and

$$\text{(4.4)}\qquad \cfrac{1}{w - a_1^2 - \cfrac{(a_1a_2)^2}{w - (a_2^2 + a_3^2) - \cfrac{(a_3a_4)^2}{w - (a_4^2 + a_5^2) - \ddots}}},$$

respectively. We shall apply Theorem 3.1 to the J-fraction (4.4) in order to obtain a convergence theorem for the S-fraction (4.3).

Let $G_p(w)$ and $H_p(w)$ denote the pth numerator and denominator of (4.3), and let $A_p(w)$ and $B_p(w)$ denote the pth numerator and denominator of (4.4). Then we have

$$\text{(4.5)}\qquad G_{2p}(w) = A_p(w), \qquad H_{2p}(w) = B_p(w), \qquad p = 0, 1, 2, \cdots .$$

Let

$$\text{(4.6)}\qquad \begin{array}{ll} \sigma_1 = a_1a_2, & \delta_1 = -a_1^2, \\ \sigma_2 = a_3a_4, & \delta_2 = -a_1^2 - a_2^2, \\ \sigma_3 = a_5a_6, & \delta_3 = -a_4^2 - a_5^2, \\ \cdots\cdots, & \cdots\cdots\cdots\cdots, \end{array}$$

so that (4.4) becomes

$$(4.7)\qquad \cfrac{1}{\delta_1 + w - \cfrac{\sigma_1^2}{\delta_2 + w - \cfrac{\sigma_2^2}{\delta_3 + w - \ddots}}}$$

From (4.6) we readily find by mathematical induction that

$$(4.8)\qquad a_{2p-1}^2 = -\frac{B_p(0)}{B_{p-1}(0)},\quad a_{2p}^2 = -\frac{\sigma_p^2 B_{p-1}(0)}{B_p(0)},\quad p = 1, 2, 3, \cdots,$$

and

$$(4.9)\qquad B_p(0) \neq 0, \qquad p = 1, 2, 3, \cdots.$$

Conversely, if (4.7) is a *J*-fraction such that (4.9) holds, then it is the even part of an *S*-fraction (4.3) whose partial numerators are given by (4.8).

The odd numerators and denominators of the *S*-fraction (4.3) can be expressed in terms of the numerators and denominators of (4.7) by means of the formulas

$$(4.10)\qquad \begin{aligned} G_{2p-1}(w) &= \frac{B_{p-1}(0)A_p(w) - B_p(0)A_{p-1}(w)}{B_{p-1}(0)},\\ H_{2p-1}(w) &= \frac{B_{p-1}(0)B_p(w) - B_p(0)B_{p-1}(w)}{B_{p-1}(0)}, \end{aligned} \qquad p = 1, 2, 3, \cdots.$$

These may be verified immediately by means of the recurrence formula $G_{2p}(w) = G_{2p-1}(w) - a_{2p-1}^2 G_{2p-2}(w)$, and the like relation for $H_{2p}(w)$, if we use (4.5) and (4.8). It will be observed that the first two polynomials (2.1), formed for the *J*-fraction (4.7), differ from (4.10) only by a constant factor:

$$(4.11)\qquad \frac{G_{2p-1}(w)}{H_{2p-1}(w)} = \frac{U_p(w)}{V_p(w)}, \qquad p = 1, 2, 3, \cdots.$$

If we make the substitution

$$(4.12)\qquad a_p^2 = -\frac{1}{k_p k_{p+1}},\quad p = 1, 2, 3, \cdots;\ k_1 = 1,$$

in (4.3), the latter becomes

$$(4.13)\qquad \cfrac{1}{k_1w+\cfrac{1}{k_2+\cfrac{1}{k_3w+\cdots}}}\,.$$

The pth numerator and denominator of (4.13) are

$$k_1k_2\cdots k_pG_p(w) \quad\text{and}\quad k_1k_2\cdots k_pH_p(w),$$

respectively.

We shall now prove the following theorem.

THEOREM 4.1. *Let* $k_1, k_2, k_3, \cdots$ *be complex numbers different from zero such that the series*

$$(4.14)\qquad \sum k_{2p+1}, \qquad \sum k_{2p+1}(k_2+k_4+\cdots+k_{2p})^2$$

are absolutely convergent, and let

$$(4.15)\qquad \lim_{p=\infty}(x_2+k_4+\cdots+k_{2p})=\infty.$$

Then, for each value of w, *the S-fraction* (4.13) *converges, or else its reciprocal converges to the value* 0. *If the series* (4.14) *converge absolutely and* (4.15) *fails to hold, then neither the S-fraction nor its reciprocal converges for a single value of* w. *If the S-fraction converges for one value of* w, *then its value is a meromorphic function of* w *to which it converges uniformly over every bounded closed region containing none of the poles of the function.*

PROOF. The polynomials $X_{p+1}(w)$ and $Y_{p+1}(w)$ for the J-fraction (4.7) are given by

$$X_{p+1}(w)=\frac{A_p(w)}{\sigma_1\sigma_2\cdots\sigma_p}, \qquad Y_{p+1}(w)=\frac{B_p(w)}{\sigma_1\sigma_2\cdots\sigma_p}.$$

By (4.5), (4.6) and (4.12) we then obtain

$$\begin{aligned} |X_{p+1}(0)|^2 &= |G_{2p}(0)\cdot k_1k_2\cdots k_{2p}|^2\cdot|k_{2p+1}|,\\ |Y_{p+1}(0)|^2 &= |H_{2p}(0)\cdot k_1k_2\cdots k_{2p}|\cdot|k_{2p+1}|. \end{aligned}$$

Since $k_1k_2\cdots k_pG_p(w)$ and $k_1k_2\cdots k_pH_p(w)$ are the pth numerator and denominator, respectively, of (4.13), we readily find by mathematical induction that

$$G_{2p}(0)\cdot k_1k_2\cdots k_{2p}=k_2+k_4+\cdots+k_{2p}, \quad H_{2p}(0)\cdot k_1k_2\cdots k_{2p}=1,$$

and consequently

$$(4.16)\qquad \begin{aligned} |X_{p+1}(0)|^2 &= |k_{2p+1}(k_2 + k_4 + \cdots + k_{2p})^2|, \\ |Y_{p+1}(0)|^2 &= |k_{2p+1}|. \end{aligned}$$

From Theorem 3.1 we therefore conclude that when the series (4.14) converge absolutely, then, by (4.11):

$$\lim_{p\to\infty} \frac{G_{2p-1}(w)}{H_{2p-1}(w)} = \frac{u(w)}{v(w)},$$

where $u(w)$ and $v(w)$ are entire functions, provided $v(w) \neq 0$. Inasmuch as $u(w) \neq 0$ when $v(w) = 0$, we conclude that for any w, the sequence of odd approximants of the S-fraction, or else the sequence of reciprocals of these approximants, must converge. By Theorem 3.1 and (4.5),

$$\lim_{p\to\infty} \frac{G_{2p}(w)}{H_{2p}(w)} = \frac{u(w)}{v(w)}$$

if and only if $\lim_{p\to\infty} |A_p(0)/B_p(0)| = \lim_{p\to\infty} |X_{p+1}(0)/Y_{p+1}(0)| = \infty$. But, by (4.16),

$$\left|\frac{X_{p+1}(0)}{Y_{p+1}(0)}\right| = |k_2 + k_4 + \cdots + k_{2p}|.$$

Therefore, if (4.15) holds, then the S-fraction or its reciprocal converges. If (4.15) fails to hold, then the S-fraction and its reciprocal diverge for every w. If the S-fraction converges for a single value of w, then its value is the meromorphic function $u(w)/v(w)$. The convergence is clearly uniform over every closed bounded region containing none of the poles of this function.

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RECIPROCALS OF J-MATRICES

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1. Introduction. We consider J-matrices

$$J = (j_{pq}), \qquad j_{pq} = 0 \quad \text{for} \quad |p - q| \geqq 2, \qquad j_{pp} = b_p,$$
$$j_{p+1,p} = j_{p,p+1} = -a_p \neq 0,$$

such that

$$(1.1) \quad I[J(x, \bar{x})] = \sum I(b_p)\,|x_p|^2 - \sum I(a_p)(x_p\bar{x}_{p+1} + \bar{x}_p x_{p+1}) \geqq 0$$

for all x_p for which the sums converge. These are the J-matrices associated with a positive definite J-fraction [4, 5, 1].[1] Let $X_p(z)$ and $Y_p(z)$ denote the solutions of the system of linear equations

$$(1.2) \quad -a_{p-1}x_{p-1} + (b_p + z)x_p - a_p x_{p+1} = 0, \; p = 1, 2, 3, \cdots; a_0 = 1,$$

under the initial conditions $x_0 = -1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. We shall prove that when at least one of the series

$$(1.3) \qquad \sum_{p=1}^{\infty} |X_p(0)|^2, \qquad \sum_{p=1}^{\infty} |Y_p(0)|^2$$

diverges, then the matrix $J + zI$ has a unique bounded reciprocal for $I(z) > 0$, and that when both the series (1.3) converge then the matrix $J + zI$ has infinitely many different bounded reciprocals. This theorem was proved by Hellinger [2] for the case where the coefficients a_p and b_p are all real.

2. Reciprocals of an arbitrary J-matrix. The general right reciprocal of $J + zI$ is (ρ_{pq}) where $\rho_{1,q}$, $q = 1, 2, 3, \cdots$, are arbitrary functions of z, and [3, p. 116]

$$(2.1) \qquad \rho_{pq}(z) = \begin{cases} \rho_{1,q}(z)Y_p(z), & p = 1, 2, 3, \cdots, q; \\ \rho_{1,q}(z)Y_p(z) + X_q(z)Y_p(z) - X_p(z)Y_q(z), & \\ & p = q + 1, q + 2, q + 3, \cdots. \end{cases}$$

We shall say that the *determinate case* or the *indeterminate case* holds for the J-matrix according as at least one of the series (1.3) diverges or both of these series converge, respectively. In the indeterminate case, both of the series

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$\sum_{p=1}^{\infty} |X_p(z)|^2, \qquad \sum_{p=1}^{\infty} |Y_p(z)|^2 \tag{2.2}$$

converge for every value of z [3, p. 120]. Hence if the functions $\rho_{1,q}(z)$ are chosen such that the series $\sum |\rho_{1,q}(z)|^2$ converges, it follows by (2.1) and Schwarz's inequality that the double series $\sum |\rho_{pq}(z)|^2$ converges and therefore the matrix $(\rho_{pq}(z))$ is bounded. If, in particular,

$$\rho_{1,q}(z) = Y_q(z)f(z) - X_q(z),$$

then

$$\rho_{pq}(z) = \begin{cases} Y_p(z)Y_q(z)\left(f(z) - \dfrac{X_q(z)}{Y_q(z)}\right), & p = 1, 2, \cdots, q; \\[2ex] Y_p(z)Y_q(z)\left(f(z) - \dfrac{X_p(z)}{Y_p(z)}\right), & p = q+1, q+2, \cdots, \end{cases} \tag{2.3}$$

so that the matrix (ρ_{pq}) is symmetric. If, for example, $f(z)$ is an entire function, then the matrix (ρ_{pq}) given by (2.3) is bounded in the indeterminate case for all values of z. Hence we have the following theorem.

THEOREM 2.1. *In the indeterminate case, the J-matrix $J+zI$ has infinitely many different reciprocals $(\rho_{pq}(z))$ which are bounded for all values of z.*

We have not used the condition (1.1), so that this theorem holds for arbitrary J-matrices.

3. The determinate case. We suppose now that (1.1) holds. Then

$$\beta_p = I(b_p) \geq 0, \qquad p = 1, 2, 3, \cdots, \tag{3.1}$$

and there exist constants g_p such that if $\alpha_p = I(a_p)$ then

$$\alpha_p^2 = \beta_p\beta_{p+1}(1 - g_{p-1})g_p, \qquad 0 \leq g_{p-1} \leq 1, \; p = 1, 2, 3, \cdots. \tag{3.2}$$

Conversely, if (3.1) and (3.2) hold, then (1.1) holds [5, p. 91].

For a fixed positive integer n, let $\xi_1, \xi_2, \cdots, \xi_n$ be arbitrary real numbers. Let (ρ_{pq}) be any right reciprocal of $J+zI$, so that, if we now take $a_0 = 0$,

$$-a_{p-1}\rho_{p-1,q} + (b_p + z)\rho_{p,q} - a_p\rho_{p+1,q} = \delta_{p,q}, \qquad p, q = 1, 2, 3, \cdots, \tag{3.3}$$

where $\delta_{p,q} = 0$ or 1 according as $p \neq q$ or $p = q$, respectively. On multiplying (3.3) by ξ_q and summing over q from 1 to n we obtain

$$-a_{p-1}\eta_{p-1}+(b_p+z)\eta_p-a_p\eta_{p+1}=\xi_p, \tag{3.4}$$

where

$$\eta_p=\sum_{q=1}^{n}\rho_{pq}\xi_q. \tag{3.5}$$

We now suppose (ρ_{pq}) is symmetric, so that (2.3) holds for some function $f(z)$. We note that

$$w=\frac{a_n\rho_{n,q}}{\rho_{n+1,q}}$$

is independent of q for $q=1, 2, 3, \cdots, n$:

$$w=a_n\frac{Y_n(z)f(z)-X_n(z)}{Y_{n+1}(z)f(z)-X_{n+1}(z)},\quad f(z)=\frac{X_{n+1}(z)w-a_nX_n(z)}{Y_{n+1}(z)w-a_nY_n(z)}. \tag{3.6}$$

For a fixed z with $I(z)>0$, the transformation

$$t=\frac{X_{n+1}(z)w-a_nX_n(z)}{Y_{n+1}(z)w-a_nY_n(z)}$$

maps the half-plane $I(w)\geqq\beta_{n+1}g_n$ upon a circular region $K_n(z)$ (cf. [1]). Hence we see by (3.6) that the value of the function $f(z)$ is in $K_n(z)$ if and only if $I(w)\geqq\beta_{n+1}g_n$. If the latter inequality holds then [1, p. 261]

$$\beta_n+y-I\left(\frac{a_n^2}{w}\right)\geqq\beta_n g_{n-1}+y,\qquad \text{where}\quad y=I(z)>0,$$

or

$$I\left(\frac{a_n^2}{w}\right)\leqq\beta_n(1-g_{n-1}). \tag{3.7}$$

Now

$$a_n\rho_{n+1,q}=\frac{a_n^2}{w}\rho_{n,q}.$$

On multiplying this by ξ_q and summing over q from 1 to n, we get

$$a_n\eta_{n+1}=\frac{a_n^2}{w}\eta_n. \tag{3.8}$$

We now multiply (3.4) by $\bar{\eta}_p$, sum over p from 1 to n, and eliminate the quantity $a_n\eta_{n+1}\bar{\eta}_n$ by (3.8). This gives immediately the relation

$$\sum_{p=1}^{n}(b_p+z)\,|\,\eta_p\,|^2-\sum_{p=1}^{n-1}a_p(\eta_p\bar{\eta}_{p+1}+\bar{\eta}_p\eta_{p+1})=\frac{a_n^2}{w}\,|\,\eta_n\,|^2+\sum_{p=1}^{n}\xi_p\bar{\eta}_p.$$

If we consider only the imaginary part and make use of the inequality (3.7) and the relations (3.2) we then obtain (cf. [1, p. 258])

$$\begin{aligned}(3.9)\qquad & y\sum_{p=1}^{n}|\,\eta_p\,|^2+\sum_{p=1}^{n-1}|\,(\beta_p(1-g_{p-1}))^{1/2}\eta_p-(\beta_{p+1}g_p)^{1/2}\eta_{p+1}\,|^2\\ &\qquad+\sum_{p=1}^{n}\xi_pI(\eta_p)\leqq 0.\end{aligned}$$

Hence, in particular,

$$(3.10)\qquad y\sum_{p=1}^{n}|\,\eta_p\,|^2+\sum_{p=1}^{n}\xi_pI(\eta_p)\leqq 0.$$

This holds under the assumption that the value of the function $f(z)$ is in the circular region $K_n(z)$.

Turning now to the quadratic form

$$R_n(\xi,\xi)=\sum_{p,q=1}^{n}\rho_{pq}(z)\xi_p\xi_q=\sum_{p=1}^{n}\xi_p\eta_p,$$

we have, by Schwarz's inequality and (3.10),

$$\begin{aligned}|\,R_n(\xi,\xi)\,|^2=\left|\sum_{p=1}^{n}\xi_p\eta_p\right|^2&\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\cdot y\sum_{p=1}^{n}|\,\eta_p\,|^2\\&\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\left(-\sum_{p=1}^{n}\xi_pI(\eta_p)\right)\\&=\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\cdot(-I[R_n(\xi,\xi)]).\end{aligned}$$

Therefore,

$$|\,R_n(\xi,\xi)\,|^2\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\cdot|\,R_n(\xi,\xi)\,|,$$

or

$$(3.11)\qquad |\,R_n(\xi,\xi)\,|\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2.$$

This holds for any particular values of n and z, $I(z)>0$, such that the value of $f(z)$ is in $K_n(z)$. Now [1, §3], $K_1(z)\supset K_2(z)\supset K_3(z)\supset \cdots$, and there is at least one function $f(z)$ which is analytic for $I(z)>0$ whose values are in *all* the circles $K_n(z)$. Hence we conclude that the following theorem is true.

THEOREM 3.1. *If* (1.1) *holds, then the matrix* $J+zI$ *has at least one reciprocal which is bounded for* $I(z)>0$.

We shall now prove the following theorem.

THEOREM 3.2. *If* (1.1) *holds, then, in the determinate case, the matrix* $J+zI$ *has just one reciprocal which is bounded for all* z *for which* $I(z)>0$.

PROOF. In the determinate case at least one of the series (2.2) diverges; and since [1, p. 262, formula (3.4)]

$$\left|\frac{X_p(z)}{Y_p(z)}\right| \leqq \frac{1}{y} \qquad \text{for } y = I(z) > 0, \tag{3.12}$$

it follows that the second of the series (2.2) diverges for $I(z)>0$. Therefore [1, p. 263, formula (3.12)], the radius $r_p(z)$ of the circle $K_p(z)$ tends to 0 as p tends to ∞. This implies that there is only one function $f_0(z)$ which for $I(z)>0$ has its values in all the circles $K_p(z)$. The reciprocal (ρ_{pq}) of $J+zI$ given by (2.3) with $f(z)=f_0(z)$ is bounded for $I(z)>0$. It is required to show that any other reciprocal is unbounded for at least one z in $I(z)>0$.

We consider an arbitrary reciprocal of $J+zI$ in $I(z)>0$. This must be given by (2.3). If $f(z)\not\equiv f_0(z)$ for $I(z)>0$, there must exist a value $z=z_0$, $I(z_0)>0$, such that

$$\left|f(z_0) - \frac{X_p(z_0)}{Y_p(z_0)}\right| \geqq k,$$

for all sufficiently large values of p, k being a positive constant. This follows from the fact that $X_p(z_0)/Y_p(z_0)$ is in the circle $K_{p-1}(z_0)$. Hence by (2.3), $|\rho_{pq}(z_0)|^2\geqq |Y_q(z_0)|^2k^2\cdot|Y_p(z_0)|^2$, for each q and for all sufficiently large values of p. Since $|Y_q(z_0)|>0$ by (3.12), and since the series $\sum|Y_p(z_0)|^2$ is divergent, it follows that the series

$$\sum_{p=1}^{\infty}|\rho_{pq}(z_0)|^2$$

is divergent. Therefore the matrix $(\rho_{pq}(z_0))$ is unbounded.

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ILLINOIS INSTITUTE OF TECHNOLOGY

BOUNDED J-FRACTIONS

H. S. WALL

1. Introduction. A J-fraction

$$(1.1)\qquad \cfrac{1}{b_1+z-\cfrac{a_1^2}{b_2+z-\cfrac{a_2^2}{b_3+z-\cdots}}},$$

in which the coefficients a_p and b_p are complex constants and z is a complex parameter, is said to be *bounded* if there exists a constant M such that

$$(1.2)\qquad |a_p| \leqq M/3, \qquad |b_p| \leqq M/3, \qquad p = 1, 2, 3, \cdots .$$

This condition can be formulated in terms of J-forms in accordance with the following theorem.

THEOREM 1.1. *The J-fraction* (1.1) *is bounded if and only if there exists a constant N such that*

$$(1.3)\qquad \begin{aligned} &\left| \sum_{p=1}^{n} b_p u_p v_p - \sum_{p=1}^{n-1} a_p(u_p v_{p+1} + u_{p+1} v_p)\right| \\ &\qquad \leqq N\left(\sum_{p=1}^{n} |u_p|^2 \cdot \sum_{p=1}^{n} |v_p|^2\right)^{1/2}, \qquad n = 1, 2, 3, \cdots, \end{aligned}$$

for all values of the variables u_p and v_p, the constant N being independent of the variables and of n.

In fact, if (1.3) holds then we find, on specializing the values of the u_p and v_p, that $|b_p| \leqq N$, $|a_p| \leqq N$, $p=1, 2, 3, \cdots$; and if (1.2) holds then, by Schwarz's inequality, (1.3) holds with $N=M$.

If (1.3) holds, then the J-form $\sum b_p u_p v_p - \sum a_p(u_p v_{p+1} + u_{p+1} v_p)$ is said to be *bounded*, and the least value of N which can be used in that inequality is called the *norm* of the J-form. We shall also call this number the *norm* of the J-fraction. When (1.2) holds then, as pointed out above, (1.3) holds with $N=M$. *Hence the norm of the J-fraction does not exceed the least number M which can be used in* (1.2).

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THEOREM 1.2. *If* (1.2) *holds, then the J-fraction converges uniformly for* $|z| \geqq M$.

For if the J-fraction is transformed by an equivalence transformation so that all the partial denominators are equal to unity, then the nth partial numerator is

$$-\frac{a_{n-1}^2}{(b_{n-1}+z)(b_n+z)}.$$

If $|z| \geqq M$, and (1.2) holds, this has modulus not greater than 1/4. Hence it follows by a well known theorem that the J-fraction converges uniformly for $|z| \geqq M$.

In the case of the J-fraction

$$\cfrac{1}{1+z-\cfrac{1}{1+z-\cfrac{1}{1+z-\cdots}}},$$

the least number M which can be used in (1.2) is $M=3$. Hence the J-fraction converges uniformly for $|z| \geqq 3$. It diverges if z is real, positive, and less than 3. On the other hand, for the J-fraction

$$\cfrac{1}{z-\cfrac{(1/4)}{z-\cfrac{(1/4)}{z-\cdots}}},$$

the least value of M which can be used in (1.2) is $M=3/2$. The norm of this J-fraction is $N=1$, and it converges for $|z| \geqq 1$. In fact, it converges if z is not on the real interval $-1<x<+1$.

The principal object of this note is to show that a J-fraction with norm N converges if z is not in a certain convex set contained in the circle $|z|=N$. Moreover, if the partial numerators a_p^2 are different from zero, the corresponding J-matrix has a unique bounded reciprocal for all z not in this convex set. It was shown by Hellinger and Toeplitz [3][1] that there is a unique bounded reciprocal for $|z|>N$.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

2. Convergence of bounded *J*-fractions. Let $a=e^{i\theta}$ be a complex number with modulus unity. Then the *J*-fractions (1.1) and

$$
\tag{2.1} \cfrac{a}{b_1a+Z-\cfrac{(a_1a)^2}{b_2a+Z-\cfrac{(a_2a)^2}{b_3a+Z-\cdots}}}\,, \qquad Z = az,
$$

are equivalent in the sense that their nth approximants are identical with one another for $n=1, 2, 3, \cdots$. Also, they obviously have one and the same norm.

Let

$$
\tag{2.2} \alpha_p(\theta) = I(a_pa), \qquad \beta_p(\theta) = I(b_pa), \qquad p = 1, 2, 3, \cdots .
$$

Then, if (1.1) is bounded, it follows from (1.3) that there exists a finite constant $Y(\theta)$ such that

$$
\tag{2.3} \sum_{p=1}^{n} [\beta_p(\theta) + Y(\theta)]x_p^2 - 2\sum_{p=1}^{n-1} \alpha_p(\theta)x_px_{p+1} \geqq 0, \quad n = 1, 2, 3, \cdots ,
$$

for all real values of $x_1, x_2, x_3, \cdots$. If $Y_0(\theta)$ is the least value of $Y(\theta)$ which can be used in (2.3), then we must have

$$
| Y_0(\theta) | \leqq N, \qquad 0 \leqq \theta < 2\pi,
$$

where N is the norm of the *J*-fraction.

From (2.3) it follows that if we put $Z=iY(\theta)+\zeta$ in (2.1), then (2.1) is a positive definite *J*-fraction in the variable ζ. Therefore, if c is a positive constant, the nth approximant of (1.1), which is the same as the nth approximant of (2.1), satisfies the inequality [1]

$$
\left| \frac{A_n(z)}{B_n(z)} \right| \leqq \frac{1}{c},
$$

provided $I(\zeta) \geqq c$, that is, provided

$$
\tag{2.4} x \sin \theta + y \cos \theta \geqq Y(\theta) + c, \qquad \text{where } z = x + iy;
$$

and $B_n(z) \neq 0$ when (2.4) holds. This can be interpreted geometrically as follows. Let K denote the set of all points $z=x+iy$ such that

$$
x \sin \theta + y \cos \theta \leqq Y(\theta) \qquad \text{for } 0 \leqq \theta < 2\pi.
$$

Then, K is a convex set of which the straight lines $x \sin \theta + y \cos \theta = Y(\theta)$ are the supporting lines; the zeros of all the denominators $B_n(z)$ of the J-fraction (1.1) are in K; the approximants of the J-fraction are uniformly bounded over any domain whose distance from K is positive. We shall let K_0 denote the convex set determined in this way corresponding to the function $Y_0(\theta)$ defined above.

By Theorem 1.2, the J-fraction converges if $|z|$ is sufficiently large. We may then conclude immediately by a theorem of Stieltjes [6] that the following theorem is true.

THEOREM 2.1. *A bounded J-fraction converges uniformly over every bounded closed region whose distance from the convex set K_0 is positive. In particular, the J-fraction converges if $|z| > N$, where N is the norm of the J-fraction.*

We note the following special cases. If the coefficients a_p and b_p are all real, then $Y_0(0) = Y_0(\pi) = 0$, so that K_0 reduces to an interval of the real axis contained in the interval $-N \leqq x \leqq +N$. If the a_p are pure imaginary and the b_p are real and positive, then the set K_0 is contained in the left half-plane, $x = R(z) \leqq 0$.

3. **Bounds for the zeros of a polynomial.** The preceding considerations furnish a method for determining bounds for the zeros of a polynomial. Let $P(z)$ be a polynomial of degree n, $n > 1$, and let $Q(z)$ be any polynomial of degree $n-1$ such that there is a continued fraction expansion of the form

$$\frac{Q(z)}{P(z)} = \cfrac{c}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{\ddots}{\qquad - \cfrac{a_{n-1}^2}{b_n + z}}}}, \tag{3.1}$$

where $a_p \neq 0$, $p = 1, 2, 3, \cdots, n-1$, and $c \neq 0$. This expansion can be easily obtained by applying the euclidean algorithm for the greatest common divisor to $Q(z)$ and $P(z)$. Let K_0 be the convex set which is associated with this J-fraction in the manner indicated in §2. Then the zeros of $P(z)$ are all contained in K_0.

One may readily obtain a rectangle containing the set K_0. In fact, if we use the notation of §2, such a rectangle is given by

$$\begin{aligned} y &\leqq Y(0), & x &\leqq Y(\pi/2), \\ y &\geqq -Y(\pi), & x &\geqq -Y(3\pi/2). \end{aligned}$$

This rectangle is obtained by minimizing four *real* quadratic forms.

By way of illustration, let $P(z)=z^3+(2+i)z^2+(3+i)z+(2i+2)$, and take $Q(z)=2z^2+iz+2$. Then,

$$\frac{Q(z)}{P(z)}=\cfrac{2}{(2+i/2)+z-\cfrac{(3i/2)^2}{-i/6+z-\cfrac{(8^{1/2}i/3)^2}{2i/3+z}}}.$$

We require lower bounds $-Y(\theta)$ for the quadratic form

$$\begin{aligned}(2\sin\theta+(1/2)\cos\theta)x_1^2-(1/6)\cos\theta x_2^2+(2/3)\cos\theta x_3^2\\ -3\cos\theta x_1x_2-(2^{5/2}/3)\cos\theta x_2x_3,\end{aligned}$$

under the condition $x_1^2+x_2^2+x_3^2=1$, and for $\theta=0,\ \pi/2,\ \pi,\ 3\pi/2$. Easily determined lower bounds are given by

$$Y(0)=19/6,\qquad Y(\pi/2)=0,\qquad Y(\pi)=11/3,\qquad Y(3\pi/2)=2.$$

Therefore, the zeros of $P(z)$ are contained in the rectangle

$$\begin{aligned} y &\leqq 19/6, & x &\leqq 0,\\ y &\geqq -11/3, & x &\geqq -2.\end{aligned}$$

The zeros of $P(z)$ are actually equal to

$$-1-i,\qquad \frac{-1-7^{1/2}i}{2},\qquad \frac{-1+7^{1/2}i}{2}.$$

The size of the rectangle depends upon the choice of the polynomial $Q(z)$. In fact, it is easy to show that the zeros of $Q(z)$ also lie in the convex set K_0. Furthermore, the computational difficulties are less for some choices of $Q(z)$ than they are for other choices. Let

$$P(z)=z^n+(p_1+iq_1)z^{n-1}+(p_2+iq_2)z^{n-2}+\cdots+(p_n+iq_n).$$

Then, if

$$Q(z)=p_1z^{n-1}+iq_2z^{n-2}+p_3z^{n-3}+iq_4z^{n-4}+\cdots,$$

the computation involved in obtaining the J-fraction expansion for $Q(z)/P(z)$ is especially simple. Moreover, from this expansion one can determine immediately the number of zeros of $P(z)$ in each of the half-planes $R(z)<0$ and $R(z)>0$. For details, we refer the reader to a recent paper of Frank [2].

4. The bounded reciprocal of a bounded J-matrix. We suppose that (1.1) is bounded, that $a_p \neq 0$, $p = 1, 2, 3, \cdots$, and consider the J-matrix

$$(4.1)\quad J + zI = \begin{pmatrix} b_1 + z, & -a_1, & 0, & 0, & 0, & \cdots \\ -a_1, & b_1 + z, & -a_2, & 0, & 0, & \cdots \\ 0, & -a_2, & b_2 + z, & -a_3, & 0, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

If the norm of (1.1) is N, and $|z| > N$, then the matrix $J + zI$ has a unique bounded reciprocal which is given by

$$(J + zI)^{-1} = \frac{I}{z} - \frac{J}{z^2} + \frac{J^2}{z^3} - \cdots .$$

This is a matrix whose elements are power series in $1/z$, convergent for $|z| > N$. In particular, the element in the first row and first column is the power series expansion of the J-fraction, and its sum is the value of the J-fraction (Hellinger and Toeplitz [3]).

We can now show that $J + zI$ has a unique bounded reciprocal for any z not in the set K_0 defined in §2. In fact, if we put $Z = iY_0(\theta) + \zeta$ in (2.1), then, as we have seen, the J-fraction is a positive definite J-fraction in the variable ζ. The corresponding J-matrix is

$$(4.2)\qquad aJ + iY_0(\theta)I + \zeta I.$$

Inasmuch as the series $\sum(1/|aa_p|)$ is divergent, the *determinate case* holds for the J-fraction [1] and consequently [7] the matrix (4.2) has a unique bounded reciprocal for $I(\zeta) > 0$. *We therefore conclude immediately that the J-matrix $J + zI$ has a unique bounded reciprocal for any z not in the set K_0 defined in* §2.

5. **Functions represented by J-fractions.** Every infinite subsequence of approximants of a positive definite J-fraction contains an infinite subsequence which converges for $I(z) > 0$ to a function which is analytic and has a negative imaginary part in this domain, and which has the form

$$(5.1)\qquad f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u},$$

where $\phi(u)$ is bounded and nondecreasing. There are functions which are analytic and have negative imaginary parts for $I(z) > 0$ which are not limits of sequences of approximants of positive J-fractions. In fact, the most general function of this description has the form

$$(5.2)\qquad \begin{aligned} &\int_{-\infty}^{+\infty} \frac{1+zu}{z-u}\, d\phi(u) + a - bz \\ &\qquad = \int_{-\infty}^{+\infty} \left(\frac{u}{1+u^2} + \frac{1}{z-u}\right)(1+u^2)\,d\phi(u) + a - bz, \end{aligned}$$

where a and b are real, $b \geqq 0$, and $\phi(u)$ is bounded and nondecreasing. This can be seen as follows. F. Riesz [5] and Herglotz [4] showed that a function $f(w)$ is analytic and has a positive real part for $|w| < 1$ if and only if it has the form

$$f(w) = \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w}\, d\sigma(t) + ia,$$

where a is real and $\sigma(t)$ is bounded and nondecreasing. If we multiply this integral by $-i$ and make the substitution

$$(5.3)\qquad w = \frac{1+iz}{1-iz},$$

mapping the unit circle upon the upper half-plane, we obtain after simple transformations

$$\int_0^{2\pi} \frac{1 - z\tan(t/2)}{\tan(t/2) + z}\, d\sigma(t) + a.$$

This can be transformed into (5.2) if we put $u = \tan(t/2)$.

We take this occasion to point out that there exists an identical continued fraction transformation of the integral (5.2). We have the following theorem.

THEOREM 5.1. *A necessary and sufficient condition for a function to be analytic and have a negative imaginary part for* $I(z) > 0$ *is that it have a continued fraction expansion of the form*

$$(5.4\qquad \cfrac{c}{z - r_0 - \cfrac{g_1(1+z^2)}{z - r_1 - \cfrac{(1-g_1)g_2(1+z^2)}{z - r_2 - \cfrac{(1-g_2)g_3(1+z^2)}{z - r_3 - \cdots}}}},$$

where $c > 0$, $0 < g_p < 1$, $-\infty < r_{p-1} < +\infty$, $p = 1, 2, 3, \cdots$, *or a terminat-*

ing continued fraction expansion of this form in which the last g_p which appears may be equal to unity. The continued fraction converges uniformly over every bounded closed region within the half-plane $I(z) > 0$, and is uniquely determined by the function expanded.

To prove this, it is only necessary to make the substitution

$$z = \frac{4w}{(1 - w)^2}$$

in the continued fraction (3.14) of [8], multiply the resulting continued fraction by $-i$, and then make the substitution (5.3) above.

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ON ANALYTIC FUNCTIONS WITH BOUNDED CHARACTERISTIC

HANS FRIED[1]

A function $f(re^{i\phi})$, regular within the unit circle, is called a function with bounded characteristic if

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\phi})|\, d\phi$$

is bounded, where $\log^+ |f(re^{i\phi})| = \max (\log|f(re^{i\phi})|, 0)$. If $f(z)$ is a function with bounded characteristic, then

$$\lim_{r\to 1} f(re^{i\phi}) = f(e^{i\phi})$$

exists almost everywhere [1].[2]

In the first part of this paper we prove the following:

THEOREM I. *Let* $\{f_n(z)\}$ $(n=1, 2, 3, \cdots)$ *and* $f(z)$ *be functions with bounded characteristics, let*

$$(1)\qquad \begin{aligned} \log A_n &= \lim_{r\to 1} \int_0^{2\pi} \log^+ |f_n(re^{i\phi})|\, d\phi, \\ \log A &= \lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\phi})|\, d\phi, \end{aligned}$$

and

(2) $|f(e^{i\phi})-f_n(e^{i\phi})| < m_n$, *for* $\phi \in E_n$, *and let* μ_n *be the measure of* E_n.

If

$$(3)\qquad \lim_{n\to\infty} m_n^{\mu_n} = 0,$$

and for every positive σ *there exists a positive integer* n_σ *such that*

$$(4)\qquad A_n < m_n^{-\sigma\mu_n} \quad for \quad n > n_\sigma,$$

then the sequence $\{f_n(z)\}$ *tends uniformly to* $f(z)$ *in any closed domain interior to the unit circle.*

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[1] Deceased December 23, 1945.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

Special cases of this theorem are the following:

A. If the numbers A_n are bounded, we need only the condition (3), as (4) follows from (3) in this case.

B. If the numbers A_n are bounded, $E_1=E_2=\cdots=E_n=\cdots$ and $\mu_1>0$, we need only the condition $\lim_{n\to\infty} m_n=0$ instead of (3) and (4). This case was proved by Ostrowski [1].

C. If $E_1=E_2=\cdots=E_n=\cdots$ and $\mu_1>0$, we can replace (3) and (4) by the following two conditions: $\lim_{n\to\infty} m_n=0$, and for every positive σ there exists a positive integer n_σ such that $A_n<m_n^{-\sigma}$ for $n>n_\sigma$. This case was proved by Milloux [2] under the less general assumptions that the functions $f_n(z)$ and $f(z)$ are bounded, and that $E_1=E_2=\cdots=E_n=\cdots$ is an arc (α,β) of the unit circle on which the functions $f_n(z)$ and $f(z)$ are continuous.

In the second part of this paper we prove the following theorem.

THEOREM II. *Let*

$$f(z)=\sum_{k=1}^{\infty}c_k z^{n_k} \tag{5}$$

be a function with bounded characteristic and let

$$\frac{n_{k+1}}{n_k}\geqq q>1 \qquad (k=1,2,\cdots); \tag{6}$$

then

$$\sum_{k=1}^{\infty}|c_k|^2 \tag{7}$$

converges.

This theorem generalizes a theorem of R. Paley [3], which proved that if $f(z)$ has an expansion (5) where the n_k satisfy (6) and

$$\lim_{r\to 1}\int_0^{2\pi}|f(re^{i\phi})|\,d\phi \tag{8}$$

is bounded, then the series (7) converges. From $\log^+|f(z)|<|f(z)|$ it follows that each function which satisfies (8) is of bounded characteristic. Therefore, Paley's theorem is a consequence of Theorem II. We shall show that Theorem II gives a negative answer to the following question asked by Bloch [4]: If $f(z)$ is a function with bounded characteristic, must the derivative of $f(z)$ have the same property?

1. Proof of Theorem I. We put $\phi_n(z)=f(z)-f_n(z)$ and $\phi_n(e^{i\phi})=f(e^{i\phi})-f_n(e^{i\phi})$. We have

$$
(9)\qquad \begin{aligned} \log^+ |\phi_n(z)| = \log^+ |f(z) - f_n(z)| &\leqq \log^+ (|f(z)| + |f_n(z)|) \\ &\leqq \log^+ |f(z)| + \log^+ |f_n(z)| + \log 2. \end{aligned}
$$

Let $\{r_k\}$ be an increasing sequence of positive numbers with

$$
\lim_{k\to\infty} r_k = 1.
$$

By Jensen's formula we have, for $r<r_k$,

$$
(10)\qquad \begin{aligned} \log |\phi_n(re^{i\phi})| + \sum_{\mu=1}^{m} \log \left| \frac{r_k^2 - \bar{a}_\mu r e^{i\phi}}{r_k(re^{i\phi} - a_\mu)} \right| \\ = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r_k^2 - r^2) \log |\phi_n(r_k e^{i\theta})|}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2} d\theta \end{aligned}
$$

where $a_1, a_2, \cdots, a_m$ are the zeros of $\phi_n(z)$ within the circle of radius r_k. As the second term on the left side of (10) is positive we have

$$
(11)\qquad 2\pi \log |\phi_n(re^{i\phi})| \leqq \int_0^{2\pi} \frac{(r_k^2 - r^2) \log |\phi_n(r_k e^{i\theta})| \, d\theta}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2}.
$$

As $\lim_{k\to\infty}\phi_n(r_k e^{i\phi}) = \phi_n(e^{i\phi})$ for $\phi \in E_n$, there exists by Egoroff's theorem, for every positive δ_1, a set $E_n' < E_n$ $(n=1, 2, \cdots)$ such that

$$
(12)\qquad \mu(E_n') > \mu_n - \delta_1,
$$

and

$$
(13)\qquad \lim_{k\to\infty} \phi_n(r_k e^{i\phi}) = \phi_n(e^{i\phi})
$$

uniformly in E_n'.

Because of (3) we can assume $m_n < 1$ $(n=1, 2, \cdots)$; hence it follows from (2) and (13), that

$$
(14)\qquad |\phi_n(r_k e^{i\phi})| < m_n + \delta_2 < 1,
$$

for $\phi \in E_n'$ and for sufficiently large k, where δ_2 is arbitrarily small. We denote by E_n'' the complement of E_n' with respect to $(0, 2\pi)$. Then (11) takes the form

$$
\begin{aligned} 2\pi \log |\phi_n(re^{i\phi})| \leqq \int_{E_n''} \frac{(r_k^2 - r^2) \log^+ |\phi_n(r_k e^{i\theta})| \, d\theta}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2} \\ + \int_{E_n'} \frac{(r_k^2 - r^2) \log |\phi_n(r_k e^{i\theta})| \, d\theta}{r_k^2 - 2r_k r \cos(\phi - \theta) + r^2}. \end{aligned}
$$

As the first integral on the right side is positive and the second negative, we have, because of (1), (9) and (14),

$$
\begin{aligned}
2\pi \log \left| \phi_n(re^{i\phi}) \right| &\leqq \frac{r_k + r}{r_k - r} \int_{B_n''} \log^+ \left| \phi_n(re^{i\theta}) \right| d\theta \\
&\quad + \frac{r_k - r}{r_k + r} \int_{B_n'} \log \left| \phi_n(re^{i\theta}) \right| d\theta \\
&\leqq \frac{r_k + r}{r_k - r} \log 2AA_n \\
&\quad + \frac{r_k - r}{r_k + r} (\mu_n - \delta_1) \log (m_n + \delta_2),
\end{aligned}
$$

for sufficiently large k. We get, therefore, for $\lim_{k\to\infty} r_k = 1$, as δ_1 and δ_2 are arbitrarily small,

$$
2\pi \log \left| \phi_n(re^{i\phi}) \right| \leqq \frac{1+r}{1-r} \log 2AA_n + \frac{1-r}{1+r} \mu_n \log m_n,
$$

or

$$
\left| \phi_n(re^{i\phi}) \right| = (2AA_n)^{(1+r)/2\pi(1-r)} m_n^{\mu_n(1-r)/2\pi(1+r)}. \tag{15}
$$

Let B be any closed domain within the unit circle. There exists an $r' < 1$, such that the interior of the circle with radius r' contains B. Now we choose

$$
\sigma < \left(\frac{1-r'}{1+r'} \right)^2. \tag{16}
$$

If we put $A' = \max (A, 1)$ and $A_n' = \max (A_n, 1)$ we get, for $re^{i\phi} \in B$, by use of (4),

$$
\left| \phi_n(re^{i\phi}) \right| \leqq (2A')^{(1+r')/2\pi(1-r')} m_n^{(\mu_n/2\pi)((1-r')/(1+r') - \sigma(1+r')/(1-r'))}
$$

for sufficiently large n.

As, because of (16), $(1-r')/(1+r') - \sigma(1+r')/(1-r') > 0$ and since, by assumption, $\lim_{n\to\infty} m_n^{\mu_n} = 0$, we have proved the theorem.

2. **Proof of Theorem II.** For the proof of Theorem II we need the following two theorems, the first due to Hardy and Littlewood [5], the second to Zygmund [6].

THEOREM A. *If* $f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$ and $n_{k+1}/n_k \geqq q > 1$ $(k = 1, 2, \cdots)$, *then the existence of* $\lim_{r\to 1} f(re^{i\phi_0}) = f(e^{i\phi_0})$ *implies the convergence of* $\sum_{k=1}^{\infty} c_k e^{in_k\phi_0}$.

THEOREM B. *If the trigonometric series*

$$\sum_{k=1}^{\infty} (a_k \cos n_k\theta + b_k \sin n_k\theta) \qquad (n_k + 1/n_k \geqq q > 1)$$

converges in a set of positive measure, then the series

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

converges.

As $f(z)$ is a function with bounded characteristic

$$\lim_{r\to 1} f(re^{i\phi}) = f(e^{i\phi})$$

exists almost everywhere. If we put $c_k = a_k - ib_k$ then by Theorem A the series

$$\sum_{k=1}^{\infty} (a_k - ib_k)(\cos n_k\phi + i \sin n_k\phi) = \sum_{k=1}^{\infty} (a_k \cos n_k\phi + b_k \sin n_k\phi) + i\sum_{k=1}^{\infty} (a_k \sin n_k\phi - b_k \cos n_k\phi)$$

converges almost everywhere. Therefore, by Theorem B, the series

$$\sum_{k=1}^{\infty} a_k^2 + b_k^2 = \sum_{k=1}^{\infty} |c_k|^2$$

converges.

Now we shall show that there exist bounded functions, whose derivatives are functions whose characteristics are not bounded.

The radius of convergence of the power series

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{n_k}}{n_k} \qquad \left(\frac{n_{k+1}}{n_k} \geqq q > 1\right)$$

is equal to 1 and the function $f(z)$ is bounded within the unit circle. The derivative

$$f'(z) = \sum_{k=1}^{\infty} z^{n_k - 1}$$

is, by Theorem II, a function whose characteristic is not bounded.

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ON THE (C, 1) SUMMABILITY OF CERTAIN RANDOM SEQUENCES

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It is known [1][1] that if a sequence $\{a_n\}$ $(n=1, 2, \cdots)$ of real numbers is summable $(C, 1)$ to a value α, and if $\sum a_n^2/n^2 < \infty$, then almost all the subsequences of $\{a_n\}$ are summable $(C, 1)$ to α. It will be shown that this statement continues to hold if "almost all" is replaced by "with probability 1" and "subsequences" by the more general term "product sequences," the meaning of which will be defined in the next paragraph. The only analytic tool used is the strong law of large numbers [2]: if $\{y_n\}$ is a sequence of independent random variables with expected values $E(y_n)=0$ and $E(y_n^2)=b_n^2$, for which $\sum b_n^2/n^2 < \infty$, then with probability 1 the sequence $\{y_n\}$ is summable $(C, 1)$ to the value 0.

DEFINITION. Let $\{a_n\}$ be a sequence of constants and let $\{x_n\}$ be a sequence of random variables such that the values of each x_n are non-negative integers. For every n let $k(n)$ be the least positive integer m such that

$$(1) \qquad \sum_{1}^{m} x_i \geqq n,$$

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[1] Numbers in brackets refer to references listed at end of paper.

and set $w_n = a_{k(n)}$. (In intuitive terms this definition of w_n is equivalent to the following. Starting with the sequence

$$x_1a_1,\ x_2a_2,\ \cdots,\ x_na_n,\ \cdots, \tag{2}$$

strike out each term for which $x_i = 0$, and then replace each remaining term x_ia_i by the block $a_i, a_i, \cdots, a_i$ with x_i terms. Then w_n is the nth term in the resulting sequence.) The sequence $\{w_n\}$ of random variables will be called the product of $\{a_n\}$ by $\{x_n\}$.

THEOREM 1. *Let $\{a_n\}$ be a sequence of real numbers summable $(C, 1)$ to the value α. Let $\{x_n\}$ be a sequence of independent, non-negative, integer-valued random variables with $E(x_n) = \bar{x}$ $(0 < \bar{x} < \infty)$, $E(x_n - \bar{x})^2 = d_n^2$, and such that*

$$\sum_1^\infty \frac{d_n^2}{n^2} < \infty, \tag{3}$$

$$\sum_1^\infty \frac{a_n^2 d_n^2}{n^2} < \infty. \tag{4}$$

Then with probability 1 *the product sequence $\{w_n\}$ is summable $(C, 1)$ to the value α.*

PROOF. It follows from the strong law of large numbers applied to the sequence $\{x_n - \bar{x}\}$ that with probability 1,

$$\begin{aligned} &\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \\ &\qquad = \lim_{n\to\infty}\left[\frac{(x_1 - \bar{x}) + \cdots + (x_n - \bar{x})}{n} + \bar{x}\right] = \bar{x} \neq 0. \end{aligned} \tag{5}$$

Likewise it follows from the same law applied to the sequence $\{a_n(x_n - \bar{x})\}$ that with probability 1,

$$\begin{aligned} &\lim_{n\to\infty} \frac{x_1a_1 + \cdots + x_na_n}{n} \\ &= \lim_{n\to\infty}\left[\frac{a_1(x_1 - \bar{x}) + \cdots + a_n(x_n - \bar{x})}{n} + \bar{x}\,\frac{a_1 + \cdots + a_n}{n}\right] = \bar{x}\alpha. \end{aligned} \tag{6}$$

But

$$\frac{x_1a_1 + \cdots + x_na_n}{x_1 + \cdots + x_n} = \left[\frac{x_1a_1 + \cdots + x_na_n}{n}\right]\left[\frac{n}{x_1 + \cdots + x_n}\right]. \tag{7}$$

Hence with probability 1,

$$\lim_{n\to\infty} \frac{x_1a_1 + \cdots + x_na_n}{x_1 + \cdots + x_n} = a. \tag{8}$$

Moreover, from (5) we note that, with probability 1, $\lim (x_1 + \cdots + x_n) = \infty$, and hence, with probability 1, $k(n)$ is defined for every n.

Now we introduce the abbreviations

$$r(n) = n - (x_1 + \cdots + x_{k(n)-1}), \tag{9}$$

$$p(n) = x_1a_1 + \cdots + x_{k(n)-1}a_{k(n)-1}, \tag{10}$$

$$q(n) = x_1 + \cdots + x_{k(n)-1}, \tag{11}$$

noting that from the definitions it follows that

$$0 < r(n) \leqq x_{k(n)}, \tag{12}$$

and that

$$\frac{w_1 + \cdots + w_n}{n} = \frac{a_{k(1)} + \cdots + a_{k(n)}}{n} = \frac{p(n) + r(n)a_{k(n)}}{q(n) + r(n)}. \tag{13}$$

There are now two cases to consider. If $a_{k(n)} \geqq p(n)/q(n)$ then it is easily seen from (13) that

$$\begin{aligned} \frac{x_1a_1 + \cdots + x_{k(n)-1}a_{k(n)-1}}{x_1 + \cdots + x_{k(n)-1}} &= \frac{p(n)}{q(n)} \leqq \frac{w_1 + \cdots + w_n}{n} \\ &\leqq \frac{p(n) + x_{k(n)}a_{k(n)}}{q(n) + x_{k(n)}} = \frac{x_1a_1 + \cdots + x_{k(n)}a_{k(n)}}{x_1 + \cdots + x_{k(n)}}. \end{aligned} \tag{14}$$

However, if $a_{k(n)} \leqq p(n)/q(n)$ then (14) holds with both inequalities reversed. Since $k(n)$ becomes infinite with n, it follows from these inequalities and (8) that with probability 1,

$$\lim_{n\to\infty} \frac{w_1 + \cdots + w_n}{n} = a, \tag{15}$$

which completes the proof of the theorem.

If in Theorem 1 we let each x_n assume the values 0 and 1 with probabilities 1/2 and 1/2, then to each sequence $\{x_n\}$ corresponds the real number $0 \leqq x \leqq 1$ with dyadic expansion

$$x = .x_1x_2 \cdots x_n \cdots, \tag{16}$$

and probability in the space of sequences $\{x_n\}$ is identical with Lebesgue measure in the unit interval. Moreover, "product sequence" becomes "subsequence," and hence Theorem 1 specializes to the result referred to at the beginning of this note.

As another special case of Theorem 1 we shall derive a theorem on repetition sequences.

THEOREM 2. *Let* $\{a_n\}$ *be a sequence of real numbers summable* $(C, 1)$ *to the value* α *and such that* $\sum a_n^2/n^2 < \infty$. *To each* $0 \leqq t \leqq 1$ *with dyadic expansion*

$$t = .t_1t_2 \cdots t_n \cdots \qquad (t_n = 0 \text{ or } 1) \tag{17}$$

let correspond the sequence $\{v_n\}$, *where* $v_1 = a_1$ *and*

$$v_{n+1} = a_{(1+t_1+\cdots+t_n)} \qquad (n = 1, 2, \cdots). \tag{18}$$

Then for almost every t *the sequence* $\{v_n\}$ *is summable* $(C, 1)$ *to the value* α.

PROOF. In Theorem 1 let each x_n take on all positive integral values, with $Pr(x=k) = 2^{-k}$ $(k=1, 2, \cdots)$. Then

$$E(x_n) = \bar{x} = \sum_1^\infty k2^{-k} = 2; \quad E(x_n - \bar{x})^2 = \sum_1^\infty (k-2)^2 2^{-k} = 2. \tag{19}$$

Thus the hypotheses in Theorem 1 on the sequence $\{x_n\}$ are satisfied, and hence, with probability 1, the product sequence $\{w_n\}$ of $\{a_n\}$ by $\{x_n\}$ is summable $(C, 1)$ to the value α. Now to each t defined by (17) let correspond the sequence $\{x_n\}$ such that

(20) $x_1 = 1$ plus the number of consecutive 0's immediately following the decimal point in (17),

and for $n = 1, 2, \cdots$

(21) $x_{n+1} = 1$ plus the number of consecutive 0's immediately following the nth 1 in (17).

It is easy to verify that this correspondence $t \leftrightarrow \{x_n\}$ is one-to-one between the interval $0 \leqq t \leqq 1$ and the space of sequences $\{x_n\}$, and that it carries Lebesgue measure in the former into probability in the latter. Moreover, if $t \leftrightarrow \{x_n\}$ then the sequence $\{v_n\}$ associated with t by (18) is identical with the product $\{w_n\}$ of $\{a_n\}$ by $\{x_n\}$. Since with probability 1 the sequence $\{w_n\}$ is summable $(C, 1)$ to α, it follows that for almost every t the sequence $\{v_n\}$ is summable $(C, 1)$ to α, which was to be proved.

We conclude with an application of Theorem 2 to random sequences of transformations (compare [3]). Let $\{U_n\}$ $(n=0, 1, 2, \cdots;$ $U_0 = I = \text{identity})$ be a sequence of transformations of a space M

into itself, let A be a subset of M, and let p be a point of M. Denote the characteristic function of A by

$$\phi_A(q) = \begin{matrix} 1 & \text{if} & q \in A, \\ 0 & \text{if} & q \notin A. \end{matrix} \tag{22}$$

If the sequence $\{\phi_A(U_n(p))\}$ is summable $(C, 1)$ to the value α let us say that the triple $(\{U_n\}, A, p)$ has the ergodic limit α. Let T be a fixed transformation of M into itself such that $(\{T^n\}, A, p)$ has the ergodic limit α, and to each t in the interval $0 \leqq t \leqq 1$ with dyadic expansion (17) let correspond the sequence of transformations $\{U_n\}$ where $U_0 = I$ and

$$U_n = \begin{matrix} TU_{n-1} & \text{if} & t_n = 1 \\ IU_{n-1} = U_{n-1} & \text{if} & t_n = 0 \end{matrix} \quad (n = 1, 2, \cdots). \tag{23}$$

If we set $a_n = \phi_A(T^{n-1}(p))$ $(n = 1, 2, \cdots)$ and $v_n = \phi_A(U_{n-1}(p))$, then $v_1 = a_1$ and (18) holds. It follows at once from Theorem 2 that for almost every t the triple $(\{U_n\}, A, p)$ has the ergodic limit α.

If a completely additive measure μ is defined on a σ-field of subsets of M with $\mu(M) = 1$, and if the triple $(\{U_n\}, A, p)$ has the ergodic limit α except for a set of p with μ-measure 0, let us say that the pair $(\{U_n\}, A)$ has ergodic limit α. Assume that this holds for $(\{T^n\}, A)$. Consider the Cartesian product of M with the unit interval $0 \leqq t \leqq 1$, and let H denote the set of all pairs (p, t) for which the sequence $\{\phi_A(U_n(p))\}$ is summable $(C, 1)$ to α, where $\{U_n\}$ is defined by (23). By the result of the preceding paragraph the intersection of H with any fixed p-line (except for a set of p of μ-measure 0) has Lebesgue measure 1. If H is measurable it follows by Fubini's theorem that the intersection of H with any fixed t-line (except for a set of t of Lebesgue measure 0) has μ-measure 1. Hence for almost all t the pair $(\{U_n\}, A)$ has the ergodic limit α.

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APPROXIMATE ISOMETRIES

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In a recent paper [1][1] Hyers and Ulam formulated the problem of approximate isometries. Thus if E_1 and E_2 are metric spaces, a transformation T on E_1 to E_2 is an ϵ isometry if $|d_1(T(x), T(x')) - d(x, x')| < \epsilon$, for all x, x' in E_1. These authors analyzed the ϵ isometries defined on a complete abstract Euclidean space E and showed that if T maps E onto itself and $T(\theta) = \theta$, then there exists an isometry [2, p. 165], U, of E onto E such that $\|T(x) - U(x)\| < 10\epsilon$. The analysis depends on the properties of the scalar product. In the present work we show, first, that similar results hold when $E_1 = E_2 = L_r(0, 1)$, $1 < r < \infty$, though, except of course for $r = 2$, a scalar product no longer exists. It is shown further that it is sufficient that E_2 belong to a restricted class of uniformly convex Banach spaces and that E_1 be a Banach space.

THEOREM 1. *Let $T(x)$ be an ϵ isometry of $L_r(0, 1)$, $1 < r < \infty$, into itself with $T(\theta) = \theta$. Then $U(x) = L_{n\to\infty} T(2^n x)/2^n$ exists for each x and $U(x)$ is an isometric, linear transformation.*

Our fundamental assumption is that

$$\big| \| T(x) - T(x') \| - \| x - x' \| \big| < \epsilon, \qquad T(\theta) = \theta. \tag{1.01}$$

The following inequality is due to Clarkson [3, 4],

$$\|\alpha + \beta\|^p + \|\alpha - \beta\|^p \leqq 2(\|\alpha\|^q + \|\beta\|^q)^{p-1}, \tag{1.02}$$

where here and later we understand that

$$p = \sup (r/(r-1)) \geqq 2 \geqq q = \inf (r, r/(r-1)).$$

Let

$$2\alpha = T(x),\ 2\beta = T(x) - T(2x).$$

Then

$$\begin{aligned} &\|T(x) - T(2x)/2\|^p \\ &\leqq 2^{1-q(p-1)}(\|T(x)\|^q + \|T(x) - T(2x)\|^q)^{p-1} - \|T(2x)/2\|^p \\ &\leqq (\|x\| + \epsilon)^p - (\|x\| - \epsilon/2)^p. \end{aligned} \tag{1.03}$$

If $\|x\| \leqq \epsilon$ then the right-hand side of equation (1:03) is surely in-

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[1] Numbers in brackets refer to the Bibliography.

ferior to $(2\epsilon)^p$. For $0 \leqq s \leqq 1$, $r \geqq 1$, the following inequalities are easy to establish,

(1.04) $$(1+s)^r \leqq 1 + (2^r - 1)s,$$

(1.05) $$(1-s)^r \geqq 1 - rs.$$

Hence for $\|x\| > \epsilon$ the right-hand side of equation (1.03) is dominated by

$$\left(\frac{2^{p+1} + p - 2}{2}\right) \epsilon \|x\|^{p-1}.$$

Accordingly in both cases

$$\|T(x) - T(2x)/2\| \leqq k\|x\|^{1/q} + 2\epsilon$$

where

$$k = \left(\frac{\epsilon(2^{p+1} + p - 2)}{2}\right)^{1/p}.$$

Write

$$\|T(2^n x)/2^n - T(x)\| \leqq k_n\|x\|^{1/q} + l_n\epsilon$$

and

$$\begin{aligned} &\|T(2^{n+1}x)/2^{n+1} - T(x)\| \\ (1.06) \qquad &\leqq \|T(2^{n+1}x)/2^{n+1} - T(2x)/2\| + \|T(2x)/2 - T(x)\| \\ &\leqq (2^{-1/p}k_n + k)\|x\|^{1/q} + (l_n/2 + 2)\epsilon. \end{aligned}$$

On setting the right-hand side of Equation (1.06) equal to $k_{n+1}\|x\|^{1/q} + l_{n+1}\epsilon$; we have the difference equations

$$k_{n+1} = 2^{-1/p}k_n + k, \; l_{n+1} = l_n/2 + 2.$$

The solutions of these equations are

$$k_n = k\sum_{j=0}^{n} 2^{-j/p} \leqq k/1 - 2^{-1/p} = A,$$

$$l_n = 2\sum_{j=0}^{n} 2^{-j} \leqq 4.$$

Hence

(1.07) $$\|T(2^{n+m}x)/2^{n+m} - T(2^n x)/2^n\| \leqq A2^{-n/p}\|x\|^{1/q} + 4(2^{-n}\epsilon).$$

Since $L_r(0, 1)$ is complete we can define $U(x)$ by

$$U(x) = L_{n\to\infty} T(2^n x)/2^n, \qquad U(\theta) = \theta.$$

Moreover in view of equations (1.01) and (1.07) we can establish directly that $U(x)$ is an isometry (and is linear [2, p. 166]). We shall make frequent use of

$$\|U(x) - T(x)\| \leq A\|x\|^{1/q} + 4\epsilon. \tag{1.08}$$

THEOREM 2. *If T is an ϵ isometry of $L_r(0, 1)$ on $L_r(0, 1)$ then U is an isometry of $L_r(0, 1)$ on $L_r(0, 1)$.*

The proof given by Hyers and Ulam for their Theorem 3 is obviously valid here. A more general situation is covered by our Theorem 5.

THEOREM 3. *If $T(x)$ is an ϵ isometry of $L_r(0, 1)$ on $L_r(0, 1)$ then $\|T(x) - U(x)\| \leq 12\epsilon$.*

We shall tacitly follow the convention that $z \in L_r(0, 1)$ has as its representative the function $z_r(s)$. Choose $x \neq \theta$ arbitrarily and write

$$U(x) = y_1, \quad T(x) = y_2.$$

We first assume that y_1 and y_2 are not collinear with θ. Then $\|y_2 - vy_1\|$ has a unique, positive minimum for some value of v, say u. For instance, this is a consequence of the Alaoglu-Birkhoff lemma [5, p. 299] *that in a uniformly convex Banach space a closed convex set, here* $\{y_2 - vy_1 \mid |v| < \infty\}$, *contains an element of least norm* (reflexivity would be sufficient for this) *and this element is unique.* Let

$$y_0 = (y_2 - uy_1)/\|y_2 - uy_1\|. \tag{3.01}$$

Then

$$y_2 = uy_1 + \|y_2 - uy_1\| y_0 = uy_1 + wy_0. \tag{3.02}$$

It is significant for our developments that $w \geqq 0$. In view of Theorem (2) a unique element, x_0, is defined by $U^{-1}y_0$.

Since $\inf \|y_0 - ky_1\| = 1$ there is a linear functional [2, p. 57] of unit norm, f_0, such that $f_0(y_0) = \|y_0\| = 1$ and $f_0(y_1) = 0$. Let $E_0 = \{y \mid f_0(y) = 0\}$. Since L_r^* is strictly convex it is easy to show that f_0 is unique and an explicit representation is

$$f_0(y) = \int_0^1 |y_0(s)|^{r-1} \operatorname{sign} y_0(s) y(s) ds. \tag{3.03}$$

An obvious argument shows every element in E is uniquely expressible as a sum of a multiple of y_0 and an element in E_0. Moreover, if

$y \in E_0$, then

$$\|y_0 + y\| \geqq |f_0(y_0 + y)| = 1. \tag{3.04}$$

We have from Equation (1.08)

$$\begin{aligned} \|T(x) - U(x)\| &= \|(u - 1)y_1 + wy_0\| \\ &\leqq A\|x\|^{1/q} + 4\epsilon \\ &= A\|y_1\|^{1/q} + 4\epsilon. \end{aligned} \tag{3.05}$$

Moreover

$$\big| \|vy_1 + wy_0\| - \|y_1\| \big| = \big| \|(Tx)\| - \|x\| \big| < \epsilon. \tag{3.06}$$

We write

$$T(2^n x_0) = 2^n x_0 + l_n y_0 + Y_n \tag{3.07}$$

where $Y_n \in E_0$. Accordingly

$$\epsilon > \big| \|2^n y_0 + l_n y_0 + Y_n - uy_1 - wy_0\| - \|2^n y_0 - y_1\| \big|. \tag{3.08}$$

Also

$$\big| \|2^n y_0 + l_n y_0 + Y_n\| - 2^n \big| < \epsilon, \tag{3.09}$$

$$\|l_n y_0 + Y_n\| \leqq A 2^{n/q} + 4\epsilon. \tag{3.10}$$

From equations (3.02), (3.09), and (3.10) it is manifest that

$$-(A2^{n/q} + 4\epsilon) \leqq l_n \leqq \epsilon, \quad \|Y_n\| \leqq 2(A2^{n/q} + 4\epsilon).$$

Actually it is sufficient for our purpose that $\|Y_n\|/2^n$, $l_n/2^n$ go to 0 as $n \to \infty$.

We remark that

$$\|2^n y_0 - y_1\| = 2^n + o(1). \tag{3.11}$$

Indeed equation (3.02) entails

$$\frac{d}{dt}\|y_0 - ty_1\| \,\Big|_{t=0} = -\int_0^1 (|y_0(s)|^{r-1} \operatorname{sign} y_0(s)) y_1(s)\, ds = 0.$$

Hence we have from equations (3.08), (3.09), and (3.11)

$$\begin{aligned} &\|(2^n + l_n)y_0 + Y_n\|^r - \|(2^n + l_n)y_0 + Y_n - uy_1 - wy_0\|^r \\ &\qquad \leqq (2r\epsilon)2^{n(r-1)} + o(2^{n(r-1)}). \end{aligned} \tag{3.12}$$

The crucial step in our demonstration is the justification of the assertion that the left-hand side of equation 3.12 is

$$2^{n(r-1)}(rw) + o(2^{n(r-1)}).$$

Write $t = 2^{-n}$, $l_t = l_n/2^n$, $Y_t = Y_n/2^n$. Then l_t and $\|Y_t\|$ go to 0 with t.

Write $V = tu$, $W = tw$ and α and β for real numbers between 0 and V and 0 and W respectively. Write also $L_t = l_t - \beta$ and

$$\psi(s, t) = y_0(s)(1 + L_t) + Y_t(s) - \alpha y_1(s),$$

$$f_t(y) = \int_0^1 |\psi(s, t)|^{r-1} \operatorname{sign} \psi(s, t) y(s) ds.$$

Denote the rectangle $|u| \leqq A$, $w \leqq B$ by Q. For each choice of t the theorem of the mean guarantees that α and β exist such that the left-hand side of equation (3.12) has the value

$$-\left(V\frac{\partial}{\partial\alpha} + W\frac{\partial}{\partial\beta}\right)\|(1+L_t)y_0 + Y_t - \alpha y_1\|^r = r(Vf_t(y_1) + Wf_t(y_0)). \tag{3.13}$$

For arbitrary positive δ and all sufficiently small t values

$$\sup(|l_t| + tB, \|Y_t\|, t\|Ay_1\|) < \delta.$$

Let

$$S = \{s \mid y_0(s) \leqq |L_t y_0(s) + Y_t(s) - \alpha y_0(s)|\}$$

and write R for the complement of S in $0 \leqq s \leqq 1$. Then

$$|f_t(y_1)| \leqq \left|\int_R |\psi(s, t)|^{r-1} \operatorname{sign} y_0(s) y_1(s) ds\right| + \left|\int_S |\psi(s, t)|^{r-1} \operatorname{sign} \psi(s, t) y_1(s) ds\right|.$$

It may be verified that

$$\int_S |\psi(s, t)|^{r-1} |y_1(s)| ds \leqq 2^{r-1} \int_S |L_t y_0(s) + Y_t(s) - \alpha y_1(s)|^{r-1} |y_1(s)| ds$$
$$\leqq 2^{r-1}(3\delta)^{r/(r-1)}\|y_1\|. \tag{3.14}$$

$$\left|\int_R |\psi(s, t)|^{r-1} \operatorname{sign} y_0(s) y_1(s) ds\right| \leqq \left|\int_0^1 |\psi(s, t)|^{r-1} \operatorname{sign} y_0(s) y_1(s) ds\right| + \left|\int_S |\psi(s, t)|^{r-1} \operatorname{sign} y_0(s) y_1(s) ds\right|.$$

The first integral on the right-hand side can be written

$$\int_0^1 (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1}) \operatorname{sign} y_0(s) y_1(s) ds + \int_0^1 |y_0(s)|^{r-1} \operatorname{sign} y_0(s) y_1(s) ds.$$

Since $y_1 \in E_0$ the last integral vanishes, and we may dominate by

$$\int_R (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1})|y_1(s)|ds + \int_S (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1})|y_1(s)|ds.$$

The first integral may be written

$$\int_R \left| \left| 1 + \frac{L_t y_0(s) + Y_t(s) - \alpha y_1(s)}{y_0(s)} \right|^{r-1} - 1 \right| |y_0(s)|^{r-1} |y_1(s)| ds.$$

For $|z| \leqq 1$ and positive k we have

$$(1 + |z|)^k \leqq 1 + k|z|,\ 1 + (2^k - 1)|z|,\ (1 - |z|)^k \geqq 1 - |z|,\ 1 - k|z|,$$

according as k is less than or greater than 1. Hence for some positive K the last integral written is bounded by

$$\begin{aligned} K \int_R |L_t y_0(s) + Y_t(s) - \alpha y_1(s)|\, |y_0(s)|^{r-2} |y_1(s)| ds \\ \geqq K\|L_t y_0 + Y_t - \alpha y_1\|^{r/(r-1)} \|y_0\|^{r(r-2)/(r-1)} \|y_1\| \\ \leqq K(3\delta)^{r/(r-1)} \|y_1\|. \end{aligned}$$

Since all the integrals over S are covered by equation (3.14),

$$|f_t(y_1)| \leqq C\delta^{r/(r-1)}, \tag{3.15}$$

uniformly in $(u, w) \in Q$ for all sufficiently small t. Hence

$$utf_t(y_1) = o(t).$$

Now

$$\begin{aligned} f_t(y_0) = \int_R + \int_S (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1}) \operatorname{sign} \psi(s, t) ds \\ + \int_0^1 |y_0(s)|^{r-1} \operatorname{sign} \psi(s, t) y_0(s) ds. \end{aligned} \tag{3.16}$$

Each of the first two integrals on the right-hand side is readily shown to be inferior in absolute value to $C_1\delta^{r/r-1}$. The last integral on the right-hand side may be written

$$\|y_0\|^r + \int_S |y_0(s)|^{r-1} (\operatorname{sign} \psi(s, t) - \operatorname{sign} y_0(s)) y_0(s) ds.$$

Evidently the integral over S is inferior in absolute value to

$$2\int_S |\psi(s, t)|^{r-1} |y_0(s)| ds \leqq 2(3\delta)^{r/(r-1)}.$$

Therefore

$$f_t(y_0) = 1 + \rho, \qquad |\rho| \leqq C_2\delta^{r/(r-1)},$$

for all sufficiently small t values uniformly for $(u, w) \in Q$. Thus

$$wtf_t(y_0) = wt + o(t). \tag{3.17}$$

Hence the right-hand side of equation (3.13) is $rwt+o(t)$. Accordingly, equation (3.12) may be written

$$2^{n(r-1)}rw + o(2^{n(r-1)}) \leqq 2^{n(r-1)}2r\epsilon + o(2^{n(r-1)}).$$

Therefore

$$w \leqq 2\epsilon. \tag{3.18}$$

We have then

$$\|y_2 - y_1\| \leqq |u - 1|\, \|y_1\| + 2\epsilon.$$

From equations (3.06) and (3.18) we infer

$$\big|\,|u| - 1\big|\, \|y_1\| \leqq 3\epsilon. \tag{3.19}$$

Hence if $u \geqq 0$ we have

$$\|y_2 - y_1\| \leqq 5\epsilon. \tag{3.20}$$

The case that y_1, y_2 and θ are collinear offers no exception. Here $y_2 = uy_1$ and, for $u \geqq 0$, equation (3.06) surely implies equation (3.20).

Suppose now that $u<0$. It may be shown from equations (3.05), (3.06), and (3.18) that the maximum value of $\|y_1\|$, denoted by B, consistent with $u<0$ is given by

$$2B = AB^{1/q} + 9\epsilon. \tag{3.21}$$

In view of equations (1.08) and (3.19) it follows that in all cases

$$\|y_2 - y_1\| \leqq \sup (5\epsilon, 2B - 5\epsilon). \tag{3.22}$$

Evidently B depends on p and ϵ alone and goes to 0 with ϵ. Since $1<q\leqq 2$ it can be verified that for $A+9\epsilon<2$, for instance,

$$B \leqq ((A + (A^2 + 72\epsilon)^{1/2})/4)^q.$$

Instead of continuing with the determination of explicit bounds for B from equation (3.21) it seems preferable to present an alternative argument which has the merit of yielding a convenient bound directly. The idea behind this argument is borrowed from [1] and consists of the observation that for all sufficiently large multiples of x, say lx, $T(lx)$ has a positive y_1 component. This is of course obvious, since we need merely satisfy $l\|x\| \geq B$. Accordingly if $u<0$, there is an integer m such that in the general case

$$T(2^m x) = -u_2 2^m y_1 + w_2 z_0, \qquad T(2^{m+1}x) = u_1 2^{m+1} y_1 + w_1 Z_0, \tag{3.23}$$

where u_1 and u_2 are *non-negative* and z_0 and Z_0 have unit norm and are determined in the same way as y_0. The argument leading to equation (3.18) shows that $0 \leq w_j \leq 2\epsilon$, $j=1, 2$. The possibility that either (or both) of $T(2^m x)$ and $T(2^{m+1}x)$ is collinear with y_1 and θ is formally included by taking the corresponding w as 0. We write z for $2^m y_1$. Thus $\|y_1\| \leq \|z\|$. Then, in view of equations (1.01) and (3.23),

$$\big|\, \|(2u_1 + u_2)z - w_2 z_0 + w_1 Z_0\| - \|z\| \,\big| < \epsilon$$

and

$$|2u_1 + u_2 - 1|\, \|z\| \leq 5\epsilon. \tag{3.24}$$

Similarly the analogues of equation (3.19) are

$$|2u_1 - 2|\, \|z\| \leq 3\epsilon, \tag{3.25}$$

$$|u_2 - 1|\, \|z\| \leq 3\epsilon. \tag{3.26}$$

There are several cases to consider, depending on whether (u_1, u_2) and $2u_1+u_2-1$ are larger or smaller than 1. The largest value of $\|z\|$ is admitted in the event $\sup (u_1, u_2) \leq 1 \leq 2u_1+u_2 \leq 3$. On combining the inequalities in (3.24), (3.25), and (3.26) we obtain in this case

$$\|z\| \leq 11\epsilon/2.$$

Since $\|y_1\| \leq \|z\|$ we infer for $u<0$ and then, by equation 3.20, for all u,

$$\|T(x) - U(x)\| \leq 2\|y_1\| + \epsilon \leq 12\epsilon. \tag{3.27}$$

The developments just concluded motivate the generalization presented below. Elements in the Banach spaces E_1 and E_2 are denoted by x and by y or z respectively. Our restrictions bear on E_2 alone. Henceforth we assume E_2 is a uniformly convex Banach space [3], that is to say $\|z_1-z_2\| \geq \gamma \sup (\|z_1\|, \|z_2\|)$, $\gamma>0$, implies $\|z_1+z_2\| \leq 2\,(1-\delta(\gamma)) \sup (\|z_1\|, \|z_2\|)$, where $\delta(\gamma)$ is strictly monotone with $\delta(0)=0$, $\delta(2)=1$. We define γ' for all positive δ as $\sup \{\gamma \mid \delta(\gamma) \leq \inf (1, \delta)\}$ and write $\gamma'=\psi(\delta)$. We drop the primes in the sequel. We require that E_2 satisfy the following restrictions, also,

$$\sum_{n=1}^{\infty} \psi(2^{-n}C) < \infty \tag{A}$$

for every positive C,

$$\text{(B)} \qquad L_{\lambda\to 0,\ \|z\|\to 0}(\|y_0 + z - \lambda y_1\| - \|y_0 + z\|)/\lambda = 0,$$

where $\|y_0\| = 1$ and z and y_1 lie in the linear space $\{y \mid f_0(y) = 0, f_0(y_0) = 1, \|f_0\| = 1\}$. (It is easy to verify that these conditions are satisfied by $L_r(0, 1)$, $1 < r < \infty$.)

THEOREM 4. *Let $T(x)$ be an ϵ isometry of E_1 into E_2 with $T(\theta_1) = \theta_2$. Then $U(x) = L_{n\to\infty}T(2^n x)/2^n$ exists for each x and $U(x)$ is an isometric (linear) transformation.*

Evidently sup $(\|T(2x) - T(x)\|, \|T(x)\|)$ can be written $\|x\| + \rho$ where $|\rho| < \epsilon$. Then

$$\|(T(2x) - T(x)) - T(x)\| \geqq \gamma(\|x\| + \rho)$$

implies

$$\|x\| - \epsilon/2 \leqq \|T(2x)/2\| \leqq (1 - \delta(\gamma))(\|x\| + \rho).$$

Hence

$$\delta(\gamma) \leqq 3\epsilon/2\|x\|.$$

Let $\delta_n = 3\epsilon/2^{n+1}\|x\|$. Then, since $|\rho| < \epsilon$,

$$\|T(2^{n+1}x) - 2T(2^n x)\| \leqq \gamma_n(2^n\|x\| + \epsilon).$$

Evidently γ_n is a monotone nonincreasing function on the positive real axis with $L_{\|x\|\to\infty}\gamma_n = 0$. In view of (A)

$$\text{(4.01)} \qquad \sum_{j=1}^{\infty} 2^{-j}\|T(2^j x) - 2T(2^{j-1}x)\| \leqq (\|x\| + \epsilon) \sum_{j=1}^{\infty} \gamma_j = k(\|x\|)(\|x\| + \epsilon).$$

It is important to observe that $k(\|x\|)$ is a monotone nonincreasing function of $\|x\|$ and

$$\text{(4.02)} \qquad L_{\|x\|\to\infty}k(\|x\|) = 0.$$

Since E_2 is complete, the demonstrated convergence of the left-hand side of equation (4.01) ensures the existence of $U(x)$ given by

$$U(x) = L_{n\to\infty}T(2^n x)/2^n.$$

It is easy to see that U is an isometry on E_1 to E_2 and that

$$\text{(4.03)} \qquad \|T(x) - U(x)\| \leqq k(\|x\|)(\|x\| + \epsilon).$$

THEOREM 5. *If T is onto E then U is onto E.*

The simple demonstration below covers more general E_2 spaces than those of our hypotheses. Let E_2' be the range of U. Then E_2' is closed in E_2 [2, p. 145]. Assume the assertion of the theorem invalid, that is to say E_2' is a proper subspace of E_2. Evidently for some positive s_0, $s \geqq s_0$ implies $k(s)(s+\epsilon) \leqq s+\epsilon-2$. It is well known [2, p. 83] that for some z_0 of unit norm, $\|z_0-z\| \geqq 1-(s_0+\epsilon)^{-1}$ for all z in E_2'. Let x satisfy $T(x)=(s_0+\epsilon)z_0$. Then $s_0 \leqq \|x\| \leqq s_0+2\epsilon$ and

$$(5.01) \quad \|T(x) - U(x)\| = (s_0+\epsilon)\|z_0 - U(x)/(s_0+\epsilon)\| \geqq s_0 + \epsilon - 1.$$

On the other hand, using equation (4.03),

$$(5.02) \qquad \|T(x) - U(x)\| \leqq s_0 + \epsilon - 2.$$

Since equations (5.01) and (5.02) are incompatible, our theorem is established. Hence incidentally E_1 and E_2 are equivalent [2, p. 180].

THEOREM 6 *If T defines an ϵ isometry of E_1 onto E_2 with $T(\theta_1)=\theta_2$ then* $\|T(x)-U(x)\| \leqq 12\epsilon$.

We continue the notation introduced in the course of the proof of Theorem 3. The results through equation (3.09) hold subject to the trivial modification of replacing $A\|y_1\|^{1/\epsilon}+4\epsilon$ by $k(\|y_1\|)(\|y_1\|+\epsilon)$. Moreover the analogues of equations (3.09) and (3.10) establish

$$- k(2^n)(2^n+\epsilon) \leqq l_n \leqq \epsilon, \quad \|Y_n\| \leqq 2k(2^n)(2^n+\epsilon).$$

In view of equation (4.02) we can assert $\|Y_n\|/(2^n-|l_n|)$ and $l_n/2^n$ go to 0 with n. Then by (B)

$$\begin{aligned}(6.01) \quad \|T(2^n x_0)\| - \|T(2^n x_0) - T(x)\| &\leqq \|2^n y_0\| + \epsilon - \|2^n y_0 - y_1\| + \epsilon \\ &\leqq 2\epsilon + o(1).\end{aligned}$$

The left-hand side of equation (6.01), for large n, is

$$(6.02) \quad \begin{aligned}&\left\{(2^n+l_n)\left(\left\|y_0+\frac{Y_n}{2^n+l_n}\right\| - \left\|y_0+\frac{Y_n-uy_1}{2^n+l_n-w}\right\|\right)\right\} \\ &\qquad + \left\{w\left\|y_0+\frac{Y_n-uy_1}{2^n+l_n-w}\right\}\right. .\end{aligned}$$

With $z=Y_n/(2^n+l_n-w)$, $\lambda=u/(2^n+l_n-w)$ it follows readily from (B) that the first brace of terms goes to 0 with n and hence $w+o(1) \leqq 2\epsilon+o(1)$ or

$$w \leqq 2\epsilon.$$

For some positive B, $k(B)(B+\epsilon)+2\epsilon \leqq B$ in view of equation (4.02).

Then a simple argument using equations (4.03) and (3.18) shows $u>0$ for $\|y_1\|>B$. The remainder of the proof follows the pattern of the proof of Theorem 3 in detail and is therefore omitted. It will be noted that the possibility that the functional f_0 (cf. Theorem 3) may not be unique does not disturb the proof.

REMARK. Since U is linear T is a 36ϵ linear transformation [6]. Moreover $U(x)$ is the unique distributive operation satisfying $\overline{L}_{\|x\|\to\infty}\|T(x)-V(x)/\|x\|<\infty$.

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BLOCH'S THEOREM FOR REAL VARIABLES

S. BOCHNER

The famous theorem of Bloch states that every function

$$F(z) = z + a_2z^2 + \cdots \tag{1}$$

of the complex variable z in the unit-circle $|z| \leqq 1$ maps some subdomain of the unit-circle univalently onto a circle of a fixed radius whose size is independent of $f(z)$. The purpose of the present paper is to point out the following generalization to n real variables, $n \geqq 2$.

THEOREM 1. *Corresponding to any integer* n, $n = 1, 2, 3, \cdots$, *and any positive constant* K, $K > 0$, *there exists a positive radius*

$$R_0 = R_0(n, K), \qquad R_0 > 0,$$

having the following property:

If the real functions

$$u_i = f_i(x_1, \cdots, x_n), \qquad i = 1, 2, \cdots, n, \tag{2}$$

are defined and solutions of the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \tag{3}$$

in a neighborhood of the sphere

$$x_1^2 + \cdots + x_n^2 \leqq 1, \tag{4}$$

if their Jacobian

$$J(x_1, \cdots, x_n) = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} \tag{5}$$

satisfies the decisive relation

$$\sum_{i,j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)^2 \leqq K \cdot | J(x_1, \cdots, x_n) |^{2/n} \tag{6}$$

in (4), *and if*

$$J(0, \cdots, 0) = 1, \tag{7}$$

then there exists in (4) *an open set S such that the functions* (2) *are a one-*

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to-one transformation of S onto a sphere of radius R_0 in the Euclidean space of the variables $(u_1, \cdots, u_n)$.

The decisive restriction (6) can be easily interpreted. If we introduce the differential form

$$ds^2 = g_{ij}(x)dx_i dx_j,$$

where

$$g_{ij}(x) = \sum_m \frac{\partial f_m}{\partial x_i} \cdot \frac{\partial f_m}{\partial x_j},$$

then in the ellipsoid $g_{ij}(x)\xi_i\xi_j = 1$ the quotient of the longest over the shortest axis shall have a finite bound which is independent of x; if, however, the determinant of g_{ij} vanishes at a point, then all numbers $\partial f_m/\partial x_i$ shall vanish there separately. This is a generalization of conformal transformations, for which all axes have the same length at any point.

For $n = 2$ the significance of this generalization has been brought to light in the work of Ahlfors[1] when he proved that Picard's theorem on exceptional values will remain in force for this type of transformation. For $n \geqq 3$, our condition may turn out to be of even greater significance. On the one hand there are no strictly conformal transformations other than trivial ones (inversions in spheres). But on the other hand, general analytic transformations in several *complex* variables of the form

$$w_l = F_l(z_1, \cdots, z_k), \qquad l = 1, \cdots, k;\ n = 2k, \tag{8}$$

do not obey our restriction for $k \geqq 2$, which fact may be said to be responsible for the complete breakdown of Picard's theorem in that case.

As for our additional requirement that each component $f_i(x)$ shall be a solution of (3), it will be evident from the course of the proof that it could be considerably toned down. However, we are drawing up the condition in this rigid form in order to point up an analogy to the case of complex variables. For one complex variable, (3) requires as in the classical case that each component of (1) separately shall be a solution of the Laplace equation. However, in lieu of the rigorous relation between the two components as implied in the Cauchy-Riemann equation we now have the much milder condition (6). For several complex variables, analyticity implies in particular the relations

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[1] Compare the account in R. Nevanlinna, *Eindeutige analytische Funktionen*, p. 343. There were also earlier efforts in that direction. See, for instance, H. Groetzsch, *Ueber möglichst konforme Abbildungen von schlichten Bereichen*, Berichte über der Verhandlungen der K. Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematische-Physische Klasse, vol. 84 (1932) pp. 114–120.

$$\frac{\partial^2 f_i}{\partial x_l^2} + \frac{\partial^2 f_i}{\partial y_l^2} = 0, \qquad l = 1, \cdots, k,$$

for each pair (x_l, y_l) separately, whereas our present condition (3) replaces this by the summary requirement

$$\sum_{l=1}^{k} \left(\frac{\partial^2 f_i}{\partial x_l^2} + \frac{\partial^2 f_i}{\partial y_l^2} \right) = 0,$$

thus creating a situation which is even more general than the case of one complex variable would indicate.

For the proof of Theorem 1 we introduce the functions

$$M(r) = \max_{x_1^2+\cdots+x_n^2 \leqq r^2} \left| J(x_1, \cdots, x_n) \right|^{1/n}$$

and

$$\phi(r) = (1 - r)M(r).$$

Since $\phi(0) = 1$ and $\phi(1) = 0$, there exists a number r^0 such that

$$\phi(r^0) = 1, \qquad \phi(r) < 1 \text{ for } r^0 < r \leqq 1.$$

Also, there exists a point

$$x^0 = (\overset{0}{x}_1, \cdots, \overset{0}{x}_n) \tag{9}$$

for which $(x_1^0)^2 + \cdots + (x_n^0)^2$ is precisely equal to $(r^0)^2$ and such that

$$J(\overset{0}{x}_1, \cdots, \overset{0}{x}_n)^{1/n} = M(\overset{0}{r}) = \frac{1}{1 - r^0}.$$

Now introduce the functions

$$g_i(\xi_1, \cdots, \xi_n) = \frac{f_i(\overset{0}{x}_m + \rho\xi_m) - f_i(\overset{0}{x}_m)}{\rho \left| J(x_m^0) \right|^{1/n}}, \quad i = 1, \cdots, n, \tag{10}$$

where

$$\rho = \frac{1 - r^0}{2},$$

and since for

$$\xi_1^2 + \cdots + \xi_n^2 \leqq 1 \tag{11}$$

we have

$$\sum_{m=1}^{n} (x_m^0 + \rho\xi_m)^2 \le \left(\frac{1+r^0}{2}\right)^2,$$

they will exist in the unit sphere (11). We also introduce their Jacobian

$$G(\xi_1, \cdots, \xi_n) = \frac{\partial(g_1, \cdots, g_n)}{\partial(\xi_1, \cdots, \xi_n)},$$

and we list the following properties:

(i) All functions $g_i(\xi)$ are again solutions of the Laplacean $\partial^2 g/\partial\xi_1^2 + \cdots + \partial^2 g/\partial\xi_n^2 = 0$.

(ii) $G(0, \cdots, 0) = 1$.

(iii) $G(\xi_1, \cdots, \xi_n) \le 2^n$ for ξ in (11).

(iv) $\sum_{i,j=1}^{n} (\partial g_i/\partial\xi_j)^2 \le K|G(\xi_1, \cdots, \xi_n)|^{2/n} \le 4K$.

We note that property (iii) follows from

$$\left|\frac{\partial(g_1, \cdots, g_n)}{\partial(\xi_1, \cdots, \xi_n)}\right| = \frac{|J(x_1, \cdots, x_n)|}{|J(x_1^0, \cdots, x_n^0)|} \le \left(\frac{M((1+r^0)/2)}{M(r^0)}\right)^n$$

$$\le \left(\frac{2}{1-r^0}(1-r^0)\right)^n = 2^n.$$

We are now going to make use of property (i). Since the first partial derivatives $\partial g_i/\partial\xi_j$ are themselves solutions of the Laplacean, it follows from Poisson's integral for harmonic functions that corresponding to n and K there exists a constant $L = L(n, K)$ such that the validity of property (iv) in (11) will imply the further property

$$\text{(v)} \qquad \sum_{i,j,k=1}^{n} \left(\frac{\partial^2 g_i}{\partial\xi_j\partial\xi_k}\right)^2 \le L$$

in

$$\text{(12)} \qquad \xi_1^2 + \cdots + \xi_n^2 \le 1/2.$$

However, properties (ii), (iv) and (v), together with

$$g_i(0, \cdots, 0) = 0, \qquad i = 1, \cdots, n,$$

lead to the following conclusion. There exists a radius $R_1 = R_1(n, K, L) = R_1(n, K)$ such that on some subset B of (12)—the subset depending on the given functions $g_i(\xi)$—the functions

$$\eta_i^0 = g_i(\xi_1, \cdots, \xi_n)$$

are a one-to-one transformation of B into the sphere

$$\eta_1^2 + \cdots + \eta_n^2 \leq R_1^2.$$

This is a "uniform" version of the theorem on implicit functions and can be derived from the Kronecker integral or in some other fashion.[3]

Returning now to the original functions (2), we see from (10) that these functions will transform a subdomain B of (4) into a sphere of radius

$$R_0 = R_1 \cdot \rho \left| J(x_n^0) \right|^{1/n}.$$

However, $\rho \left| J(x_n^0) \right|^{1/n} = ((1-r_0)/2) \cdot (1/(1-r_0)) = 1/2$, and this completes the proof of Theorem 1.

THEOREM 2. *If all assumptions of Theorem* 1 *are fulfilled in the sphere* $x_1^2 + \cdots + x_n^2 \leq \lambda^2$, *then the map contains a sphere of radius* $R = \lambda \cdot R_0(n, K)$.

This follows from Theorem 1, if we replace $f_i(x)$ by $\lambda f_i(x/\lambda)$.

Hence we conclude:

THEOREM 3. *If all assumptions of Theorem* 1 *are fulfilled in the entire space, then the map contains spheres of arbitrarily large radius.*

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[3] See, for instance, the first chapter in Carathéodory's *Variationsrechnung*.

A NOTE ON THE AREA OF A NONPARAMETRIC SURFACE

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1. **Introduction.** Recently, Besicovitch [1][1] has exhibited a surface of the form $z=f(x, y)$ for which sequences of inscribed polyhedra, corresponding to triangulations into nearly isosceles right triangles, do not converge in area to the area of the surface. On the other hand, for surfaces defined by functions $f(x, y)$ which are absolutely continuous in the sense of Tonelli this paper shows (see Theorem I) that in a statistical sense sequences of inscribed polyhedra that arise from triangulations which are successive refinements and which consist of right triangles of any preassigned shape (exactly isosceles, for example) will converge in area to the area of the surface. For surfaces of a more general class (defined by functions which are absolutely continuous in the sense of Young, see below) analogous but somewhat weaker theorems (see Theorems II and III) are proved.

In order to state the results more precisely let us consider the following definitions. Let Q_0 denote the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Let $\bar{S}$ denote the continuous surface defined by the continuous function $z=f(x, y)$ for $-\infty < x < +\infty$, $-\infty < y < +\infty$. Suppose that $f(x, y)$ is periodic of period one in x and in y. The results of this paper are valid for a continuous function defined only on Q_0 since by extension of definition and a suitable change of scale (see Saks [8, p. 170]) the above conditions can be satisfied.

Let $D_n^0(u, v)$, for $0 \leqq u \leqq 1$ and $0 \leqq v \leqq 1$, denote a subdivision of Q_0 into rectangles formed by the lines $x=u+i/n$ and $y=v+i/n$ where i takes on all integral values (positive, negative, and zero) which give lines across Q_0. Let (x', y'), (x', y''), (x'', y'), and (x'', y'') denote the vertices of a generic rectangle with $x' < x''$ and $y' < y''$. Let

$$T_1(f; x', y') = \frac{1}{2}\left\{1 + \left[\frac{f(x'', y') - f(x', y')}{x'' - x'}\right]^2 + \left[\frac{f(x', y'') - f(x', y')}{y'' - y'}\right]^2\right\}^{1/2}, \tag{1}$$

$$T_2(f; x', y') = \frac{1}{2}\left\{1 + \left[\frac{f(x'', y'') - f(x', y'')}{x'' - x'}\right]^2 + \left[\frac{f(x'', y'') - f(x'', y')}{y'' - y'}\right]^2\right\}^{1/2}. \tag{2}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Now $(x''-x')(y''-y')T_i(f; x', y')$, $i=1, 2$, are the areas of two triangles inscribed in the surface $S: z=f(x, y)$ and defined over the rectangle $x'\leqq x\leqq x''$, $y'\leqq y\leqq y''$. Thus

$$
\begin{aligned}
(3)\quad & A[P_0^n(f; u, v)] \\
& = \sum_{D_n^0(u,v)} [T_1(f; x', y') + T_2(f; x', y')](x'' - x')(y'' - y'),
\end{aligned}
$$

where the summation is over the vertices (x', y') of $D_n{}^0(u, v)$, is the area of a polyhedron $P_0{}^n(f; u, v)$ inscribed in S and defined over Q_0.

The Lebesgue area of a surface S: $z=f(x, y)$, $(x, y)\in Q_0$, can be defined as follows:

$$
(4)\qquad A(S) = \text{g.l.b.} \liminf_n E(P_n)
$$

where $E(P_n)$ is the area of the polyhedron P_n in the elementary sense and the greatest lower bound is taken over all sequences of polyhedra $P_n: z=p_n(x, y)$, $(x, y)\in Q_0$, such that $p_n(x, y)$ converges uniformly on Q_0 to $f(x, y)$. We shall assume that the reader is familiar with the definitions of bounded variation in the sense of Tonelli (BVT) and absolute continuity in the sense of Tonelli (ACT) (see, for example, Saks [8, p. 169]). We shall further assume familiarity with the concept of absolute continuity of a function on a Borel set (see Young [9, p. 52]).

Let $f(x, y)$ be continuous and BVT on a rectangle R: $a\leqq x\leqq b$, $c\leqq y\leqq d$. Then we shall say that $f(x, y)$ is absolutely continuous in the sense of Young on R, or simply ACY on R: $a\leqq x\leqq b$, $c\leqq y\leqq d$, if the following conditions hold:

(a) $R=B'+B''$, where B' and B'' are Borel sets; .

(b) for each x, $a\leqq x\leqq b$, let B_x' be the intersection of B' with the segment $x=x$, $c\leqq y\leqq d$. B_y'' is defined similarly. For almost every x, $f(x, y)$ is absolutely continuous on B_x' as a function of y, and, for almost every y, $f(x, y)$ is absolutely continuous on B_y'' as a function of x. The term ACY is justified by the fact that L. C. Young [9] first formulated this condition and made an important application of it.

The following theorems will be proved in this paper.

THEOREM I. *If $f(x, y)$ is ACT on Q_0 then for every increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that n_i divides n_{i+1} for all i we have*

$$
\lim_i A[P_0^{n_i}(f; u, v)] = A(S)
$$

for almost every $(u, v)\in Q_0$.

THEOREM II. *If $f(x, y)$ is ACY on Q_0 then for almost every $(u, v) \in Q_0$*

$$\liminf_{n} A[P_0^n(f; u, v)] = A(S).$$

THEOREM III. *If $f(x, y)$ is ACY on Q_0 then for any sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers with $n_i \to +\infty$, there exists a subsequence $\{n_\alpha\}$ of $\{n_i\}$ such that for almost every $(u, v) \in Q_0$*

$$\lim_{\alpha} A[P_0^{n_\alpha}(f; u, v)] = A(S).$$

In the study of the area of nonparametric surfaces various people have imposed conditions of regularity on the triangulations considered. Inspection of the proofs which follow will show that the subdivision $D_n{}^0(u, v)$ may be replaced by a subdivision $D_{mn}{}^0(u, v)$ arising from m intervals in the x-direction and n in the y-direction, and the theorems analogous to those above but stated for squares $Q(u, v)$ (cf. §2) would still be valid. In order to give theorems valid on Q_0 Lemma 1 (cf. §2) requires that m/n and n/m both be bounded (also see Radó [5]).

If in the above definition of the Lebesgue area of a surface S we further require that the polyhedra be inscribed in S we obtain a quantity $A^*(S)$, and clearly $A^*(S) \geqq A(S)$. The question of when $A(S) = A^*(S)$ has been called the *Problem of Geöcze*. It has been proved (see Huskey [2], for example) that $A(S) = A^*(S)$ if $f(x, y)$ is ACT on Q_0. Theorem I implies more than this, namely, that almost every sequence constructed as above converges in area to $A(S)$ if $f(x, y)$ is ACT on Q_0.

The fundamental, and so far unsolved, problem here is whether just continuity of $f(x, y)$ implies $A(S) = A^*(S)$. Radó [5] has shown in this general case that $A^*(S) \leqq 2^{1/2}A(S)$. It has been shown (Huskey [3]) that if $f(x, y)$ is ACY on Q_0 then $A(S) = A^*(S)$. Theorems II and III show that somewhat more than this is true.

By the inequality

$$A(S) \leqq \liminf_{n} A[P_0^n(f; u, v)] \leqq \liminf_{\alpha} A[P_0^{n_\alpha}(f; u, v)] = A(S)$$

we see that Theorem III implies Theorem II. However, independent proofs are given here. It is hoped that this presentation will encourage some reader to attack successfully the general case described above.

2. **Preliminary.** Let $Q(u, v)$ denote the square $u \leqq x \leqq u+1$, $v \leqq y \leqq v+1$. Let $D_n(u, v)$ denote a subdivision of $Q(u, v)$ into n^2 congruent squares. Let $S(u, v)$ denote the surface defined by $z = f(x, y)$,

$(x, y) \in Q(u, v)$. Let $P_n(f; u, v)$ denote the polyhedron inscribed in $S(u, v)$ and which corresponds to the subdivision $D_n(u, v)$ (see §1). By boundary triangles of $P_n(f; u, v)$ we mean triangles which have at least one vertex on the edge of $S(u, v)$.

LEMMA 1. *The sum of the areas of the boundary triangles of $P_n(f; u, v)$ or of $P_0^n(f; u, v)$ converge to zero as $n \to +\infty$. (See Huskey* [3, p. 338].)

Due to periodicity $P_n(f; u, v)$ can be replaced by a trianglewise identical polyhedron $P_n(f; u', v')$ having only boundary triangles overlapping the edges of S and whose other triangles are identical with those of $P_0^n(f; u, v)$. It is easy to show that $A[S(u, v)] = A(S)$, see Saks [8, p. 179]. Thus, in view of the above lemma, it is clear that Theorems I, II, and III will be established if we prove that, under the assumptions stated therein,

$\lim_i A[P_{n_i}(f; u, v)] = A(S)$ for almost every $(u, v) \in Q_0$;
$\liminf_n A[P_n(f; u, v)] = A(S)$ for almost every $(u, v) \in Q_0$; and
$\lim_\alpha A[P_{n_\alpha}(f; u, v)] = A(S)$ for almost every $(u, v) \in Q_0$.

LEMMA 2. *If the measurable function $g_n(x, y)$ is periodic in x of period $1/n$ and, likewise, periodic in y of period $1/n$, then for any increasing sequence $\{n_i\}_{i=1}^{\infty}$ such that n_i divides n_{i+1} for all integers i, we have* $\liminf g_{n_i}(x, y) = a$ *constant for almost every* (x, y).

PROOF. (This lemma seems to be well known but knowing of no convenient reference we give the following proof.) Let $\{n_i\}$ be given as described and let $\phi(x, y) = \liminf_i g_{n_i}(x, y)$. $\phi(x, y)$ is periodic in x and in y of period $1/n_i$ for every i. (1) We first assume $\phi(x, y)$ is continuous and it is easy to show that $\phi(x, y)$ is constant for all points (x, y). (2) We now assume that $\phi(x, y)$ is summable on Q_0 and let $\phi_h(x, y) = h^{-2}\int_0^h\int_0^h \phi(x+u, y+v)\,du\,dv$. $\phi_h(x, y)$ is continuous, periodic of period $1/n_i$ in x and in y for all i, and is therefore a constant. $\phi_h(x, y) \to \phi(x, y)$ as $h \to 0$ for almost every (x, y). Therefore, $\phi(x, y)$ is a constant almost everywhere. (3) To prove the theorem we note that $\phi(x, y)$ is measurable and let

$$\phi_N(x, y) = \min\,[\max\,[\phi(x, y), -N], N] \qquad \text{where } N \geqq 0.$$

Now $|\phi_N(x, y)| \leqq N$ is summable and periodic as above so $\phi_N(x, y)$ is a constant almost everywhere. $\phi_N(x, y) \to \phi(x, y)$ as $N \to +\infty$, so $\phi(x, y)$ is a constant almost everywhere. QED.

3. **Proof of Theorem I.** Let h be a positive number and set

$$f^h(x, y) = \frac{1}{h^2}\int_0^h\int_0^h f(x+\alpha, y+\beta)\,d\alpha\,d\beta$$

which is an integral mean of $f(x, y)$. It is well known that $A(S^h) \to A(S)$ as $h \to 0$ where S^h: $z = f^h(x, y)$, $(x, y) \in Q_0$ (see Saks [8, p. 179]).

Let $V_c^d(x, f)$ denote the total variation of $f(x, y)$ for $x = x$ and $c \leqq y \leqq d$. $V_a^b(y, f)$ has a similar definition. It is easy to show (see Huskey [2, p. 251]; or Radó and Reichelderfer [6, p. 535]) that

$$\int_a^b V_c^d(x, f - f^h)dx + \int_c^d V_a^b(y, f - f^h)dy \to 0 \quad \text{as } h \to 0. \tag{5}$$

Jessen [4] has shown that if $f(x)$ is summable and of period one in x then for any increasing sequence of positive integers $\{m_i\}_{i=1}^{\infty}$ such that m_i divides m_{i+1} for all i it is true that

$$\lim \sum_{k=0}^{m_i - 1} (1/m_i) f(x + k/m_i) = \int_0^1 f(x)dx$$

for almost every x.

LEMMA 3.

$$\left| A[P_n(f; \xi, \eta)] - A[P_n(f^h; \xi, \eta)] \right|$$
$$\leqq \sum_{i,j=0}^{n-1} \frac{1}{n} V_{\eta + j/n}^{\eta + (j+1)/n}\left(\xi + \frac{i}{n}, f - f^h\right) + V_{\xi + i/n}^{\xi + (i+1)/n}\left(\eta + \frac{i}{n}, f - f^h\right).$$

The proof for Theorem 2.7 in Huskey [2, p. 253] can be easily altered to prove this lemma.

Using this lemma with Jessen's theorem and the fact that $A(S)$ is finite, we have for all (ξ, η) not in a set e_1 of measure zero

$$\begin{aligned} &\lim \sum_{k=0}^{m_i - 1} \left(\frac{1}{m_i}\right)\left[V_\eta^{\eta+1}\left(\xi + \frac{k}{m_i}, f - f^h\right) + V_\xi^{\xi+1}\left(\eta + \frac{k}{m_i}, f - f^h\right)\right] \\ &\qquad = \int_\xi^{\xi+1} V_\eta^{\eta+1}(x, f - f^h)dx + \int_\eta^{\eta+1} V_\xi^{\xi+1}(y, f - f^h)dy. \end{aligned} \tag{6}$$

Now, to prove the theorem, consider

$$\begin{aligned} &\left| A(S) - A[P_{m_i}(f; \xi, \eta)] \right| \\ &\qquad \leqq \left| A(S) - A(S^h) \right| + \left| A(S^h) - A[P_{m_i}(f^h; \xi, \eta)] \right| \\ &\qquad\quad + \left| A[P_{m_i}(f^h; \xi, \eta)] - A[P_{m_i}(f; \xi, \eta)] \right|. \end{aligned} \tag{7}$$

For $\epsilon > 0$ given, choose h so that the first term on the right of (7) is less than $\epsilon/2$, and so that the right-hand side of (6) is less than $\epsilon/4$. Next, choose (ξ, η) in $Q_0 - e_1$ and then choose i so that the remaining two terms of (7) are less than $\epsilon/2$. Note that

$\lim A[P_{m_i}(f^k;\ \xi,\ \eta)]=A(S^k)$ for every $(\xi,\ \eta)$ (see Saks [8, p. 177]). This completes the proof of Theorem I.

4. **Proof of Theorem II.** $A[P_n(f;\ \xi,\ \eta)]$ is periodic in ξ of period $1/n$ and in η of period $1/n$. Therefore, except for $(\xi,\ \eta)$ in a set e_3 of measure zero, we have

$$\liminf_i A[P_{m_i}(f;\ \xi,\ \eta)] = A \text{ (a constant)} \tag{8}$$

where m_i divides m_{i+1} for all i.

By definition and periodicity of $A[P_n(f;\ \xi,\ \eta)]$,

$$\begin{aligned}
&\iint_{Q(\xi,\eta)} [T_1(f;\ \alpha,\ \beta) + T_2(f;\ \alpha,\ \beta)]d\alpha d\beta \\
&\quad = \sum_{D_n(\xi,\eta)} n^2 \int_0^{1/n}\int_0^{1/n} (1/n^2)[T_1(f;\ x' + \alpha,\ y' + \beta) \\
&\qquad\qquad + T_2(f;\ x' + \alpha,\ y' + \beta)]d\alpha d\beta \\
&\quad = n^2 \int_0^{1/n}\int_0^{1/n} A[P_n(f;\ \xi + \alpha,\ \eta + \beta)]d\alpha d\beta \\
&\quad = \iint_{Q(\xi,\eta)} A[P_n(f;\ \alpha,\ \beta)d\alpha d\beta.
\end{aligned} \tag{9}$$

This formula is essentially a result of Radó (see [5, p. 500]).

L. C. Young [9] has shown that if $f(x,\ y)$ is ACY on Q_0 then

$$\iint_{Q_0} 2T_i(f;\ \alpha,\ \beta)d\alpha d\beta \to A(S), \qquad i = 1,\ 2, \tag{10}$$

as $n\to\infty$ in any manner (cf. §1).

Now, almost everywhere in Q_0,

$$A = \liminf_i A[P_{m_i}(f;\ \xi,\ \eta)] \geqq \liminf_n A[P_n(f;\ \xi,\ \eta)],$$

$n=1,\ 2,\ 3,\ \cdots$. So, using formulas (8), (9), and (10), the lemma of Fatou, and the property ACY of $f(x,\ y)$, we have for almost every $(\xi,\ \eta)$

$$\begin{aligned}
A(S) &= \lim_i \iint_{Q_0} A[P_{m_i}(f;\ \alpha,\ \beta)]d\alpha d\beta \geqq \int_0^1\int_0^1 A dx dy \\
&\geqq \liminf A[P_n(f;\ \xi,\ \eta)] \geqq A(S),
\end{aligned}$$

which proves the theorem.

5. Proof of Theorem III.

LEMMA 4. *If* (a) *E is a measurable set in the plane and of finite measure,* (b) *$\{g_n(x, y)\}_{n=0}^{\infty}$ is a sequence of non-negative real-valued functions measurable and summable on E,* (c) $\liminf_n g_n(x, y) \geqq g_0(x, y)$ *almost everywhere on E,* (d) $\limsup_n \iint_E g_n(x, y)dxdy \leqq \iint_E g_0(x, y)dxdy$, *then there exists a subsequence $\{g_{n_i}(x, y)\}_{i=1}^{\infty}$ such that $g_{n_i}(x, y) \to_i g_0(x, y)$ almost everywhere on E.*

For a proof of this lemma see Radó and Reichelderfer [6, p. 551]. Using equations (4), (9), and (10) we readily see that $A[P_{n_i}(f; \xi, \eta)]$ satisfies all the above conditions on Q where $\{n_i\}$ is any sequence of integers such that $n_i \to \infty$. ($g_0(x, y)$ corresponds to $A(S)$.) This completes the proof of Theorem III.

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RING AND LATTICE CHARACTERIZATIONS OF COMPLEX HILBERT SPACE

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Introduction. In an earlier paper [1][2] by the authors it was suggested that at least the ring characterization of real Hilbert space given therein might be extended to the complex case by making use of a device employed by B. H. Arnold [2] in so extending a theorem of Eidelheit. It is the purpose of the present note to show that this can indeed be done not only for the ring characterization but for the lattice one as well.

The difficulty in the complex case is that the complex field admits a great many discontinuous automorphisms. It is overcome by making use of the device of Arnold mentioned above to show that in the infinite-dimensional case only continuous automorphisms present themselves (see Lemma 2 below). It is shown by an example that the infinite-dimensionality is essential and that accordingly the theorems of [1] cannot be extended to the complex case in quite their full generality.

1. Two preliminary lemmas.

Lemmas 1 and 2 below constitute our formulation of Arnold's argument.

LEMMA 1. *Let X be an infinite-dimensional normed linear space (real or complex). Then there exists an infinite sequence $x_1, x_2, x_3, \cdots$ of elements of X such that given any bounded infinite sequence $\lambda_1, \lambda_2, \lambda_3, \cdots$ of scalars there exists a member l of the conjugate space $\overline{X}$ of X such that $l(x_i) = \lambda_i$ for $i = 1, 2, \cdots$.*

PROOF. As is well known, it is possible to construct infinite sequences $x_1, x_2, \cdots; l_1, l_2, \cdots$ where each x_i is in X and each l_i is in $\overline{X}$ so that $l_i(x_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$. Furthermore, it is clear that we may suppose that $\|l_i\| = 1/2^i$. Let $\lambda_1, \lambda_2, \cdots$ be an arbitrary bounded sequence of scalars with l.u.b.$|\lambda_i| = M$. For each

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[1] It should be pointed out that Professor Kakutani is in no way responsible for the appearance of the results of this paper in their present form. They are the result of a continuation of our joint work based upon a suggestion which Professor Kakutani made in a letter just before he left for Japan and before the final draft of our earlier paper had been written. The second named author is entirely responsible for postponing the following up of this suggestion and publishing the results separately in the present paper.

[2] Numbers in brackets refer to the bibliography.

$x = h_1(x)x_1 + h_2(x)x_2 + \cdots + h_n(x)x_n$ in $x_1 + x_2 + \cdots$ define $\bar{l}(x)$ as $h_1(x)\lambda_1 + h_2(x)\lambda_2 + \cdots + h_n(x)\lambda_n$. Then $\bar{l}$ is obviously linear.[3] Furthermore

$$|\bar{l}(x)| \leqq \|h_1\|\|x\||\lambda_1| + \|h_2\|\|x\||\lambda_2| + \cdots + \|h_n\|\|x\||\lambda_n|$$
$$\leqq M\|x\|[1/2 + 1/4 + \cdots + 1/2^n] < M\|x\|.$$

Thus $\bar{l}$ is bounded. Hence by the Hahn-Banach extension theorem or the Bohnenblust-Sobczyk extension of it to complex spaces [3], $\bar{l}$ coincides on $x_1 + x_2 + \cdots$ with a member l of $\bar{X}$. Since it is obvious that $\bar{l}(x_i) = \lambda_i$ for $i = 1, 2, \cdots$ the truth of the lemma follows.

LEMMA 2. *Let T be a one-to-one semi-linear*[4] *transformation of an infinite-dimensional complex normed linear space X onto a second such space Y. Then if T carries closed maximal subspaces into closed maximal subspaces the automorphism involved in the semi-linearity of T is a continuous one.*

PROOF. Let $x_1, x_2, \cdots$ be a sequence of elements of X having the property described in Lemma 1. Let $\mu_1, \mu_2, \cdots$ be an arbitrary bounded sequence of complex numbers and let $\mu_1', \mu_2', \cdots$ be their transforms by the automorphism in question. In order to prove that the automorphism is continuous it is clearly sufficient to show that $\{\mu_i'\}$ is a bounded sequence. Suppose that it is not. Then we may select a subsequence $\lambda_1', \lambda_2', \cdots$ such that $|\lambda_i'| > i\|T(x_i)\|$ for $i = 1, 2, \cdots$. Now let l be a member of $\bar{X}$ such that $l(x_i) = \lambda_i$ for $i = 1, 2, \cdots$, let M be the null space of l and let $\bar{x}$ be an element of X with $l(\bar{x}) = 1$. It is clear that for $i = 1, 2, \cdots$ we have $x_i = z_i + \lambda_i\bar{x}$ where $z_i \in M$. Hence $T(x_i)/\lambda_i' = T(z_i/\lambda_i) + T(\bar{x})$. Since $\|T(x_i)/\lambda_i'\| < 1/i$ it follows that $T(-z_i/\lambda_i) \to T(\bar{x})$ as $i \to \infty$. Hence $T(\bar{x})$ is in the closure of $T(M)$. Since M is a closed maximal subspace, $T(M)$ is, by hypothesis, closed. Hence $T(\bar{x}) \in T(M)$. Since T is one-to-one this means that $\bar{x}$ is in M and we have a contradiction. Thus the lemma is proved.

COROLLARY. *Under the hypotheses of Lemma 2 the automorphism in question is either the identity or the automorphism carrying every complex number into its conjugate and hence T is either linear or "conjugate linear."*

2. **The lattice characterization.** Theorem 1 of [1] after slightly

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[3] In this paper linear means additive and homogeneous.

[4] A semi-linear transformation is a transformation T such that $T(\lambda x + \mu y) = \lambda' T(x) + \mu' T(y)$ where x and y are arbitrary elements of the vector space, λ and μ are scalars and $\lambda \to \lambda'$ is an automorphism of the scalar field independent of x, y, λ and μ.

strengthening the dimensionality restriction may be stated for complex spaces as follows:

THEOREM 1. *Let X be an infinite-dimensional complex Banach space, and let Λ be its lattice of closed subspaces. If there exists an operation $M\rightarrow M'$ from Λ to Λ such that:*

(1) *if M_1 and M_2 are in Λ and $M_1\subseteq M_2$ then $M_1'\supseteq M_2'$,*

(2) *if $M\in\Lambda$ then $M''=M$,*

(3) *if $M\in\Lambda$ then $M'\cap M=0$,*

then there exists an isomorphism between X and a (not necessarily separable) complex Hilbert space such that the correspondence $M\leftrightarrow M'$ is carried over into the correspondence between orthogonal complements in the Hilbert space. In other words it is possible to introduce an inner product (x, y) in X with the properties listed in the standard Hilbert space axioms such that the new norm $|||x|||$ in X defined by the equation $|||x||| = (x, x)^{1/2}$ is equivalent to the given norm $\|x\|$ and furthermore such that for each M in Λ, M' is the set of all y in X for which $(x, y)=0$ for all x in M.

PROOF. Let M be a one-dimensional subspace of X. It is clear that M is closed and it follows from the fact that $M\rightarrow M'$ is a lattice anti-isomorphism that M' is a maximal closed linear subspace of X. Then let M^0 be the one-dimensional subspace of $\overline{X}$, the conjugate of X, consisting of those members l of $\overline{X}$ such that $l(m)=0$ for all m in M'. It is easy to see that the operation $M\rightarrow M^0$ thus defined sets up a one-to-one linear independence preserving correspondence between the one-dimensional subspaces of X on the one hand and those of $\overline{X}$ on the other. Now in the real case Lemma A of [4] allowed us to conclude the existence of a linear transformation of X on $\overline{X}$ such that $T(M)=M^0$ for all one-dimensional subspaces M of X. A slight modification of the proof of this lemma enables us to conclude the existence of a semi-linear such transformation in the present case. We need only replace our reference to the theorem that a collineation between two real projective planes can be represented analytically by a linear transformation by a reference to the more general theorem relating collineations between projective planes over a field Φ to semi-linear transformations.

Now every maximal closed subspace of X is clearly of the form M' where M is one-dimensional and it is readily verified that $T(M')$ is the set of all l in $\overline{X}$ taking M into zero and hence is maximal and closed. It follows then from Lemma 2 that T is either linear or conjugate linear. We show now that the former alternative is impossible. Supposing T to be linear let x and y be linearly independent and set

$T(x)=f$, $T(y)=g$. Then for each complex number μ, $T(x+\mu y)=f+\mu g$ and $(f+\mu g)(x+\mu y)=f(x)+\mu(g(x)+f(y))+\mu^2 g(y)$. Since $M\cap M'=0$, $f(x)$, $g(y)$ and $(f+\mu g)$ $(x+\mu y)$ are all different from zero. But since $g(y)\neq 0$ the equation $f(x)+\mu(g(x)+f(y))+\mu^2 g(y)=0$ has a solution for μ. Since $(f+\mu g)(x+\mu y)\neq 0$ this is a contradiction.

We now use the conjugate linear transformation T to define an inner product in X. For each two elements x and y let $(x, y)=f(x)$ where $f=T(y)$. Clearly (x, y) is linear in x and conjugate linear in y. To show that (x, y) is Hermitian symmetric (that is, $(y, x)=(x, y)^*$)[5] we observe first that $(x, y)=0$ if and only if $(y, x)=0$. In fact $(x, y)=0$ if and only if $M_x\subseteq M_y'$ where M_x and M_y are the one-dimensional subspaces of X generated by x and y respectively. But by properties (1) and (2) of the operation $M\rightarrow M'$ we have $M_x\subseteq M_y'$ if and only if $M_y\subseteq M_x'$, that is, if and only if $(y, x)=0$. Now let x and y be arbitrary elements of X with $(x, y)\neq 0$. Then there exist complex numbers λ and μ such that (a) $\lambda(x, x)+(x, y)=0$ and (b) $\mu(y, y)+(x, y)=0$. (Since $g(z)$ where $g=T(z)$ is never zero for $z\neq 0$ it follows that for $z\neq 0$, $(z, z)\neq 0$.) Equation (a) may be transformed into equation (c): $\lambda(x, x)^*+(y, x)^*=0$ by means of the following sequence of equations each of which is obviously equivalent to its predecessor: $\lambda(x, x)+(x, y)=0$, $(x, \lambda^* x)+(x, y)=0$, $(x, \lambda^* x+y)=0$, $(\lambda^* x+y, x)=0$, $\lambda^*(x, x)+(y, x)=0$, $\lambda(x, x)^*+(y, x)^*=0$. Similarly (b) may be transformed into (d): $\mu(y, y)^*+(y, x)^*=0$. Comparing (c) and (d) on the one hand and (a) and (b) on the other we conclude that $\lambda(x, x)^*=\mu(y, y)^*$ and that $\lambda(x, x)=\mu(y, y)$. Thus $(x, x)/(x, x)^*=(y, y)/(y, y)^*$. It follows that if (z, z) is real for one element z it is real for every w with $(x, w)\neq 0$ and hence for every w. But a simple adjustment of T always enables us to realize this condition. Hence we may suppose that $(x, x)^*=(x, x)$ and $(y, y)^*=(y, y)$. We now conclude from (a) and (c) that $(y, x)^*=(x, y)$ so that (x, y) is Hermitian symmetric. The proof of the positive definiteness of (x, x) (after a suitable adjustment of T) and the fact that the norm which it defines has the properties required is word for word the same as the corresponding proof in the real case [1, page 52] and will not be repeated here. We only remark that it is readily verified, using the Bohnenblust-Sobczyk extension of the Hahn-Banach theorem, that the theorems of Banach referred to are true for complex as well as real spaces.

Just as in the real case we conclude at once, from Theorem 1 and certain obvious properties of finite-dimensional spaces, the truth of:

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[5] In this paper whenever A denotes a complex number A^* will denote its complex conjugate.

THEOREM 2. *Let X be a complex Banach space and let Λ be its lattice of closed subspaces. Then X is isomorphic to a complex Hilbert space (not necessarily separable or infinite-dimensional) if and only if there exists an operation $M \rightarrow M'$ from Λ onto Λ which satisfies* (1), (2), *and* (3) *of Theorem* 1.

3. **The ring characterization.** The device for reducing the ring theorem to the lattice theorem used in the real case works equally well here and we may state the following theorem.

THEOREM 3.[6] *Let X be a complex Banach space and let R be the ring of continuous linear transformations of X into itself. Then X is isomorphic to a complex Hilbert space (not necessarily separable or infinite-dimensional) if and only if there is an operation $T \rightarrow T'$ from R onto R which has the following properties:*

(1) *If T_1 and T_2 are members of R then*

$$(T_1T_2)' = T_2'T_1' \quad \text{and} \quad (T_1 + T_2)' = T_1' + T_2'.$$

(2) *If T is a member of R then $T'' = T$.*

(3) *If T is a member of R such that $T'T = 0$ then $T = 0$.*

In case X is infinite-dimensional the analogue for rings of Theorem 1 is true. In fact if we follow through the construction of the lattice operation from the ring operation used in proving Theorem 3 from Theorem 2 (see [1, pp. 53 and 54]) and the construction of an inner product from the lattice operation we see without great difficulty that if S_x and S_y are members of R whose ranges are the one-dimensional subspaces of X defined by the elements x and y of X then $(T(x), y) = 0$ if and only if $S_y' T S_x = 0$ and $(x, T'(y)) = 0$ if and only if $S_x' T' S_y = 0$. Thus since $(S_y' T S_x)' = S_x' T' S_y$ it follows that $(T(x), y) = 0$ if and only if $(x, T'(y)) = 0$ and hence that $(x, T'(y)) = 0$ if and only if $(x, T^*(y)) = 0$ where T^* is the adjoint of T calculated with respect to the inner product (x, y). Thus for every y, $T'(y)$ and $T^*(y)$ have identical orthogonal complements so that one is a scalar multiple of the other. It follows easily from this that $T' = T^*$, for all T in R. In fact for each x in X, $(1+T)'(x) = \lambda(1+T)^*(x) = \lambda(1+T^*)(x) = \lambda x + \lambda T^*(x)$ and $(1 + T)'(x) = (1 + T')(x) = x + \mu T^*(x)$. Thus $\lambda x + \lambda T^*(x) = x + \mu T^*(x)$ and if $T^*(x)$ and x are linearly independent we can conclude that $\lambda = \mu = 1$. If $T^*(x)$ and x are linearly dependent choose S so that $S^*(x)$ and x are not. Then $(S+T)^*(x) = (S+T)'(x)$ so that $S^*(x) + T^*(x) = S'(x) + T'(x)$. Also $S^*(x) = S'(x)$ so that

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[6] For a proof of a weaker form of this theorem in which it is assumed in addition that $(\lambda T)^* = \lambda^* T^*$ see [5].

$T^*(x)=T'(x)$. Thus in any case $T^*(x)=T'(x)$ so that $T^*=T'$. Thus we may state:

THEOREM 4. *Let X be an infinite-dimensional complex Banach space and let R be the ring of all continuous linear transformations of X into itself. If there exists an operation $T\to T'$ from R onto R which has properties* (1), (2) *and* (3) *of Theorem 3 so that X is isomorphic to a not necessarily separable complex Hilbert space then this isomorphism may be set up in such a way that the correspondence $T\to T'$ goes over into the correspondence between an operator and its adjoint. In other words, it is possible to introduce an inner product (x, y) in X with the properties listed in the standard Hilbert space axioms such that the new norm $|||x|||$ in X defined by the equation $|||x|||=(x, x)^{1/2}$ is equivalent to the given norm $\|x\|$ and furthermore such that for all x and y in X, $(T(x), y)=(x, T'(y))$.*

4. **Some counter examples.** The discussion given in paragraph 3 of [1] applies without essential change in the complex case so that we may conclude here as there that our theorems need not be true if we fail to assume completeness for X.

Finally we show that the infinite-dimensionality assumption in Theorems 1 and 4 cannot be omitted. To this end let α be any automorphism of the complex field other than the identity and the conjugacy automorphism. For each complex number λ let $\lambda'=\alpha^{-1}(\alpha(\lambda))^*$. Then $\lambda\to\lambda'$ is clearly an involutory automorphism of the complex field. Furthermore if $\lambda_1\lambda_1'+\lambda_2\lambda_2'+\cdots+\lambda_n\lambda_n'=0$ then

$$\alpha(\lambda_1)\alpha(\lambda_1')+\alpha(\lambda_2)\alpha(\lambda_2')+\cdots+\alpha(\lambda_n)\alpha(\lambda_n')=0$$

or

$$\alpha(\lambda_1)\alpha(\lambda_1)^*+\alpha(\lambda_2)\alpha(\lambda_2)^*+\cdots+\alpha(\lambda_n)\alpha(\lambda_n)^*=0.$$

Hence $\alpha(\lambda_1)=\alpha(\lambda_2)=\cdots=\alpha(\lambda_n)=0$ so that $\lambda_1=\lambda_2=\cdots=\lambda_n=0$. Now let R_n be the vector space of n-uples of complex numbers and for each two members $x=\lambda_1, \lambda_2, \cdots, \lambda_n$; $y=\mu_1, \mu_2, \cdots, \mu_n$ of R_n let $(x, y)=\lambda_1\mu_1'+\lambda_2\mu_2'+\cdots+\lambda_n\mu_n'$. Then (x, y) is linear in x, semilinear with respect to $'$ in y and such that $(y, x)=(x, y)'$ and $(x, x)=0$ if and only if $x=0$. Now for each linear transformation T of R_n into itself define T' by the equation $(T(x), y)=(x, T'(y))$ for all x and y in R_n. Then, as is easily seen, T' is a linear transformation of R_n into itself and the operation $T\to T'$ has the properties (1), (2), and (3) listed in the statement of Theorem 3. On the other hand it is not difficult to show that there is no inner product in X for which $T'=T^*$. Let λ be a non-real complex number such that $\alpha(\lambda)$ is real. Then

$\lambda' = \lambda$. Hence $\lambda \to \lambda'$ is not the conjugacy operation. Since it is obviously not the identity there exists a non-real complex number μ such that μ' is real. Let T be μ times the identity. Then T' will be μ' times the identity and for any inner product T^* will be $\bar{\mu}$ times the identity. Since μ' is real and $\bar{\mu}$ is not, T' and T^* cannot be equal.

Similarly using this same "semi-inner product" to define orthogonality in R_n one can prove for $n > 1$ the existence of an operation on subspaces having the properties listed in the statement of Theorem 1 and not coinciding with the operation of taking orthogonal complements for any inner product in R_n.

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A NOTE ON LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS

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In this paper the coefficients a_{ij} in the equations

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0 \qquad (i = 1, \cdots, m) \tag{1}$$

are constant rational integers and all letters denote integers. If $m = n-1$ and the rank is $n-1$ then the complete solution in integers is well known. Thus, if E_j is the determinant obtained by deleting the jth column from the matrix of the coefficients, and if $e = (E_1, \cdots, E_n)$, then the solution is

$$x_j = (-1)^j t E_j / e \qquad (j = 1, \cdots, n), \tag{2}$$

in which t is an arbitrary integer.

E. T. Bell recently conjectured that if $m < n-1$ and if the rank r is m then the solution is similarly obtained from the system formed by (1) and the equations

$$\xi_{i1}x_1 + \cdots + \xi_{in}x_n = 0 \qquad (i = 1, \cdots, n - m - 1), \tag{3}$$

in which the ξ_{ij} are arbitrary integers. In this paper this conjecture is proved by induction. Since this solution is written down directly from (1) and is fully displayed these results are more usable than those in the literature.[1]

If $r = 1$ it can be assumed without limitation that $a_1 \cdots a_n \neq 0$, $(a_1, \cdots, a_n) = 1$, and at least one of $x_1, \cdots, x_n$ is not zero. If $n = 3$ there are integers $t, y_1, y_2, y_3, d, A_1, A_2, k_1, k_2$ such that

$$x_1 = ty_1, \qquad x_2 = ty_2, \qquad x_3 = ty_3, \qquad (y_1, y_2, y_3) = 1, \tag{4}$$

$$a_1 = dA_1, \qquad a_2 = dA_2, \qquad (A_1, A_2) = 1, \qquad k_1A_2 - k_2A_1 = 1. \tag{5}$$

Since $(d, a_3) = 1$ there is an integer s such that

$$y_3 = ds, \qquad A_1y_1 + A_2y_2 + a_3s = 0. \tag{6}$$

Then since $(A_1, A_2) = 1$ there is an integer r such that

$$y_1 - a_3k_2s = A_2r, \qquad y_2 + a_3k_1s = -A_1r. \tag{7}$$

These conditions are also sufficient. Hence the complete solution is

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[1] Th. Skolem, *Diophantische Gleichungen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 5, no. 4, 1938; D. N. Lehmer, Proc. Nat. Acad. Sci. U.S.A. vol. 4 (1919).

$$(8)\qquad \begin{aligned} x_1 &= \quad t\cdot\frac{1}{d}\begin{vmatrix} dk_2s & -r \\ a_2 & a_3 \end{vmatrix}, \\ x_2 &= -t\cdot\frac{1}{d}\begin{vmatrix} dk_1s & -r \\ a_1 & a_3 \end{vmatrix}, \end{aligned} \qquad x_3 = t\cdot\frac{1}{d}\begin{vmatrix} dk_1s & dk_2s \\ a_2 & a_3 \end{vmatrix}.$$

Since $(d, a_3)=1$ there are integers p and P such that $-r+pa_3=dP$. Hence d is the greatest common divisor of the determinants in (8). Therefore for the particular values $-dk_1s$, $-dk_2s$, r of ξ_{11}, ξ_{12}, ξ_{13}, (8) is an instance of (2).

Similarly, if $n=4$ and $d_1=(a_1, a_2)$, $d_2=(a_3, a_4)$ then the solution of $a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$ is equivalent to that of $A_1y_1+A_2y_2=-d_2S$ and $A_3y_3+A_4y_4=d_1S$. By the preceding discussion the complete solution for each of these equations is known. The expressions for S are then equated. Thus it is proved that there are integers $k_1, \cdots, k_4$, r_1, r_2, t_1, t_2, s such that the solution is obtained by applying (2) to the matrix

$$(9)\qquad \begin{bmatrix} -r_2A_1 & -r_2A_2 & d_2k_3t_1s & d_2k_4t_1s \\ d_1k_1t_2s & d_1k_2t_2s & -r_1A_3 & -r_1A_4 \\ d_1A_1 & d_1A_2 & d_2A_3 & d_2A_4 \end{bmatrix}.$$

If $n\geqq 5$ the notations $x_j=ty_j$ $(j=1, \cdots, n)$, $(y_1, \cdots, y_n)=1$, $a_j=\delta\alpha_j$ $(j=1, \cdots, n-2)$, $a_j=\zeta A_j$ $(j=n-1, n)$, $(\alpha_1, \cdots, \alpha_{n-2})=1$, $(A_{n-1}, A_n)=1$, $\alpha_j=d_1A_j$ $(j=1, \cdots, n-3)$, $(A_1, \cdots, A_{n-3})=1$, $\alpha_{n-2}=d_2A_{n-2}$, $\zeta=d_2\eta$, $(A_{n-2}, \eta)=1$ are used. The solution of $a_1x_1+\cdots+a_nx_n=0$ is equivalent to that of $A_{n-1}y_{n-1}+A_ny_n=\delta S$ and $\alpha_1y_1+\cdots+\alpha_{n-2}y_{n-2}=-\zeta S$. In particular, if $n=5$ then y_4, y_5, S are obtained as in (8) and y_1, y_2, y_3, S from (9) with A_4 replaced by η. If, to simplify the notation, f_{ij} denotes the element in the ith row and jth column of this new matrix then $x_1, \cdots, x_5$ are obtained by applying (2) to the matrix

$$(10)\qquad \begin{bmatrix} -r_4d_1A_1 & -r_4d_1A_2 & -r_4d_2A_3 & d_2\eta q_4S_4t_4 & d_2\eta q_5S_4t_4 \\ \delta f_{11} & \delta f_{12} & \delta f_{13} & f_{14}A_4 & f_{14}A_5 \\ \delta f_{21} & \delta f_{22} & \delta f_{23} & f_{24}A_4 & f_{24}A_5 \\ \delta d_1A_1 & \delta d_1A_2 & \delta d_2A_3 & d_2\eta A_4 & d_2\eta A_5 \end{bmatrix}.$$

In general, the matrix for $x_1, \cdots, x_n$ is obtained from the matrix for $n-1$ variables in precisely the same way as (10) is obtained from (9).

If $1<r<n-1$ it can be assumed without limitation that the first equation satisfies the conditions $a_1\cdots a_n\neq 0$, $(a_1, \cdots, a_n)=1$,

of the preceding discussion. If E_i denotes $a_{i1}x_1+\cdots+a_{in}x_n$ $(i=1, \cdots, r)$ then a given solution of $E_1=0$ is obtained by applying (2) to an appropriate matrix whose ith row is $\alpha_{i1}, \cdots, \alpha_{in}$ $(i=1, \cdots, n-2)$ and whose last row is $a_{11}, \cdots, a_{1n}$. By hypothesis these values of $x_1, \cdots, x_n$ also satisfy $E_2=0$. Therefore the determinant obtained by placing the row $a_{21}, \cdots, a_{2n}$ under this matrix is zero. Hence, if A_i is defined as $\alpha_{i1}x_1+\cdots+\alpha_{in}x_n$ $(i=1, \cdots, n-2)$, then the rank s of the functions $A_1, \cdots, A_{n-2}$, E_1, E_2 is less than n. Therefore there is a set of s functions from this list on which each remaining function is linearly dependent, with integer coefficients. If indeed the subset includes both E_1 and E_2 then the notation can be assigned so that the subset is $A_1, \cdots, A_{s-2}$, E_1, E_2. Then there are integers $d, d_1, \cdots, d_s$ such that $d\neq 0$ and

$$(11)\qquad dA_{n-2} = d_1A_1 + \cdots + d_{s-2}A_{s-2} + d_{s-1}E_1 + d_sE_2.$$

Now the result of applying (2) to the original matrix is the same as the result of applying it to the matrix which is obtained by replacing $\alpha_{n-2,j}$ by $d\alpha_{n-2,j}$ $(j=1, \cdots, n)$. By (11) in this new matrix $d\alpha_{n-2,j}$ may be replaced by the sum $d_1\alpha_{1j}+\cdots+d_{s-2}\alpha_{s-2,j}+d_{s-1}a_{1j}+d_sa_{2j}$, and hence by d_sa_{2j}, and hence by a_{2j}. Again, if the subset includes E_2 but not E_1, or if it includes neither E_1 nor E_2, or if it includes E_1 but not E_2, then in a similar way the matrix can be replaced by a matrix having $a_{11}, \cdots, a_{1n}$ as last row and $a_{21}, \cdots, a_{2n}$ as another row. This process can be continued until $a_{i1}, \cdots, a_{in}$ $(i=1, \cdots, r)$ appear.

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WATER WAVES ON SLOPING BEACHES

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1. Introduction. Consider the time-dependent velocity potential $\phi(x, y, t)$ of the motion of an incompressible inviscid liquid in two space dimensions x, y. The Newtonian equations and the condition of incompressibility lead to the following mathematical model: Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{1.1}$$

and Bernoulli's law

$$\frac{\partial\phi}{\partial t}+\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial t}\right)^2\right)\Big/2+gy+p=\text{const.}, \tag{1.2}$$

relating the pressure p and the gravity potential gy per unit mass to the velocity potential; here the x-axis is horizontal, the y-axis vertical and upwards and g is the modulus of gravity acceleration. At the boundary of air and water the pressure is supposed to have a constant value p_0; thus (1.2) relates implicitly the surface elevation y to the velocity potential. A considerable simplification is introduced by assuming the motion to be small of first order so that in (1.2) the quadratic terms may be cancelled. Then the motion becomes a small perturbation of the equilibrium position in which the surface will be thought of as given by $y=0$. For small elevations of the surface one concludes from (1.2) that, but for terms of higher order,

$$y=y(x, t)=-\frac{\partial\phi}{\partial t}(x, 0, t)\Big/g, \tag{1.3}$$

where the constant on the right hand has been absorbed in a properly modified ϕ. To this equation is added the condition that a particle at the surface remains at the surface; that is (using the "substantial" time derivative of (1.2)),

$$dy(x, t)/dt=-\frac{\partial^2\phi}{\partial t^2}\Big/g=\partial\phi/\partial y, \tag{1.3.1}$$

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where terms of second order, such as $(\partial^2\phi/\partial x\partial t)(\partial\phi/\partial x)$, have been dropped in comparison with $\partial^2\phi/\partial t^2$.

Among these infinitesimal motions the periodic motions are of particular interest. Set

$$(1.4)\qquad \phi(x, y, t) = \phi_1(x, y)\cos \nu t - \phi_2(x, y)\sin \nu t,$$

where $\nu > 0$ is the so-called circular frequency. From (1.1) we take

$$(1.5)\quad (\partial^2/\partial^2x + \partial^2/\partial^2y)\phi_1(x, y) = 0, \quad (\partial^2/\partial^2x + \partial^2/\partial^2y)\phi_2(x, y) = 0;$$

from (1.3)

$$(1.6)\quad y = y(x, t) = (\nu/g)\phi_1(x, 0)\sin \nu t + (\nu/g)\phi_2(x, 0)\cos \nu t;$$

from (1.3.1)

$$(1.6.1)\quad \begin{aligned}&(\nu^2/g)(\phi_1(x, 0)\cos \nu t - \phi_2(x, 0)\sin \nu t)\\ &\qquad\qquad = (\partial\phi_1/\partial y)\cos \nu t - (\partial\phi_2/\partial y)\sin \nu t.\end{aligned}$$

Special cases occur if one of the two functions ϕ_1, ϕ_2, say ϕ_2, is taken identically zero. Then the crests or troughs and the nodes of the oscillation retain their fixed positions in space. Crests or troughs of the surface elevation occur where $\partial\phi_1(x, 0)/\partial x = 0$, nodes where $\phi_1(x, 0) = 0$. Such a periodic oscillation is called a "standing wave" in contrast to the more general case of a "progressive wave," where the crests and troughs and the nodes change position with time. For the horizontal velocity component v_{nod} of the surface nodes we write (1.6) in the form $\tan \nu t = -\phi_2(x, 0)/\phi_1(x, 0)$ or $\nu t = -\text{arc tan}\,(\phi_2/\phi_1)$; differentiation with respect to t yields

$$v_{\text{nod}} = \nu/((d \text{ arc tan } \phi_1/\phi_2)/dx).$$

Similarly the horizontal component v_{cr} of the crest or trough velocity is found to be

$$v_{\text{cr}} = \nu\Big/\left(\left(d \text{ arc tan } \frac{d\phi_1}{dx}\Big/\frac{d\phi_2}{dx}\right)\Big/dx\right).$$

Both $\phi_1(x, y)$ and $\phi_2(x, y)$ are solutions of the same problem

$$(1.7)\qquad \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$$

throughout the liquid,

$$(1.8)\qquad \nu^2\phi(x, 0)/g = \frac{\partial\phi}{\partial y}(x, 0)$$

at the surface.

One of the fascinating problems concerning waves is the discovery of the influence of the depth of the water on the shape of the wave profile. Take first the classical case of infinite depth. To (1.7) and (1.8) is added the condition that the magnitude of the motion decreases to zero as $y \to -\infty$. Out of the manifold of all harmonic functions the real and imaginary parts of

$$\exp\left(-i(x+iy)\nu^2/g\right)$$

are taken as solutions ϕ_1 and ϕ_2 of (1.7) satisfying the boundary condition (1.8). With these is constructed the progressive wave

$$\phi(x, y, t) = R \exp\left(-i(x+iy)\nu^2/g - it\right) = \exp\left(\nu^2 y/g\right) \cos\left(\nu^2 x/g + \nu t\right),$$

which is periodic in x as well as in t; the x-period is $2\pi g/\nu^2$, the t-period $2\pi/\nu$. The wave profile travels unchanged in form in the direction of diminishing x as t increases; the velocity of the profile is $v_{\text{nod}} = v_{\text{cr}} = -g/\nu$ and depends therefore on the frequency.

Airy was the first to give a treatment of the infinitesimal gravity waves in a channel of constant finite depth; his results are the foundation of our knowledge on gravity waves. Again as in the case of infinite depth, the profile of the progressive wave moves, unchanged in form, with constant velocity. But the wave velocity depends on both the frequency and the depth of the channel, and so does the space period of the wave. Airy's formulae show with great clarity the passage from waves in water of finite depth to the well known limit theory of shallow water waves in whose treatment the dependence of the pressure and velocity on the depth is assumed rather than derived.

The problem whose solution forms the object of this paper is the construction of progressive water waves when the depth, instead of being a constant, has a constant slope. This problem was first attacked by E. T. Hanson[1] in 1926. He assumed the angle of the bottom with the surface to be of the form $\pi/(2q)$ with integral q, and gave the expression of one standing wave $\phi_1(x, y)$ in terms of exponential functions, without, however, constructing the progressive wave which requires the derivation of a second standing wave $\phi_2(x, y)$. M. Miche[2] quotes an explicit form of this second function ϕ_2, which he calls "houle," to distinguish it from ϕ_1, called "clapotis." For the derivation of his formula he refers the reader to a paper to appear in the Journal de mathématiques pures et appliquées; also his result is restricted to angles of form $\pi/(2q)$.

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[1] E. T. Hanson, Proc. Roy. Soc. London. Ser. A vol. 111 (1926) p. 491.

[2] M. Miche, *Mouvements ondulatoires de la mer en profondeur constante ou décroissante*, 1945, p. 88. This booklet also contains a most interesting report of various investigations of M. Miche, relating to the breaking of waves. See also M. Miche, *Annales des ponts et chaussées*, 1944, pp. 25–78, 131–164, 270–292, 369–406. Unfortunately, by the time of printing of the present paper, the author had had no opportunity to consult these latter articles.

We shall give in this paper an explicit representation in closed form for a progressive wave for all angles $p\pi/(2q)$ between bottom and surface, provided p and q are integers, p is odd, and $p<2q$. The functions involved in our formulae are exponential functions and exponential integrals of certain special types. (Unfortunately, the number of individual terms occurring in the solution becomes large as p or q become large.) In the course of our investigation we are led to the discovery of a strange and recondite relationship of these waves with the quadratic reciprocity law of number theory. In fact, this law appears as an elementary consequence of the continuous dependence of the finite standing wave on the angle between surface and bottom. We do not go here more deeply into the discussion of certain number-theoretical problems immediately suggested by this relationship, nor do we investigate the continuity of the waves as functions of the angle, for which there appears to exist a more direct approach than through our explicit expressions for these waves. A further topic left untouched is the extension of the principles of construction discussed here to the study of the three-dimensional wave problem where the wave crests are no longer parallel to the shore.

The complexity of our formulae for the waves for large p or q and the scarcity of suitable tables of exponential integrals make the evaluation of these waves a problem of considerable difficulty. It seems important to compare the "exact" theory of infinitesimal waves with other approximations such as the shallow water theory, in order to gauge the differences of the results furnished by the various theories. These problems, for angles $\pi/(2q)$, are attacked by M. Miche (loc. cit.) and by J. J. Stoker in a paper under press.

2. Reformulation of the standing wave problem. The units of length and time will now be chosen so that

$$g = 1 \quad \text{and} \quad \nu = 1.$$

Set

$$x + iy = z = e^{i\theta}$$

and let

$$\beta = \pi p/(2q), \qquad p \text{ odd and relatively prime to } q, \text{ and } 0 < p < 2q.$$

Denote by S_β the angular sector $0 > \theta > -\beta$. The problem of the standing wave for S_β consists in finding a harmonic function ϕ in S_β satisfying the following boundary and limit conditions, to be stated first for the case $p \leqq q$:

1. $\partial\phi/\partial y - \phi = 0$ on $y = 0$, $x > 0$;

2. The normal derivative $\partial\phi/\partial n = 0$ on the bottom of S_β;

3. $\phi - A \cos x - B \sin x \to 0$ for suitable constants, not both zero, A and B, as $x \to \infty$, $y = 0$;

4. The square of the particle velocity, $(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2$, remains bounded as z tends to ∞ in S_β;

5. The flow through any vertical section

$$\int_{-x \tan \beta}^{0} \frac{\partial\phi}{\partial x}(x, y)dy$$

remains bounded as $x \to \infty$.

To these conditions must be added one concerning the behavior of ϕ and its derivatives at the shore (0, 0). We shall see that there is one standing wave where

6a. ϕ and $\partial\phi/\partial x$, $\partial\phi/\partial y$ remain bounded as (x, y) tends to (0, 0), and another where

6b. ϕ behaves like $\log(x^2 + y^2)$ near (0, 0).

A solution of the above problem, if it exists, may be considered as the real part of an analytic function $W(z)$ in S_β. It can easily be concluded that $W(z)$ remains bounded as $z \to \infty$. For this purpose consider the conjugate harmonic ψ. Since the normal derivative of ϕ vanishes on the bottom of S_β, so does the tangential derivative of the conjugate ψ, whence ψ is a constant there. But

$$\begin{aligned}\psi(x, 0) - \psi(x, -x \tan \beta) &= \int_{-x \tan \beta}^{0} \frac{\partial\psi}{\partial y}(x, y)dy \\ &= \int_{-x \tan \beta}^{0} \frac{\partial\phi}{\partial x}(x, y)dy,\end{aligned}$$

so that $\psi(x, 0)$ remains bounded at the surface as $x \to \infty$, by virtue of the 5th condition. From the 4th condition it follows furthermore that $|W(z)|$ can grow at most like const. $|z|$ as $z \to \infty$ within S_β. But then the Phragmén-Lindelöf principle applies and yields the boundedness of $|W(z)|$ for $z \to \infty$ in S_β. Indeed, for the application of this principle it is not at all necessary to demand the condition 4 which states that $|W'(z)|$ remains bounded as $z \to \infty$; but it suffices to demand, besides 1, 2, 3,

4′. $|W(z)| < \exp|z|^{1+\epsilon}$ as $z \to \infty$ in S_β, for all $\epsilon > 0$, together with

5′. $|W(z)|$ remains bounded on the surface as $z = x \to \infty$.

For then the principle states that $|W(z)|$ remains bounded as $z \to \infty$ in

S_β. If $p<q$, then 4′ and 5′ imply 4 and 5. For it is then possible to extend $W(z)$ analytically by reflection on the bottom of S_β where condition 2 holds; and the double angle $2\beta=\pi p/q$ is still less than π. In the process of extension any bound for the absolute value of $W(z)$ on the surface of S_β becomes a bound for $|W(z)|$ on the bottom of $S_{2\beta}$. Accordingly, $|W(z)|$ remains bounded on the boundary of $S_{2\beta}$ as $z\rightarrow\infty$ and furthermore $|W(z)|<\exp|z|^{1+\epsilon}$ as $z\rightarrow\infty$ in $S_{2\beta}$, $\epsilon>0$. Moreover, $W(z)$ can be continued analytically across $y=0$, since this is true for the function $F(z)=idW/(dz)-W$ whose real part vanishes on $y=0$. A bound of $|F(z)|$ for $y\leqq 0$ becomes at the same time a bound of $|F(z)|$ for $y>0$. Let $|W(z)|\leqq M$ for $y\leqq 0$, x sufficiently large; then $|dW/dz|\leqq M/|y|$ by Cauchy's integral theorem. Describe a horizontal-vertical square Q of side 4 about a point x_0 of the surface as center and consider $F_1(z)=F(z)(z-x_0-2)(z-x_0+2)$. On Q we have $|F_1(z)|\leqq 54M$; hence we conclude for $|y|\leqq 2$ that $|F_1(x_0+iy)|\leqq 54M$, where $|F(x_0+iy)|\leqq 54M/(|iy-2|\,|iy+2|)\leqq 14M$, and, finally, from the definition of $F(z)$ and the bound M for $|W(z)|=|W(x_0+iy)|$ that $|dW/dz|\leqq 15M$ for x_0 sufficiently large, $-2\leqq y\leqq 0$. For $y<-2$ Cauchy's estimate gives immediately $|dW/dz|\leqq M$; so that $|dW/dz|$ remains bounded in S_β. Hence 4 and 5 are consequences of 1, 2, 3, 4′, 5′.

If p is no longer less than q, but $p<2q$, we shall still be able to exhibit a solution $W(z)$ of the wave problem satisfying 1, 2, 3, 4′, 5′; but we shall then show explicitly that $|W(z)|$ remains bounded on the bottom of S_β as $z\rightarrow\infty$, a fact which, as we have just seen, follows from the Phragmén-Lindelöf principle for $p<q$.

Let λ be an arbitrary complex number of modulus 1, and let $\partial\phi/\partial s$ stand for the derivative in the direction of the vector λ. Then

$$\partial RW/\partial s=\partial\phi/\partial s=R\partial W/\partial s=R\{\lambda dW/(d\lambda s)\}=R\{\lambda dW/dz\}.$$

Accordingly, the condition 1 may be rephrased as

$$R\{(id/dz-1)W(z)\}=0 \tag{2.1}$$

on $y=0$. Similarly for condition 2: set

$$\epsilon^{1/2}=e^{-\beta i},\qquad \epsilon=e^{-2\beta i}=e^{-p\pi i/q}; \tag{2.2}$$

then this condition becomes

$$R\{i\epsilon^{1/2}dW/dz\}=0\quad\text{on}\quad\theta=-\beta=-\pi p/(2q). \tag{2.3}$$

We furthermore see that reflection on the bottom of S_β yields

$$R\{(-i\epsilon d/dz-1)W(z)\}=0\quad\text{on}\quad\theta=-2\beta=-\pi p/q. \tag{2.4}$$

3. Derivation of an ordinary differential equation. Let $f(\xi)$ and $g(\xi)$ be polynomials with real coefficients. Then, by (2.1),

$$R\{f(d/dz)(id/dz - 1)W(z)\} = 0 \tag{3.1}$$

on $y=0$, and by (2.4),

$$\begin{aligned} g(\partial/\partial s)R\{(-i\epsilon d/dz - 1)W(z)\} \\ = R\{g(\epsilon d/dz)(-i\epsilon d/dz - 1)W(z)\} = 0 \end{aligned} \tag{3.2}$$

on $\theta = -2\beta$. Suppose we succeed in solving the identical equation

$$f(\xi)(i\xi - 1) = g(\epsilon\xi)(-i\epsilon\xi - 1); \tag{3.3}$$

then

$$f(d/dz)(id/dz - 1)W(z)$$

becomes a function $E(z)$, regular within the sector $0 > \theta > -2\beta$, and whose real part equals zero on the boundary of $S_{2\beta}$ (with the possible exception of the origin). Now all such functions $E(z)$ are known explicitly. Our problem thus becomes that of solving an ordinary differential equation

$$f(d/dz)(id/dz - 1)W(z) = E(z) \tag{3.4}$$

for an appropriate $E(z)$, and of choosing from the manifold of its solutions one, $W(z)$, such as to satisfy the boundary conditions (2.1) and (2.3) as well as certain conditions concerning the behavior of $W(z)$ near $z=0$ and $z=\infty$.

4. Derivation of $f(\xi)$.

LEMMA. *There exist polynomials $f(\xi)$ and $g(\xi)$ satisfying* (3.3) *and of degree $q-1$. They are uniquely determined by requiring the degree to be $q-1$ and $f(0)=1$. Furthermore $f(\xi) = g(\xi)$.*

The following proof is valid only if p is an odd number. This is the reason why our explicit construction of waves fails if p is even.

Let

$$\begin{aligned} f(\xi) &= a_0 + a_1\xi + \cdots, \qquad a_0 = 1, \\ g(\xi) &= b_0 + b_1\xi + \cdots. \end{aligned} \tag{4.1}$$

The identity to be solved is

$$(i\xi - 1)(1 + a_1\xi + \cdots) = (-i\epsilon\xi - 1)(b_0 + b_1\xi + \cdots). \tag{4.2}$$

Compare coefficients and find

$$\begin{aligned} 1 &= b_0, \\ i - a_1 &= -i\epsilon b_0 - b_1\epsilon, \\ &\cdots\cdots\cdots\cdots\cdots, \\ ia_{n-1} - a_n &= -i\epsilon^n b_{n-1} - b_n\epsilon^n, \\ &\cdots\cdots\cdots\cdots\cdots. \end{aligned} \tag{4.3}$$

The imaginary part of the left-hand side is a_{n-1} and may be considered known from the preceding equation of the recursion. The imaginary part of the right-hand side involves b_{n-1} and b_n. Now b_{n-1} also may be assumed known from the previous equation. Consequently b_n is determined by equating imaginary parts, provided $I(\epsilon^n) \neq 0$, that is, for all n with $0 < n < q$, since p and q have no common divisors. Once b_n is known, we take the value of a_n from the same equation by equating real parts. Thus the recursion furnishes in a unique way all coefficients a_n, b_n for $n < q$. There is a further condition to be satisfied to establish (4.2), namely, (4.3) for $n = q$, which becomes

$$ia_{q-1} - a_q = -i\epsilon^q b_{q-1} - b_q\epsilon^q. \tag{4.4}$$

Since $\epsilon^q = -1$ for odd p, this implies

$$a_{q-1} = b_{q-1};$$

if this is so, we may assume all further coefficients $a_q, b_q, a_{q+1}, b_{q+1}, \cdots$ to vanish. Now observe that $a_0 = b_0 = 1$, and that the nth recursion formula (4.3) may be satisfied by setting $a_n = b_n$, provided it is true that $a_{n-1} = b_{n-1}$. Indeed, the uniquely determined solution of the recursion formulae can be found from the recursion

$$a_n(1 - \epsilon^n) = a_{n-1}i(1 + \epsilon^n).$$

This gives for a_n the nonvanishing value

$$a_n = a_{n-1}i(1 + \epsilon^n)/(1 - \epsilon^n), \tag{4.5}$$

which is indeed real since the vector $1 - \epsilon^n$ is perpendicular to $1 + \epsilon^n$. This holds provided $\epsilon^n \neq 1$, that is, as long as $np/q \neq 2k$ with integral k; hence certainly for odd p and $0 < n < q$. Notice that (4.5) can also be written as

$$a_n = a_{n-1} \cot (np\pi/2q) = \prod_{k=1}^{n} \cot (kp\pi/2q), \qquad \text{for } n \geqq 1. \tag{4.5.1}$$

If we had permitted p to be even, then q would necessarily be odd and we would have $\epsilon^q = 1$; in this event (4.4) would yield

$$ia_{q-1}(1+\epsilon^q) = 2a_{q-1}i = a_q - b_q \neq 0,$$

which cannot be solved with real a_{q-1}, a_q, b_q.

Thus we have found

$$f(\xi) \equiv g(\xi) \equiv 1, \qquad \text{for } q = 1,$$

$$f(\xi) \equiv g(\xi) \equiv 1 + \sum_{n=1}^{q-1} i^n \xi^n \prod_{k=1}^{n} \frac{1+\epsilon^k}{1-\epsilon^k}, \qquad \text{for } q \geqq 2.$$

LEMMA. *Let $q>1$. The roots of $f(\xi)$ are*

$$-i\epsilon,\ -i\epsilon^2,\ \cdots,\ -i\epsilon^{q-1};$$

they are conjugate to and, as a set, identical with

$$i/\epsilon,\ i/\epsilon^2,\ \cdots,\ i/\epsilon^{q-1}.$$

In fact, since by (3.3)

$$(4.6) \qquad (i\xi - 1)f(\xi) = (-i\epsilon\xi - 1)f(\epsilon\xi),$$

it follows that the root of the right-hand side $\xi = i/\epsilon$ must also be a root of $f(\xi)$ since it is distinct from $-i$. But as all coefficients of $f(\xi)$ are real, the conjugate $-i\epsilon$ is also a root of $f(\xi)$. Substitute $\xi = -i\epsilon$ on the right; then either $\epsilon^2 = -1$ or $f(-i\epsilon^2) = 0$. Hence either $q = 2$, or $-i\epsilon^2$ is a root of $f(\xi)$. In the latter case put $\xi = -i\epsilon^2$ in (4.6) and find either $\epsilon^3 = -1$, $q = 3$, or $f(-i\epsilon^3) = 0$. Repeating this procedure we find that the complete set of roots of $f(\xi)$ is given by $\xi = -i\epsilon^n$, $n = 1, \cdots, q-1$. Hence

$$f(\xi) = c\prod_{n=1}^{q-1}(\xi + i\epsilon^n),$$

where

$$c^{-1} = f(0)/c = \prod_{n=1}^{q-1}(i\epsilon^n) = i^{q-1}\epsilon^{q(q-1)/2} = (-1)^{(p-1)(q-1)/2},$$

in view of (2.2). Thus

$$(4.7) \qquad f(\xi) = (-1)^{(p-1)(q-1)/2}\prod_{n=1}^{q-1}(\xi + i\epsilon^n).$$

Upon combining factors which contain conjugate roots of $f(\xi)$, we have

$$f(-i) = (-1)^{(p-1)(q-1)/2}(-i + i\epsilon)(-i + i\epsilon^{q-1}) \cdots .$$

If q is odd, we have $(q-1)/2$ such pairs of factors; if q is even, we have

$(q-2)/2$ such pairs and the factor $-i+i\epsilon^{q/2}$. Now suppose first that both roots entering such a pair (for example, $-i\epsilon$, $-i\epsilon^{q-1}$) have positive real part. Then the argument of the product of such a pair remains the same if both roots are replaced by $+1$; that is, the argument of the product of such a pair of factors is $\pi/2$. If, on the other hand, both roots involved lie on the left of the imaginary axis, the argument of such a product remains the same if both roots are replaced by -1, which gives precisely $-\pi/2$; and no root $i/\epsilon^n = \pm i$, since $0<n<q$.

Accordingly, for odd q,

$$f(-i) = (-1)^{(p-1)(q-1)/2}(-i)^{\alpha(p,q)}e^{-\pi i(q-1)/4}r,$$

where $r>0$ and $\alpha(p, q)$ is the number of roots of $f(\xi)$ with positive real part. It is easily seen that this formula still holds good for even q. To determine r, we note that

$$\begin{aligned} r^2 = \left|\prod_1^{q-1}(1-\epsilon^n)\right|^2 &= \prod_1^{q-1}(1-\epsilon^n)(1-\epsilon^{-n}) \\ &= \left|\frac{d}{dx}\left(\frac{x^{2q}-1}{x+1}\right)\right|_{x=1} = q, \end{aligned}$$

so that

$$f(-i) = (-1)^{(p-1)(q-1)/2}(-i)^{\alpha(p,q)}e^{-\pi i(q-1)/4}q^{1/2}. \tag{4.8}$$

The number of terms with positive real part in the set i/ϵ^j is the same as the number of terms among ϵ^{-j} with negative imaginary parts. Note that

$$e^{\pi ipj/q}\cdot e^{\pi ip(q-j)/q} = -1.$$

Hence, if $e^{\pi ipj/q}$ has a negative imaginary part, so does $e^{\pi ip(q-j)/q}$. Thus, for odd q, α is even and $(-i)^{\alpha(p,q)} = (-1)^{N(p,q)}$, where $N(p, q)$ is defined (for q odd or even) by

(4.9) $N(p, q)$ = number of terms with negative imaginary parts in the set $e^{\pi ipj'/q}$ with j' integral and $1 \leq j' < q/2$.

For even q, setting $j=q/2$ makes $e^{\pi ipj/q}$ equal to i^p, while for $j\neq q/2$ we have the same fact that both $e^{\pi ipj/q}$ and $e^{\pi ip(q-j)/q}$ have negative imaginary parts or both have positive imaginary parts. Accordingly,

$$(-i)^{\alpha(p,q)} = \begin{cases}(-1)^{N(p,q)}, & \text{for } p \equiv 1 \pmod 4,\ q \text{ even},\\ i(-1)^{N(p,q)+1}, & \text{for } p \equiv 3 \pmod 4,\ q \text{ even}.\end{cases}$$

Hence, from (4.8) we can conclude

$$(4.10)\quad f(-i) = \begin{cases} (-1)^{N(p,q)}e^{-\pi i(q-1)/4}q^{1/2}, & \text{for } q \text{ even, } p \equiv 1 \pmod 4, \\ (-1)^{N(p,q)}ie^{-\pi i(q-1)/4}q^{1/2}, & \text{for } q \text{ even, } p \equiv 3 \pmod 4, \\ (-1)^{N(p,q)}e^{-\pi i(q-1)/4}q^{1/2}, & \text{for } q \text{ odd.} \end{cases}$$

Another property of $f(\xi)$ is

$$(4.11)\qquad a_j = a_{q-j-1}(-1)^{(p-1)(q-1)/2}, \qquad j = 1, 2, \cdots, q-1.$$

In fact, the roots of $f(\xi)$ are simple and contain the reciprocal of each of them, whence

$$\xi^{q-1}f(\xi^{-1}) = (-1)^{(p-1)(q-1)/2}f(\xi).$$

5. **Real independence and semi-independence.** The following definitions are variants of the notion of linear independence; they are introduced in order to simplify the formal part of our computations.

DEFINITION. n complex-valued functions $F_1(x), \cdots, F_n(x)$ on a range of the variable x shall be called "really independent" on this range if the relation

$$R\left\{\sum_{\nu=1}^{n} A_\nu F_\nu(x)\right\} = 0$$

with complex coefficients A_ν implies

$$A_1 = A_2 = \cdots = A_n = 0.$$

If and only if the $F_\nu(x)$ are really independent, the $2n$ functions $RF_\nu(x)$ and $IF_\nu(x)$ are linearly independent in the ordinary sense employing real coefficients.

LEMMA. *Let* $F_1(x), \cdots, F_n(x)$ *be really independent. Then a relation with complex coefficients* a_ν, b_ν,

$$(5.1)\qquad R\left\{\sum_{1}^{n} (a_\nu F_\nu(x) + b_\nu \bar{F}_\nu(x))\right\} = 0,$$

implies

$$(5.2)\qquad a_\nu + \bar{b}_\nu = 0, \qquad \nu = 1, \cdots, n.$$

In fact,

$$0 = R\left\{\sum (a_\nu F_\nu(x) + b_\nu \bar{F}_\nu(x))\right\} = R\left\{\sum (a_\nu + \bar{b}_\nu)F_\nu(x)\right\}.$$

This identity shows that (5.1) follows from (5.2), no matter whether the F_ν are really independent or not.

DEFINITION. If $F_1(x), \cdots, F_n(x)$ are complex, $F(x)$ real, and such that

$$RF_1(x), \cdots, RF_n(x), IF_1(x), \cdots, IF_n(x), F(x)$$

are linearly independent, we shall call $F_1(x), \cdots, F_n(x), F(x)$ "really semi-independent."

LEMMA. *A relation for complex coefficients a_ν, b_ν, a,*

$$(5.3) \qquad \begin{aligned} a_\nu + \bar{b}_\nu &= 0, \qquad \nu = 1, \cdots, n, \\ a + \bar{a} &= 0, \end{aligned}$$

implies

$$(5.4) \qquad R\left\{ \sum_{\nu=1}^{n} (a_\nu F_\nu(x) + b_\nu \bar{F}_\nu(x)) + aF(x) \right\} = 0.$$

Conversely, (5.4) *implies* (5.3) *if the $F_\nu(x)$ and $F(x)$ are really semi-independent.*

Notation. In order to simplify the printing, we shall often make use of the following notation. Let γ_1 and γ_2 be two complex quantities. We write

$$\gamma_1 \cong \gamma_2,$$

in order to express that γ_1 and γ_2 are complex conjugate.

6. **Integration of** (3.4) **for** $E(z) \equiv 0$. Writing (3.4) in the form

$$(6.1) \qquad \prod_{j=1}^{q} (d/dz - \sigma_j) W(z) = 0,$$

where

$$(6.2) \qquad \sigma_j = i/\epsilon^j = ie^{\pi ipj/q}, \qquad j = 1, \cdots, q,$$

we obtain

$$W(z) = \sum_{j=1}^{q} c_j e^{\sigma_j z}.$$

Note that

$$(6.3) \qquad \sigma_j \cong \sigma_{q-j}, \qquad j = 1, \cdots, q-1,$$

whence

$$e^{\sigma_j z} \cong e^{\sigma_{q-j} z}$$

for real x. For odd q the $e^{\sigma_j z}$ $(j=q, q-1, \cdots, q-(q-1)/2=(q+1)/2)$ are $(q+1)/2$ really independent functions, for even q the $e^{\sigma_j z}$ $(j=q, q-1, \cdots, q/2+1)$ and $e^{\sigma_{q/2} z}$ are $q/2+1$ really semi-independent functions. Both statements follow from the nonvanishing of the Wronskian of $e^{\sigma_1 z}, \cdots, e^{\sigma_q z}$, which in turn is a consequence of the nonvanishing of the Vandermondian of the q distinct numbers $\sigma_1, \cdots, \sigma_q$. In fact a linear form

$$R\left\{\sum_{1}^{n}(A_\nu e^{\sigma_\nu z}+B_\nu e^{s_\nu z})\right\},$$

where $s_\nu=\bar{\sigma}_\nu$, can be written as

$$(1/2)\cdot\sum_{1}^{n}((A_\nu+\bar{B}_\nu)e^{\sigma_\nu z}+(\bar{A}_\nu+B_\nu)e^{s_\nu z}).$$

The c_j are determined by the boundary conditions (2.1) and (2.3). We find

$$(6.4)\qquad 0=R\{i\,dW/dz-W\}=R\left\{\sum_{1}^{q}(i\sigma_j-1)c_je^{\sigma_j z}\right\},\quad\text{on}\quad y=0,$$

and

$$(6.5)\qquad 0=R\{i\epsilon^{1/2}dW/dz\}=R\left\{\sum_{1}^{q}i\epsilon^{1/2}\sigma_jc_je^{\sigma_j z}\right\},\quad\text{on}\quad z=|z|\,\epsilon^{1/2}.$$

Applying the lemmas about real dependence resp. semi-independence in case q is odd resp. even, we obtain as necessary and sufficient conditions for the validity of (6.4)

$$(6.6)\qquad c_j(i\sigma_j-1)\cong-c_{q-j}(i\sigma_{q-j}-1),\qquad j=1,\cdots,q-1.$$

Note that, for $j=q$, we have $i\sigma_q-1=0$. In order to exploit (6.5), observe that by (6.2)

$$(6.7)\qquad \sigma_{q-j+1}\epsilon^{1/2}\cong\sigma_j\epsilon^{1/2},\qquad j=1,\cdots,q.$$

Hence (6.5) implies

$$(6.8)\qquad i\epsilon^{1/2}\sigma_{q-j+1}c_{q-j+1}\cong-i\epsilon^{1/2}\sigma_jc_j,$$

which reduces to

$$\epsilon\sigma_{q-j+1}c_{q-j+1}=\sigma_{q-j}\bar{c}_j$$

or

$$(6.9)\qquad c_{q-j+1}=\bar{c}_j,\qquad j=1,\cdots,q.$$

The equations (6.6) and (6.9) determine the c_j but for a real factor independent of j. We conclude directly

$$(6.10)\qquad \bar{c}_{q-j} = c_j(i\sigma_j - 1)/(i\sigma_j + 1), \qquad j = 1, \cdots, q-1,$$

or

$$(6.11)\qquad c_{j+1} = c_j(i\sigma_j - 1)/(i\sigma_j + 1), \qquad j = 1, \cdots, q-1.$$

The equations (6.11) are recursion formulae. They do not exhaust the implications of (6.6) and (6.9). Let q be even, and apply (6.10) with $j = q/2$ and (6.2):

$$\bar{c}_{q/2} = c_{q/2}(1 + \epsilon^{-q/2})/(\epsilon^{-q/2} - 1).$$

Now $\epsilon^{-q/2} = e^{\pi i p/2}$ and p is odd. Hence

$$(6.12)\qquad \bar{c}_{q/2} = c_{q/2}\,i^{-p}, \qquad \text{for } q \text{ even}.$$

Let q be odd, and put $j = (q+1)/2$ in (6.9):

$$(6.13)\qquad \bar{c}_{(q+1)/2} = c_{(q+1)/2}, \quad \text{or} \quad c_{(q+1)/2} \text{ is real for } q \text{ odd}.$$

Formulae (6.12) resp. (6.13) together with (6.11) show that the c_j are determined but for a constant real factor.

To compute the c_j, we observe that they are related to the coefficients a_{j-1} of $f(\xi)$ in a simple manner. In fact, we had (4.5),

$$a_j = a_{j-1}i(1 + \epsilon^j)/(1 - \epsilon^j),$$

and (6.11) which can be written

$$c_{j+1} = c_j(1 + \epsilon^j)/(1 - \epsilon^j), \qquad j = 1, \cdots, q-1.$$

Hence

$$c_{j+1}/a_j = -\,ic_j/a_{j-1}, \qquad j = 1, \cdots, q-1,$$

and

$$(6.14)\qquad c_j = a_{j-1}(-\,i)^j d, \qquad j = 1, \cdots, q,$$

where d does not depend on j. The absolute value of d is irrelevant, but its argument is of importance. Take q even and apply (6.12); we find

$$c_{q/2} = e^{\pi i p/4} r,$$

with a real factor r. Substitute this value of $c_{q/2}$ in the formula (6.14) for $j = q/2$ and remember that all the coefficients a_j are real. We find

$$(6.15)\qquad d = e^{\pi i(p+q)/4}, \qquad \text{for } q \text{ even}.$$

Next take q odd. Then, by (6.13), (6.14),

$$(-i)^{(q+1)/2}d = \text{real},$$

and

$$d = i^{(q+1)/2}, \qquad \text{for } q \text{ odd.} \tag{6.16}$$

From (6.14), (4.1), (4.10) we compute

$$\sum_1^q c_j = -if(-i)d,$$

$$\sum_1^q c_j = \begin{cases} (-1)^{N(p,q)}q^{1/2}, & \text{for } q \text{ odd},\\ (-1)^{N(p,q)+(p-1)/4}q^{1/2}, & \text{for } q \text{ even, } p \equiv 1 \pmod 4,\\ (-1)^{N(p,q)+(p+1)/4}q^{1/2}, & \text{for } q \text{ even, } p \equiv 3 \pmod 4. \end{cases} \tag{6.17}$$

Furthermore, by (6.14) and (4.11),

$$c_q = \begin{cases} (-i)^{(q-1)/2}, & \text{for } q \text{ odd},\\ ie^{-\pi i(p+q)/4}, & \text{for } q \text{ even.} \end{cases} \tag{6.18}$$

Accordingly, our solution $W(z)$, designated henceforth by $W_0(z)$, is

$$W_0(z) = \begin{cases} (-i)^{(q-1)/2}e^{-iz} + i^{(q+1)/2}\sum_{j=1}^{q-1}(-i)^j a_j \exp\left[ize^{i\pi jp/q}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q \text{ odd},\\ ie^{-\pi i(p+q)/4}e^{-iz} + e^{\pi i(p+q)/4}\sum_{j=1}^{q-1}(-i)^j a_j \exp\left[ize^{i\pi jp/q}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q \text{ even}, \end{cases} \tag{6.19}$$

with

$$a_j = \prod_{n=1}^{j} \cot(np\pi/2q), \qquad a_0 = 1.$$

In order to see that the coefficients c_j actually satisfy (6.6) and (6.9), we notice that these relations contain at most $2q-1$ linearly independent real equations, while on the other hand their consequences (6.11) and (6.12) resp. (6.13) are precisely $2q-1$ linearly independent relations. Thus the number of linearly independent equations (6.6) and (6.9) is $2q-1$, and they follow from (6.11) and (6.12) resp. (6.13).

7. **Behavior of $W_0(z)$ for $z\to\infty$.** For the applicability of (6.19) to the wave problem the behavior of $W_0(z)$ at infinity is of importance.

Now the moduli of the terms under the sum in (6.19) tend to zero or infinity as $z=x\rightarrow\infty$, according to whether $R\{ie^{i\pi pj/q}\}<0$ or >0. Furthermore, it is readily seen that the modulus of the sum cannot remain finite unless all terms tend to zero. Now let $1<p<2q$ and consider the angles

$$p\pi/q,\ 2p\pi/q,\ \cdots,\ (q-1)p\pi/q, \tag{7.1}$$

and their conventional representation in the unit circle. Among them there is at least one whose second side lies in the third or fourth quadrant. This is evident if $p>q$, because $p\pi/q$ has this property. For $p<q$ we remember that there are at most $(q-1)$ positions available in the first and second quadrants, namely those of

$$\pi/q,\ 2\pi/q,\ \cdots,\ (q-1)\pi/q.$$

If our contention were wrong, they would have to be filled each by a member of the set (7.1), since no two of them can occupy the same position by virtue of the fact that p and q are relatively prime. In particular there would be one, say $kp\pi/q$, which occupies the same position as $(q-1)\pi/q$. Hence $kp=q-1+nq$ with even n. Here $k\neq q-1$ since otherwise $p-1$ becomes divisible by q in contradiction to $p<q$, $q>1$. But now $(k+1)p\pi/q$ is among the angles (7.1) and its second side is identical with that of

$$(q-1+p)\pi/q=(1+(p-1)/q)\pi,$$

which lies in the third or fourth quadrant.

From the foregoing we gather that, for $1<p<2q$, there exists a j with $1\leqq j\leqq q-1$ for which $R\{ie^{i\pi jp/q}\}>0$. Accordingly $W_0(x)$ does not remain finite as $x\rightarrow\infty$ in contradiction of the condition 5′ of §2.

For $p=1$, however, $R\{ie^{i\pi jp/q}\}<0$ for $j=1, 2, \cdots, q-1$, and $W_0(x)$ remains finite as $x\rightarrow\infty$. Hence $RW_0(z)$ represents a standing wave for $p=1$, it is free of singularities and behaves for $z\rightarrow\infty$ like a simple standing wave in water of infinite depth. More precisely,

$$RW_0(z)\sim R\{e^{-\pi i(q-1)/4}e^{-iz}\}, \qquad \text{for } q \text{ odd or even,}$$

and the complex factor $e^{-\pi i(q-1)/4}$ is decisive in the location of the nodes of this simple wave. These nodes lie at $x=(-(q-1)/4+n)\pi$ with integral n. Accordingly, an increase of one unit in q makes the nodes of the simple wave move one-eighth of a wave length toward shore, if only angles β of the form $\pi/2q$ are considered.

The standing wave $RW_0(z)$ was given by E. T. Hanson.[3] As we have proved, the angles β of the form $\pi/2q$ between surface and floor of the ocean are the only ones for which it will satisfy the condition of finiteness at infinity.

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[3] Loc. cit.

For the construction of a progressive wave, Hanson's standing wave furnishes only one component for $p=1$. For this construction requires the exhibition of a second standing wave of the same period which at large distance from the shore becomes a simple wave whose nodes lie midway between those of the first component.

Plan of the following investigation. The assumption $E(z)=\text{const.}$ underlying the preceding section is a special case of (3.4). More generally we can take

$$E(z) = iz^{2kq/p}, \qquad k = 0, \pm 1, \pm 2, \cdots .$$

All these choices may be expected to lead to a solution $W(z)$ of (2.1) and (2.3), that is, to a $W(z)$ satisfying the boundary conditions. But, generally speaking, these $W(z)$ do not satisfy the condition of finiteness at infinity, and some have too large a singularity at the origin. Our plan is to obtain suitable superpositions of the various solutions which insure finiteness at infinity, indeed two independent such superpositions, so as to enable us to prescribe the location of the nodes of the simple standing wave to which they become asymptotically equal. It is remarkable that this procedure proves successful, in that the number of conditions to be satisfied is larger than the number of available constants. Consequently, it becomes necessary to investigate in detail the asymptotic behavior of the solutions to be constructed, and the algebraic character of the conditions which enforce finiteness at infinity.

The result will be that for all angles $\beta=\pi p/2q$, $1\leqq p<2q$, p odd, there exist two standing waves, ϕ_1 and ϕ_2, one which remains finite at the shore (and whose derivatives of first order (the velocity components) remain finite), the other having a logarithmic singularity at $z=0$ (resp. having a velocity which becomes infinite like $1/|z|$).

8. Properties of exponential integrals. Unfortunately, the construction of waves runs in a similar, but not identical, pattern when $p\equiv 1 \pmod 4$ and when $p\equiv 3 \pmod 4$. This necessitates two parallel sets of formulae for certain parts of the following investigation.

We begin with $p\equiv 3 \pmod 4$. Let $\alpha\geqq 0$ and consider the integral

$$\int_{\alpha}^{e^{i}\infty} e^{-t}t^{\lambda}dt, \tag{8.1}$$

where

$$\lambda = (2kq/p) - 1, \qquad k = 0, \pm 1, \pm 2, \cdots \tag{8.2}$$

The integral is a multi-valued function of s, but a single-valued function of $\log s$, once $\log \sigma_j$ and the path of integration on the Riemann surface of $\log s$ have been fixed. We shall take $\log s$ real for $s=x>0$ (on the surface) and set, generally,

$$\zeta = \log s \tag{8.3}$$

and

$$\tau_j = \log \sigma_j \tag{8.4}$$

with

$$\tau_j = -p\pi i/2 + jp\pi i/q, \qquad j = 1, \cdots, q. \tag{8.5}$$

This choice of the τ_j has been made with a view to establishing for the τ_j the same relations of conjugation that hold for the σ_j. In fact, the analogues of (6.3) and (6.7) are

$$\tau_{q-j} = p\pi i/2 - jp\pi i/q = \bar{\tau}_j, \qquad 1 \leqq j < q, \tag{8.6}$$

and

$$\tau_j - p\pi i/(2q) \cong p\pi i/2 - p(j-1/2)\pi i/q = \tau_{q-j+1} - p\pi i/(2q), \qquad 1 \leqq j \leqq q. \tag{8.7}$$

Accordingly, let the path of integration start at that point α which belongs to a real value of $\log \alpha$ and end at that point $\sigma_j s$ whose log is $\zeta+\tau_j$, and define

$$L_{\lambda,\sigma_j}(\zeta) = e^{\sigma_j s}\int_{\alpha}^{\sigma_j s} e^{-t}t^{\lambda}dt. \tag{8.8}$$

For real $\log s$, that is, for $s=x>0$ (on the surface), this function satisfies, because of (8.6) and (8.7), (8.3) and (8.4),

$$L_{\lambda,\sigma_j}(\zeta) \cong L_{\lambda,\sigma_{q-j}}(\zeta). \tag{8.9}$$

For $\zeta = \log s = \text{real} - ip\pi/2q$, that is, for $s = r\epsilon^{1/2}$, $r>0$ (on the bottom),

$$L_{\lambda,\sigma_j}(\zeta) \cong L_{\lambda,\sigma_{q-j+1}}(\zeta). \tag{8.10}$$

We find the differential equation

$$(d/ds)L_{\lambda,\sigma_j}(\zeta) - \sigma_j L_{\lambda,\sigma_j}(\zeta) = \sigma_j^{\lambda+1}s^{\lambda} = s^{-1}e^{(\zeta+\tau_j)2hq/p}, \tag{8.11}$$

valid for all ζ. Thus for real ζ, $s=x>0$,

$$(d/ds)L_{\lambda,\sigma_j}(\zeta) = \sigma_j L_{\lambda,\sigma_j}(\zeta) + x^{\lambda}(-1)^{hq}, \tag{8.12}$$

and for $\zeta = \text{real} - ip\pi/2q$, $s = r\epsilon^{1/2}$,

$$(8.13)\qquad (d/dz)L_{\lambda,\sigma_j}(\zeta) = \sigma_j L_{\lambda,\sigma_j}(\zeta) + r^\lambda(-1)^{k(q+1)}\epsilon^{-1/2}.$$

Next take $p = 1 \pmod 4$. Set

$$(8.4')\qquad \tau_j = \log(-\sigma_j),$$

where, again, (8.5) is assumed:

$$\tau_j = -p\pi i/2 + jp\pi i/q, \qquad j = 1, \cdots, q.$$

Then, we have the same relations of conjugations as before, namely (8.6) and (8.7). The definition paralleling (8.8) becomes

$$(8.8')\qquad L_{\lambda,\sigma_j}(\zeta) = e^{\sigma_j z}\int_\alpha^{-\sigma_j z} e^t t^\lambda dt.$$

It introduces a single-valued function of ζ for $\alpha \geqq 0$, $\log \alpha$ real, and $\lambda = (2kq/p) - 1$ as above. We conclude the same relations (8.9) and (8.10) of conjugation for the new definition (8.8′). Furthermore, the differential equation corresponding to (8.11) becomes

$$(8.11')\qquad (d/dz)L_{\lambda,\sigma_j}(\zeta) = \sigma_j L_{\lambda,\sigma_j}(\zeta) + (-\sigma_j)^{\lambda+1}z^\lambda,$$

and its special cases corresponding to surface and bottom are again the equations (8.12) and (8.13).

Now consider, no matter what p, the function $W_\lambda(z)$ defined by

$$(8.14)\qquad W_\lambda(z) = \sum_{j=1}^{q} c_j L_{\lambda,\sigma_j}(\zeta).$$

We gather from (8.12) that on the surface where $z = x > 0$

$$i(d/dz)W_\lambda(z) - W_\lambda(z) = \sum_1^q c_j(i\sigma_j - 1)L_{\lambda,\sigma_j}(\zeta) + x^\lambda(-1)^{kq} i\sum_1^q c_j,$$

and on the bottom where $z = r\epsilon^{1/2}$, $r > 0$,

$$i\epsilon^{1/2}(d/dz)W_\lambda(z) = \sum_{j=1}^{q} i\epsilon^{1/2}c_j\sigma_j L_{\lambda,\sigma_j}(\zeta) + r^\lambda(-1)^{k(q+1)} i\sum_1^q c_j.$$

Now, since $\sum_1^q c_j$ is real by (6.17), we have on the surface

$$R\{i(d/dz)W_\lambda(z) - W_\lambda(z)\} = R\sum_1^q c_j(i\sigma_j - 1)L_{\lambda,\sigma_j}(\zeta),$$

and on the bottom

$$R\{i\epsilon^{1/2}(d/dz)W_\lambda(z)\} = R\sum_1^q i\epsilon^{1/2}c_j\sigma_j L_{\lambda,\sigma_j}(\zeta).$$

But then the equations (6.6) and (6.8) are sufficient, because of the conjugation relations (8.9) and (8.10), to insure

$$
(8.15)\qquad \begin{aligned} R\left\{\left(i\frac{d}{dz}-1\right)W_\lambda(z)\right\} &= 0 \qquad \text{on the surface,}\\ R\left\{ie^{1/2}(d/dz)W_\lambda(z)\right\} &= 0 \qquad \text{on the bottom.}\end{aligned}
$$

This shows that $RW_\lambda(z)$ is a harmonic function satisfying the boundary conditions on surface and bottom.

It is of interest that the functions $L_{\lambda,\sigma_j}(\zeta)$ are linearly independent as functions of z for $j=1, \cdots, q$; this implies their real independence or semi-independence (see §5) on the surface as well as on the bottom. But since this fact is of no influence on the explicit formulae which we shall give for the waves, we suppress a proof thereof.

9. **Construction of a second standing wave for $p=1$.** We are now in a position to complete the construction of the progressive wave for $p=1$ by establishing a second standing wave which behaves differently at infinity from Hanson's standing wave given by (6.19).

This second standing wave is

$$
(9.1)\qquad RW_{-1}(z) = R\sum_{j=1}^{q} c_j L_{-1,\sigma_j}(\zeta).
$$

To show this we study the behavior of $W_{-1}(z)$ for large $|z|$ in S_β.

Notice that a change of the value of the positive lower limit α in the definition (8.8′) of the integral affects $W_{-1}(z)$ merely by adding a real constant times $W_0(z)$ to it. For simplicity's sake take $\alpha=1$. We have

$$
L_{-1,\sigma_j}(\zeta) = e^{\sigma_j z}\int_1^{-\sigma_j z} e^t t^{-1}dt = \int_1^{-\sigma_j z} e^{t+\sigma_j z}t^{-1}dt,
$$

with the values (8.5)

$$
\log - \sigma_j = \tau_j = -\pi i/2 + j\pi i/q, \qquad j = 1, \cdots, q.
$$

Hence the arguments of $-\sigma_j z$ lie between $-\pi i/2+\pi i/2q$ and $\pi i/2$, as z varies in the sector S_β. Thus the variable of integration may be restricted to the half-plane $R(t)\geqq 0$.

With this in mind, let $\gamma=\gamma_1+\gamma_2 i$, $\gamma_1\geqq 0$, $|\gamma|\geqq 1$. There exists an $M>0$ such that

$$
(9.2)\qquad \left|\int_1^{\gamma} e^{t-\gamma}t^{-1}dt\right| \leqq M.
$$

For we have

$$
\begin{aligned}
(9.3)\qquad \int_1^{\gamma} e^{t-\gamma}t^{-1}dt &= e^{t-\gamma}t^{-1}\Big|_1^{\gamma} + \int_1^{\gamma} e^{t-\gamma}t^{-2}dt \\
&= \gamma^{-1} - e^{1-\gamma} + \int_1^{\gamma} e^{t-\gamma}t^{-2}dt
\end{aligned}
$$

and

$$\left|\int_1^{\gamma} e^{t-\gamma}t^{-2}dt\right| \leqq e\int_1^{\gamma} \left| t\right|^{-2} \left| dt\right| \leqq e(1 - \left|\gamma\right|^{-1} + \pi/2).$$

This latter estimate is obtained by making the path of integration proceed first on the unit circle and then on a ray through the origin and γ. Consequently, (9.2) holds with any M for which

$$M \geqq \left|\gamma\right|^{-1} + e^{1-\gamma_1} + e(1+\pi/2),$$

for example, for $M = 1 + e(2+\pi/2)$.

Hence $W_{-1}(z)$ remains bounded in the sector S_β as long as $|z| \geqq 1$. Moreover, as $z = x \to \infty$, $L_{-1,\sigma_j}(\zeta) \to 0$ for $j = 1, \cdots, q-1$; for from (9.3) with $-\sigma_j x = \gamma_1 + \gamma_2 i$ we gather $\gamma_1 > 0$ and

$$
\begin{aligned}
\left|\int_1^{\gamma} e^{t-\gamma}t^{-1}dt\right| &\leqq \left|\gamma^{-1} - e^{1-\gamma} + \left(\int_1^{\gamma_1/2} + \int_{\gamma_1/2}^{\gamma}\right) e^{t-\gamma}t^{-2}dt\right| \\
&\leqq \left|\gamma\right|^{-1} + e^{1-\gamma_1} + e^{-\gamma_1/2}(1 - 2\gamma_1^{-1}) \\
&\quad + \left|\gamma_1^{-1} - 2\gamma_1^{-1}\right|.
\end{aligned}
$$

The terms of the last expression tend individually to zero as $x \to \infty$. On the other hand, consider

$$L_{-1,\sigma_q}(\zeta) = \int_1^{ix} e^{t-ix}t^{-1}dt,$$

as $x \to \infty$. By a deformation of the path of integration in the left half-plane we obtain in a familiar way

$$(9.4)\qquad \int_1^{i\infty} e^{t}t^{-1}dt = \int_1^{\prime\,-\infty} e^{t}t^{-1}dt + \pi i = r + \pi i,$$

where the sign $\int'$ denotes the Cauchy principal value of the *real* integral. Accordingly,

$$W_{-1}(x) - c_q e^{-ix}\int_1^{ix} e^{t}t^{-1}dt \to 0 \quad \text{as} \quad x \to \infty;$$

this leads to

$$W_{-1}(x) - c_q e^{-ix}(r + \pi i) \to 0 \quad \text{as} \quad x \to \infty,$$

since the integral (9.4) converges. Thus $W_{-1}(x) - rW_0(x)$ behaves like $c_q e^{-ix}\pi i$ as $x \to \infty$, while, by (6.18) and (6.19), $W_0(z)$ behaves like $c_q e^{-iz}$. Obviously the nodes of $R\{c_q e^{-ix}\}$ lie midway between those of $R\{c_q e^{-ix}\pi i\}$.

Near the origin we find, since $\sum c_j$ is real, in view of (6.17), that

$$RW_{-1}(z) \sim q^{1/2} \log |z|.$$

10. **Asymptotic behavior of exponential integrals.** We proceed to the study of the asymptotic behavior of the functions $L_{\lambda,\sigma_j}(\zeta)$. For simplicity of notation, we set, for $p \equiv 3 \pmod 4$,

$$\gamma = z\sigma_j, \qquad H_\lambda(\gamma) = e^\gamma \int_\alpha^\gamma e^{-t} t^\lambda dt$$

where now k is supposed not to be divisible by p and

$$\lambda = (2kq/p) - 1, \qquad k = 1, 2, \cdots. \tag{10.1}$$

Furthermore $1 < p < 2q$, $\alpha = 0$. Since $\lambda > 0$, integration by parts yields

$$\int_0^\gamma e^{-t} t^\lambda dt = -e^{-t} t^\lambda \Big|_0^\gamma + \lambda \int_0^\gamma e^{-t} t^{\lambda-1} dt,$$

$$\begin{aligned} H_\lambda(\gamma) = &-\gamma^\lambda - \lambda\gamma^{\lambda-1} - \cdots - \lambda(\lambda-1)\cdots(\lambda-s+1)\gamma^{\lambda-s} \\ &+ \lambda(\lambda-1)\cdots(\lambda-s)e^\gamma \int_0^\gamma e^{-t} t^{\lambda-s-1} dt, \end{aligned}$$

a formula which holds for all integral s with $0 < s < \lambda$. In particular, we shall take the value $s = [\lambda]$, defined as the largest integer less than λ. We find

$$\begin{aligned} W_\lambda(z) &= \sum_1^q c_j L_{\lambda,\sigma_j}(\zeta) \\ &= \sum c_j \sigma_j^\lambda z^\lambda (-1 - \lambda(\sigma_j z)^{-1} - \cdots - \lambda \cdots (\lambda - s + 1)(\sigma_j z)^{-s}) \\ &\quad + \lambda(\lambda-1)\cdots(\lambda-s) \sum c_j e^{\sigma_j z} \int_0^{\sigma_j z} e^{-t} t^{\lambda-s-1} dt. \end{aligned} \tag{10.2}$$

Now

$$\sum c_j \sigma_j^\lambda = \sum c_j e^{\pi i j \lambda} = \sum c_j e^{\pi i j 2kq/p} \sigma_j^{-1},$$

and by (6.14) and (8.5) this becomes

$$\sum c_j\sigma_j^{\lambda} = d\sum_1^q a_{j-1}(-i)^j e^{(-p/2+jp/q)\pi i\cdot 2kq/p}\sigma_j^{-1}$$
$$= d\sum a_{j-1}(-i)^{j-1}\epsilon^j(-1)^{kq+1} = f(-i\epsilon)d\epsilon(-1)^{kq+1} = 0,$$

for by the lemma of §4 the roots of $f(\xi)$ are $-i\epsilon, \cdots, -i\epsilon^{q-1}$.

Similarly $\sum_1^q c_j\sigma_j^{\lambda-1} = f(-i\epsilon^2)\cdot\text{const.} = 0$, where the constant denotes a number which does not depend on the index j of summation. Proceeding in a like manner, we get

$$\sum_1^q c_j\sigma_j^{\lambda-s} = f(-i\epsilon^{s+1})\cdot\text{const.} = 0 \tag{10.3}$$

for all $s > 0$ with $s \leqq q-2$. This is compatible with $s = [\lambda] = [2kq/p-1]$ for all k with

$$1 \leqq k \leqq (p-1)/2. \tag{10.4}$$

In fact, then

$$[2kq/p] \leqq [(p-1)q/p] \leqq [q - q/p] \leqq q - 1,$$
$$[\lambda] = [2kq/p - 1] \leqq q - 2.$$

For k satisfying (10.4) we conclude from (10.2)

$$\begin{aligned} W_\lambda(z) &= W_{-1+2kq/p}(z) \\ &= \Gamma(2kq/p)(\Gamma(2kq/p - [2kq/p]))^{-1}\sum_1^q c_jL_{-1+2kq/p-[2kq/p]}(\zeta). \end{aligned} \tag{10.5}$$

Set

$$W_k^*(z) = (\Gamma(2kq/p))^{-1}W_\lambda(z). \tag{10.6}$$

By (8.15), W_k^* satisfies the boundary conditions on surface and bottom.

In order to study the behavior of $W_k^*(z)$ for $z = x \to \infty$ and for $z = r\epsilon^{1/2}$, $r \to \infty$, consider one of the integrals on the right of (10.5) (it will suffice to consider only the first of the following alternatives in case $p < q$):

$$e^{\sigma_j z}\int_0^{\sigma_j z} e^{-t}t^{\lambda-[\lambda]-1}dt \text{ resp. } e^{\sigma_j r\epsilon^{1/2}}\int_0^{\sigma_j r\epsilon^{1/2}} e^{-t}t^{\lambda-[\lambda]-1}dt.$$

Suppose first that j is such that

$$\pi/2 \geqq -i\tau_j \geqq -\pi/2 \text{ resp. } \pi/2 \geqq -i\tau_j - p\pi/2q \geqq -\pi/2. \tag{10.7}$$

Then

$$\lim_{x\to\infty}\int_0^{\sigma_j x} e^{-t}t^{\lambda-[\lambda]-1}dt = \Gamma(\lambda-[\lambda])$$

resp.

$$\lim_{r\to\infty}\int_0^{\sigma_j r e^{1/n}} e^{-t}t^{\lambda-[\lambda]-1}dt = \Gamma(\lambda-[\lambda]),$$

whence

$$e^{\sigma_j x}\int_0^{\sigma_j x} e^{-t}t^{\lambda-[\lambda]-1}dt = \Gamma(\lambda-[\lambda])e^{\sigma_j x} - e^{\sigma_j x}\int_{\sigma_j x}^{\infty} e^{-t}t^{\lambda-[\lambda]-1}dt$$

resp.

$$e^{\sigma_j r e^{1/n}}\int_0^{\sigma_j r e^{1/n}} e^{-t}t^{\lambda-[\lambda]-1}dt$$

$$= \Gamma(\lambda-[\lambda])e^{\sigma_j r e^{1/n}} - e^{\sigma_j r e^{1/n}}\int_{\sigma_j r e^{1/n}}^{\infty} e^{-t}t^{\lambda-[\lambda]-1}dt.$$

Here

$$\left| e^{\sigma_j x}\int_{\sigma_j x}^{\infty}\right| = \left|\int_0^{\infty} e^{-u}\,|u+\sigma_j x|^{\lambda-[\lambda]-1}du\right|$$

$$\leqq |\sigma_j x|^{\lambda-[\lambda]-1}\int_0^{\infty} e^{-u}du \to 0,$$

resp.

$$\left| e^{\sigma_j r e^{1/n}}\int_{\sigma_j r e^{1/n}}^{\infty}\right| \to 0,$$

since $\lambda-[\lambda]-1<0$. Thus

$$e^{\sigma_j x}\left(\int_0^{\sigma_j x} e^{-t}t^{\lambda-[\lambda]-1}dt - \Gamma(\lambda-[\lambda])\right) \underset{x\to\infty}{\longrightarrow} 0, \text{ resp.}$$

(10.8)

$$e^{\sigma_j r e^{1/n}}\left(\int_0^{\sigma_j r e^{1/n}} e^{-t}t^{\lambda-[\lambda]-1} - \Gamma(\lambda-[\lambda])\right) \underset{r\to\infty}{\longrightarrow} 0,$$

if (10.7) holds.

Observe that $\int_0^{\gamma} e^{-t}t^{\lambda-[\lambda]-1}dt$ is multiplied by $e^{2\pi i\lambda}$ as $\log\gamma$ is increased by $2\pi i$. Therefore we can generalize the previous result. Suppose n_j defined equal to n, provided there exists an integral number n such that

$$(10.9)\quad \begin{array}{ll} \text{either} & n2\pi + \pi/2 \geqq -i\tau_j \geqq n2\pi - \pi/2 \\ \text{or} & n2\pi + \pi/2 \geqq -i\tau_j - p\pi/(2q) \geqq n2\pi - \pi/2. \end{array}$$

(Notice that $p<2q$ so that these two conditions never lead to different values of n_j.) Then

$$(10.10)\quad \begin{array}{l} \lim\limits_{x\to\infty} \left\{e^{\sigma_j x}\left(\int_0^{\sigma_j x} e^{-t}t^{\lambda-[\lambda]-1}dt - e^{4\pi i n_j k q/p}\Gamma(\lambda - [\lambda])\right)\right\} = 0 \text{ resp.} \\ \lim\limits_{r\to\infty} \left\{e^{\sigma_j r e^{1/\pi}}\left(\int_0^{\sigma_j r e^{1/\pi}} e^{-t}t^{\lambda-[\lambda]-1}dt - e^{4\pi i n_j k q/p}\Gamma(\lambda - [\lambda])\right)\right\} = 0. \end{array}$$

On the other hand, if one or both of the conditions (10.9) cannot be satisfied with integral n, but instead we have with some n

$$(10.11)\quad \begin{array}{l} n2\pi + \pi/2 < -i\tau_j < n2\pi + 3\pi/2 \quad \text{resp.} \\ n2\pi + \pi/2 < -i\tau_j - p\pi/2q < n2\pi + 3\pi/2, \end{array}$$

then we must have, for example,

$$\left|e^{\sigma_j x}\int_0^{\sigma_j x}\right| = \left|\int_0^{-\sigma_j x} e^{-u}(u + \sigma_j x)^{\lambda-[\lambda]-1}du\right| \leqq \left|\int_0^{\infty}\right| + \left|\int_{-\sigma_j x}^{\infty}\right|$$
$$\leqq |x|^{\lambda-[\lambda]-1} + |e^{\sigma_j x}|\,\Gamma(\lambda - [\lambda]);$$

hence

$$(10.11.1)\quad \begin{array}{l} \lim\limits_{x\to\infty} e^{\sigma_j x}\int_0^{\sigma_j x} e^{-t}t^{\lambda-[\lambda]-1}dt = 0, \text{ resp.} \\ \lim\limits_{r\to\infty} e^{\sigma_j r e^{1/\pi}}\int_0^{\sigma_j r e^{1/\pi}} e^{-t}t^{\lambda-[\lambda]-1}dt = 0. \end{array}$$

These estimates show that

$$(10.12)\quad \begin{array}{l} \lim\limits_{x\to\infty} \left\{W_k^*(x) - \sum\limits_{j=1}^{q} c_j\delta_{jk}e^{\sigma_j x}\right\} = 0, \\ \lim\limits_{r\to\infty} \left\{W_k^*(re^{1/\pi}) - \sum\limits_{1}^{q} c_j\delta_{jk}e^{\sigma_j r e^{1/\pi}}\right\} = 0, \end{array}$$

where δ_{jk} stands for

$$(10.13)\quad \delta_{jk} = \begin{cases} e^{4\pi i n_j k q/p} & \text{if (10.9) holds,} \\ 0 & \text{if (10.11) holds} \end{cases}$$

(the value of n_j in the second case is immaterial for the validity of

(10.12) because of the vanishing of the exponential factor). The possible values of n_j in (10.9) are

$$n_j = (p+1)/4;\ (p-3)/4, \cdots, 0, -1, \cdots, -(p-3)/4. \tag{10.14}$$

The first of these occurs only for $j=q$, that is, $n_q = (p+1)/4$. All the other possible values of n_j must be successive integers since two successive terms $-i\tau_j - p\pi/2q$, $-i\tau_j$ or $-i\tau_j$, $-i\tau_{j+1} - p\pi/2q$ differ by $p\pi/2q < \pi$. On the other hand, the smallest value of n_j occurs either in $n_j 2\pi - \pi/2 \leqq -i\tau_j \leqq n_j 2\pi + \pi/2$ or in $n_j 2\pi - \pi/2 \leqq -i\tau_j - p\pi/2q \leqq n_j 2\pi + \pi/2$ when this n_j is the smallest integer following that one, n', which would occur if j could become equal to zero; this would give

$$2n'\pi + \pi/2 = -p\pi/2, \qquad n' = -(p+1)/4.$$

The alternative in the relations (10.9) to (10.14) is dictated by the possibility that $p > q$. Had we demanded that $p < q$, then the Phragmén-Lindelöf principle would have permitted us to fix our attention solely on the surface, as was explained in §2.

Next we give the parallel developments for $p = 1$ (mod 4). Integrate by parts for $\lambda > 0$:

$$\int_0^{-\gamma} e^t t^\lambda dt = e^t t^\lambda \Big|_0^{-\gamma} - \lambda \int_0^{-\gamma} e^t t^{\lambda-1} dt = \cdots,$$

$$\begin{aligned} e^\gamma \int_0^{-\gamma} e^t t^\lambda dt = {} & (-\gamma)^\lambda - \lambda(-\gamma)^{\lambda-1} + \cdots \\ & + (-1)^s(-\gamma)^{\lambda-s}\lambda(\lambda-1)\cdots(\lambda-s+1) \\ & + (-1)^{s+1} e^\gamma \int_0^{-\gamma} e^t t^{\lambda-s-1} dt\, \lambda(\lambda-1)\cdots(\lambda-s), \end{aligned}$$

where $s = [\lambda]$. Then, by (8.8′),

$$\begin{aligned} W_\lambda(z) = {} & \sum_{j=1}^{q} c_j L_{\lambda,\sigma_j}(\zeta) \\ = {} & \sum_1^q \Big[c_j(-\sigma_j z)^\lambda (1 + \lambda(\sigma_j z)^{-1} + \cdots \\ & + \lambda \cdots (\lambda - s + 1)(\sigma_j z)^{-s}) \\ & + c_j(-1)^{s+1} e^{\sigma_j z} \int_0^{-\sigma_j z} e^t t^{\lambda-s-1} dt\, \lambda \cdots (\lambda - s) \Big]. \end{aligned} \tag{10.2′}$$

Now

$$\sum_1^q c_j(-\sigma_j)^\lambda = \sum_1^q c_j e^{\tau_j\lambda} = d\sum_1^q a_{j-1}(-i)^j e^{(-p/2+jp/q)2kq\pi i/p}(-\sigma_j)^{-1}$$

$$= \text{const.} \sum a_{j-1}(-i\epsilon)^{j-1} = \text{const.}\ f(-i\epsilon) = 0,$$

. . . . ,

$$\sum_1^q c_j(-\sigma_j)^{\lambda-s} = \text{const.} \sum_1^q a_{j-1}(-i)^{j-1}\epsilon^{(j-1)(s+1)}$$

$$= \text{const.}\ f(-i\epsilon^{s+1}) = 0,$$

since (10.3) and (10.4) apply again. Hence, for $k=1, 2, \cdots, (p-1)/2$, $\lambda=(2kq/p)-1$,

$$W_k^*(z) = (\Gamma(\lambda+1))^{-1}W_\lambda(z)$$

$$= (-1)^{[2kq/p]}(\Gamma(\lambda-[\lambda]))^{-1}\sum_{j=1}^q c_j L_{\lambda-[\lambda]-1,\sigma_j}(\zeta) \tag{10.6'}$$

satisfies the boundary conditions on surface and bottom.

For large x, resp. large r, consider

$$e^{\sigma_j x}\int_0^{-\sigma_j x} e^t t^{\lambda-[\lambda]-1}dt \quad \text{resp.} \quad e^{\sigma_j r\epsilon^{1/2}}\int_0^{-\sigma_j r\epsilon^{1/2}} e^t t^{\lambda-[\lambda]-1}dt.$$

For such j as make

$$-\pi/2 < -i\tau_j < \pi/2 \quad \text{resp.} \quad -\pi/2 < -i\tau_j - p\pi/2q < \pi/2,$$

we have $R(-\sigma_j)>0$, and, with $\gamma=-\sigma_j x$,

$$e^{-\gamma}\int_0^\gamma e^t t^{\lambda-[\lambda]-1}dt = \int_0^\gamma e^{t-\gamma}t^{\lambda-[\lambda]-1}dt \to 0, \tag{10.11.1'}$$

as $x\to\infty$; or resp. (10.11.1′) holds with $\gamma=-\sigma_j r\epsilon^{1/2}$ and $R(-\sigma_j\epsilon^{1/2})>0$ as $r\to\infty$. Similarly, (10.11.1′) still holds if there is an integral n for which

$$\begin{aligned} &-\pi/2 + 2n\pi < -i\tau_j < \pi/2 + 2n\pi \quad \text{resp.} \\ &-\pi/2 + 2n\pi < -i\tau_j - p\pi/2q < \pi/2 + 2n\pi. \end{aligned} \tag{10.11'}$$

On the other hand, if there is an integral n such that

$$\begin{aligned} &\pi/2 + 2n\pi \leqq -i\tau_j \leqq 3\pi/2 + 2n\pi \quad \text{resp.} \\ &\pi/2 + 2n\pi \leqq -i\tau_j - p\pi/2q \leqq 3\pi/2 + 2n\pi, \end{aligned} \tag{10.9'}$$

then let n_j be defined by $n_j=n$ in (10.9′). We have $R\gamma\leqq 0$, where

$$\gamma = -\sigma_j x \quad \text{resp.} \quad \gamma = -\sigma_j r\epsilon^{1/2},$$

$$e^{-\gamma}\int_0^{\gamma} e^t t^{\lambda-[\lambda]-1}dt = e^{-\gamma}\left(\int_0^{-\infty}+\int_{-\infty}^{\gamma}\right),$$

$$\int_0^{-\infty} e^t t^{\lambda-[\lambda]-1}dt = -\int_0^{\infty} e^{-|t|}\,|t|^{\lambda-[\lambda]-1}d\,|t|\,e^{\pi i(1+2n_j)(\lambda-[\lambda]-1)}$$
$$= e^{(2n_j+1)\pi i 2kq/p}(-1)^{[2kq/p]}\Gamma(\lambda-[\lambda]),$$

$$\left|e^{-\gamma}\int_{-\infty}^{\gamma} e^t t^{\lambda-[\lambda]-1}dt\right| = \left|\int_{-\infty}^{\gamma} e^{t-\gamma}t^{\lambda-[\lambda]-1}dt\right|$$
$$= \left|\int_{-\infty}^{0} e^{u}(u+\gamma)^{\lambda-[\lambda]-1}du\right|$$
$$\leq |\gamma|^{\lambda-[\lambda]-1} \to 0,$$

as $x\to\infty$ resp. $r\to\infty$.

Hence

$$\text{(10.10')}\qquad \lim_{x\to\infty}\Big[e^{\sigma_j x}\int_0^{-\sigma_j x} e^t t^{\lambda-[\lambda]-1}dt - (-1)^{[2kq/p]}\Gamma(\lambda-[\lambda])e^{\sigma_j x}e^{(2n_j+1)\pi i 2kq/p}\Big] = 0,$$

resp. a similar limit equation for $r\to\infty$, if $x\sigma_j$ is replaced by $r\sigma_j\epsilon^{1/2}$. Accordingly, by (10.6′)

$$\text{(10.12')}\qquad \begin{aligned}&\lim_{x\to\infty}\left\{W_k^*(x) - \sum_1^q c_j\delta_{jk}e^{\sigma_j x}\right\} = 0, \text{ resp.}\\ &\lim_{r\to\infty}\left\{W_k^*(r\epsilon^{1/2}) - \sum_1^q c_j\delta_{jk}e^{\sigma_j r\epsilon^{1/2}}\right\} = 0,\end{aligned}$$

were now

$$\text{(10.13')}\qquad \delta_{jk} = \begin{cases} e^{2\pi i kq(2n_j+1)/p} & \text{if (10.11') holds,}\\ 0 & \text{otherwise.}\end{cases}$$

Here the possible values of n_j range from $(p-1)/4$, occurring for $j=q$ when $-\sigma_q=i$, $\tau_q=p\pi/2$, to $-(p-1)/4$, occurring for the smallest j for which (10.9′) holds. For this value exceeds by 1 the value $-(p+3)/4$ which would occur if $j=0$ were admissible; for then

$$3\pi/2 + 2n'\pi = -p\pi/2, \qquad n' = -(p+3)\pi/4.$$

11. **Extension to the case** $\lambda=-1$, $p\equiv 3 \pmod 4$. If k in (8.2) is set equal to zero, α can no longer be chosen as zero if the integral (8.1) is to converge. We set $\alpha=1$ and study the asymptotic behavior

of $L_{-1,\sigma_j}(\zeta)$ defined by

$$(11.1)\qquad L_{-1,\sigma_j}(\zeta) = \begin{cases} e^{\sigma_j z}\displaystyle\int_1^{\sigma_j z} e^{-t}t^{-1}dt, & \text{for } p \equiv 3 \pmod 4, \\ e^{\sigma_j z}\displaystyle\int_1^{-\sigma_j z} e^{-t}t^{-1}dt, & \text{for } p \equiv 1 \pmod 4. \end{cases}$$

Here again the generalities about the multivalued character of the integral apply. If ζ increases by $2\pi i$, $L_{-1,\sigma_j}(\zeta)$ changes in the following manner:

$$(11.2)\qquad L_{-1,\sigma_j}(\zeta + 2\pi i) = L_{-1,\sigma_j}(\zeta) + 2\pi i e^{\sigma_j z}.$$

Furthermore, there exists the conjugation

$$(11.3)\qquad L_{-1,\sigma_j}(\zeta) \cong L_{-1,s_j}(\bar{\zeta}),$$

where $s_j = \sigma_j$. Let $\gamma \neq 0$, $-\pi/2 \leqq \arg\gamma \leqq \pi/2$, $\log\gamma = \log|\gamma| + i\arg\gamma$. Then

$$e^\gamma \int_1^\gamma e^{-t}t^{-1}dt = e^\gamma\left(\int_1^\infty - \int_\gamma^\infty\right)$$

$$= e^\gamma \int_1^\infty e^{-t}t^{-1}dt - \int_0^\infty e^{-t}(t+\gamma)^{-1}dt,$$

$$(11.4)\qquad \left| e^\gamma \int_1^\gamma e^{-t}t^{-1}dt - e^\gamma \int_1^\infty e^{-t}t^{-1}dt \right| \leqq |\gamma|^{-1},$$

$$(11.5)\qquad e^\gamma \int_1^\gamma e^{-t}t^{-1}dt - e^\gamma \int_1^\infty e^{-t}t^{-1}dt \to 0$$

as $\gamma \to \infty$, $|\arg\gamma| \leqq \pi/2$. But if $\pi/2 + \delta < \arg\gamma < 3\pi/2 - \delta$, $\delta > 0$, consider $e^\gamma\int_1^\gamma e^{-t}t^{-1}dt$ where the integral is extended along a path leading in the upper half-plane from 1 to $\gamma/|\gamma|$ and thence to γ on a straight line. We have, with $\kappa = \gamma/|\gamma| = -\kappa_1 + i\kappa_2$, $\kappa_1 > \sin\delta > 0$,

$$\left| e^\gamma \int_1^\gamma e^{-t}t^{-1}dt \right| \leqq \left| e^\gamma \int_1^\kappa e^{-t}t^{-1}dt \right| + \left| \int_\kappa^\gamma e^{-(t-\gamma)}t^{-1}dt \right|,$$

$$\left| \int_\kappa^\gamma e^{-(t-\gamma)}t^{-1}dt \right| \leqq \left| \int_1^{|\gamma|} e^{-\kappa(t-|\gamma|)}t^{-1}dt \right|$$

$$\leqq \left| \int_1^{|\gamma|/2} \right| + \left| \int_{|\gamma|/2}^{|\gamma|} \right|$$

$$\leqq e^{-\kappa_1|\gamma|/2}\log(|\gamma|/2) + 2(\kappa_1|\gamma|)^{-1}.$$

Hence $e^\gamma\int_1^\gamma e^{-t}t^{-1}dt \to 0$ as $\pi/2 + \delta \leqq \arg\gamma \leqq 3\pi/2 - \delta$ and $|\gamma| \to \infty$. Consequently, for n_j as defined by (10.9) and $\zeta = \log x$ real resp. $\zeta = \log r - ip\pi/2q$, $\log r$ real,

$$L_{-1,\sigma_j}(\zeta) - 2\pi i n_j e^{\sigma_j x} - e^{\sigma_j x}\int_1^\infty e^{-t}t^{-1}dt \to 0, \qquad \text{if } R\sigma_j \geqq 0, \tag{11.6}$$

resp.

$$L_{-1,\sigma_j}(\zeta) - 2\pi i n_j e^{\sigma_j r\epsilon^{1/2}} - e^{\sigma_j r\epsilon^{1/2}}\int_1^\infty e^{-t}t^{-1}dt \to 0, \tag{11.7}$$

$$\text{if } R(\sigma_j\epsilon^{1/2}) \geqq 0,$$

while always

$$\lim_{x\to\infty} L_{-1,\sigma_j}(\zeta) = 0, \qquad \text{if } R\sigma_j < 0,\ \zeta = \log x, \tag{11.8}$$

resp.

$$\lim_{r\to\infty} L_{-1,\sigma_j}(\zeta) = 0, \qquad \text{if } R(\sigma_j\epsilon^{1/2}) < 0,\ \zeta = \log (r\epsilon^{1/2}). \tag{11.9}$$

The formulae (11.6) and (11.7) lead to corresponding relations for

$$W_{-1}(\zeta) = \sum_1^{s} c_j L_{-1,\sigma_j}(\zeta). \tag{11.10}$$

12. **Extension to the case $\lambda = -1$, $p = 1$ (mod 4).** Let $R-\sigma_j > 0$. Then with some integral n and $\gamma = -\sigma_j x$, $R\gamma > 0$,

$$e^{-\gamma}\int_1^\gamma e^t t^{-1}dt = 2\pi i n e^{-\gamma} + \int_1^\gamma e^{t-\gamma}t^{-1}dt,$$

where the last integral can be taken along a straight line and

$$\left|\int_1^\gamma e^{t-\gamma}t^{-1}dt\right| \leqq \left|\int_1^{\gamma/|\gamma|}\right| + \left|\int_{\gamma/|\gamma|}^{\gamma}\right|$$

$$\leqq e^{-R\gamma}\cdot\text{const.} + e^{-R\gamma}\int_1^{R\gamma} e^t t^{-1}dt.$$

Consequently, if γ tends to ∞ on a ray through the origin which lies in $R\gamma > 0$, we have $e^{-\gamma}\int_1^\gamma e^t t^{-1}dt \to 0$. Now suppose n_j defined by (10.9′), that is,

$$\pi/2 + 2n_j\pi \leqq -\tau_j i \leqq 3\pi/2 + 2n_j\pi$$

resp.

$$\pi/2 + 2n_j\pi \leqq -\tau_j i - p\pi i/2q \leqq 3\pi/2 + 2n_j\pi.$$

We have in the first alternative

$$e^{\sigma_j x}\int_1^{-\sigma_j x} e^t t^{-1}dt = e^{\sigma_j x}\int_1^{-\infty} e^t t^{-1}dt + e^{\sigma_j x}\int_{-\infty}^{-\sigma_j x} e^t t^{-1}dt,$$

where

$$\int_1^{-\infty} e^t t^{-1}dt = \int_1^{\prime -\infty} e^t t^{-1}dt + (2n_j + 1)\pi i,$$

with $\int'$ understood as Cauchy's principal value, and where the path of $\int_{-\infty}^{-\sigma_j x}$ remains in the negative half-plane. Now as $x \to \infty$,

$$\left| e^{\sigma_j x}\int_{-\infty}^{-\sigma_j x} e^t t^{-1}dt \right| = \left| \int_{-\infty}^{-\sigma_j x} e^{t+\sigma_j x} t^{-1}dt \right|$$
$$= \left| \int_{-\infty}^{0} e^t (t - \sigma_j x)^{-1}dt \right| \leq x^{-1} \to 0,$$

so that

$$(12.1)\quad e^{\sigma_j x}\left[\int_1^{-\sigma_j x} e^t t^{-1}dt - (2n_j + 1)\pi i - \int_1^{\prime -\infty} e^t t^{-1}dt\right] \to 0,$$
$$\text{for } x \to \infty,\ R\sigma_j \geqq 0.$$

For the second alternative of (10.9′) we have similarly

$$(12.2)\quad e^{\sigma_j r e^{1/2}}\left[\int_1^{-\sigma_j r e^{1/2}} e^t t^{-1}dt - (2n_j + 1)\pi i - \int_1^{\prime -\infty} e^t t^{-1}dt\right] \to 0,$$
$$\text{for } r \to \infty,\ R\sigma_j e^{1/2} \geqq 0.$$

13. Construction of the finite standing wave. It is now possible to exhibit a solution of our boundary problem in the form

$$(13.1)\qquad \Omega_0(z) = W_0(z) - \sum_{k=1}^{(p-1)/2} A_k W_k^*(z);$$

$\Omega_0(z)$ remains bounded as z tends to infinity on the surface and bottom of S_β, and remains bounded as $z \to 0$. Moreover, the derivative $d\Omega_0(z)/dz$ will also remain bounded near $z = 0$ as is obvious from the definition (6.19) of $W_0(z)$ and the definitions (10.6) resp. (10.6′) and (10.2) resp. (10.2′). Our problem is to determine the A_k in such a way as to eliminate by superposition the various infinities introduced for large $|z|$ by each of the summands in (13.1). It suffices if this can be done on the surface for $p<q$, and on the surface and bottom for $q<p<2q$. Let $p\equiv 3 \pmod 4$. Fix the attention first on the surface. The estimates (10.10) and (10.12) lead to the equation

$$\lim_{x\to\infty}\left[\Omega_0(x)-\sum_j{}' c_j e^{\sigma_j x}\left(1-\sum_{k=1}^{(p-1)/2}\delta_{jk}A_k\right)\right]=0, \tag{13.2}$$

where it is necessary to extend the sum $\sum'$ only over those values j for which the first alternative of (10.9) applies. Now suppose that for those values of n, for which the first alternative (10.9) applies, *but with the exception of* $j=q$, we can solve the system of linear equations

$$\sum_{k=1}^{(p-1)/2}\delta_{jk}A_k=1, \qquad j\neq q, \tag{13.3}$$

in the unknowns A_k. Then (13.2) reduces to

$$\lim_{x\to\infty}\left[\Omega_0(x)-c_q e^{-ix}\left(1-\sum_{k=1}^{(p-1)/2}\delta_{qk}A_k\right)\right]=0. \tag{13.4}$$

In order that $\Omega_0(x)$ should behave like a simple wave at infinity, we shall have to show in addition that

$$-1+\sum_{k=1}^{(p-1)/2}\delta_{qk}A_k\neq 0. \tag{13.5}$$

It follows directly that $\Omega_0(x)$ stays bounded also on the bottom if the A_k satisfy the system of equations (13.3).

Now (10.14) shows that (13.3) represents exactly $(p-1)/2$ distinct equations, namely

$$\sum_{k=1}^{(p-1)/2}A_k e^{4\pi ikn q/p}=1, \quad n=-(p-3)/4,\cdots,+(p-3)/4. \tag{13.6}$$

A similar analysis in the case $p\equiv 1 \pmod 4$ shows the corresponding equations to be

$$\sum_{k=1}^{(p-1)/2}A_k e^{2\pi ikq(2n+1)/p}=1, \tag{13.6'}$$

$$n=-(p-1)/4,\cdots,(p-5)/4.$$

We proceed to give the values of the A_k explicitly. There occurs in this deduction a remarkable analogy to the computation of the coefficients a_j of the polynomial $f(\xi)$ introduced in §4, with the role of p and q interchanged. Consider the polynomial of degree $(p-1)/2$,

$$F(\xi) = \sum_{0}^{(p-1)/2} A_k \xi^k, \qquad \text{with} \quad A_0 = -1,$$

which vanishes at the $(p-1)/2$ distinct points

$$\xi = e^{4\pi i n q/p}, \tag{13.7}$$
$$n = -(p-3)/4, \cdots, (p-3)/4, \text{ for } p \equiv 3,$$

resp.

$$\xi = e^{2\pi i q(2n+1)/p}, \tag{13.7'}$$
$$n = -(p-1)/4, \cdots, (p-5)/4, \text{ for } p \equiv 1 \pmod 4.$$

We have

$$(13.8)\qquad \begin{aligned} F(\xi) &= \prod_{|n| \leq (p-3)/4} (\xi - \eta^n), & \eta &= e^{4\pi i q/p}, \text{ for } p \equiv 3 \pmod 4, \\ F(\xi) &= -\prod_{n \geq -(p-1)/4}^{(p-5)/4} (\xi - \eta^{2n+1}), & \eta &= e^{2\pi i q/p}, \text{ for } p \equiv 1 \pmod 4. \end{aligned}$$

Evidently (13.6) resp. (13.6′) hold. To test (13.5) we write for $p \equiv 3 \pmod 4$

$$-1 + \sum_{k=1}^{(p-1)/2} \delta_{qk} A_k = F(e^{4\pi i q(p+1)/(4p)}) = F(e^{(p+1)q\pi i/p}) \neq 0, \tag{13.9}$$

since $e^{(p+1)q\pi i/p}$ coincides with none of the roots (13.7). Similarly for $p \equiv 1 \pmod 4$

$$(13.9')\qquad \begin{aligned} -1 + \sum_{1}^{(p-1)/2} \delta_{qk} A_k &= F(e^{2\pi i (q/p)(2(p-1)/4+1)}) \\ &= F(e^{(p+1)q\pi i/p}) \end{aligned}$$

which does not vanish for a similar reason. It is easy to evaluate the A_k explicitly. We have for $p \equiv 3 \pmod 4$

$$(\xi - \eta^{(p-3)/4}) F(\eta \xi) = \eta^{(p-1)/2} F(\xi)(\xi - \eta^{-(p+1)/4}),$$

whence

$$\sum_{0}^{(p-1)/2} A_k \xi^{k+1} (\eta^k - \eta^{(p-1)/2}) = \sum_{0}^{(p-1)/2} A_k \xi^k (\eta^k - 1) \eta^{(p-3)/4}.$$

But

$$\eta^{p/2} = 1, \qquad \eta^{p/4} = (-1)^q$$

and

$$A_{k-1} = (-1)^q A_k(\eta^{k-3/4} - \eta^{-3/4})/(\eta^{k-1} - \eta^{-1/2})$$

so that

$$\begin{aligned} A_k &= (-1)^q A_{k-1}(\sin (2k-1)q\pi/p)/\sin 2kq\pi/p \\ &= (-1)^{kq+1}\prod_{\nu=1}^{k} \frac{\sin (2\nu-1)q\pi/p}{\sin 2\nu q\pi/p}; \qquad A_0 = -1. \end{aligned} \tag{13.10}$$

Similarly for $p \equiv 1 \pmod 4$

$$(\xi - \eta^{(p-1)/2})F(\eta^2\xi) = \eta^{p-1}(\xi - \eta^{-(p-1)/2})F(\xi),$$
$$\sum A_k\xi^{k+1}(\eta^{2k} - \eta^{p-1}) = \sum A_k\xi^k(\eta^{(p-1)/2}\eta^{2k} - \eta^{(p-1)/2}).$$

But here

$$\eta^p = 1, \quad \eta^{p/2} = (-1)^q, \quad A_k = (-1)^q A_{k-1}\eta^{-1/2}(\eta^{2k-1} - 1)/(\eta^{2k} - 1);$$
$$A_k = (-1)^q A_{k-1}(\sin (2k-1)q\pi/p)/\sin 2kq\pi/p,$$

which again leads to the result (13.10).

We find jointly for $p \equiv 3$ and $p \equiv 1 \pmod 4$ from (13.8)

$$(-1)^{(p+1)/2}F(e^{\pi iq(p+1)/p}) = (e^{\pi iq(p+1)/p} - e^{\pi iq(p-3)/p}) \cdots,$$

and by elementary simplification

$$(-1)^{(p+1)/2}F(e^{\pi iq(p+1)/p}) = e^{\pi iq(p^2-1)/(4p)} \prod_{1}^{(p-1)/2} (e^{2\pi iqk/p} - e^{-2\pi iqk/p}),$$

and by a reasoning similar to that employed in §4,

$$(-1)^{(p+1)/2}F(e^{\pi iq(p+1)/p}) = e^{\pi iq(p^2-1)/4p}i^{(p-1)/2}(-1)^{N(2q,p)}p^{1/2}. \tag{13.11}$$

Here $N(2q, p)$ represents again the number of terms in the set $e^{2\pi iq/p}, e^{4\pi iq/p}, \cdots, e^{\pi i(p-1)q/p}$ whose imaginary part is negative. Accordingly, by (13.4) and (13.9) resp. (13.9′),

$$\lim_{s\to\infty} [\Omega_0(x) + c_q e^{-is}F(e^{(p+1)q\pi i/p})] = 0, \tag{13.12}$$

or

$$\begin{aligned} 0 = \lim_{s\to\infty} [\Omega_0(x) + c_q e^{-is}e^{\pi iq(p^2-1)/4p} \\ \cdot i^{(p-1)/2}(-1)^{N(2q,p)+(p+1)/2}p^{1/2}]. \end{aligned} \tag{13.13}$$

14. Construction of a second standing wave for $p \equiv 3 \pmod 4$. Second interpolating function $G(\xi)$. Consider the polynomial

$$G(\xi) = \sum_{1}^{(p-1)/2} B_k\xi^k \tag{14.1}$$

where

$$(14.2)\qquad \begin{aligned} B_k &= -\,i(A_k/2)(\eta^k-\eta^{-k})/(\eta^k+\eta^{-k}-2)\\ &= -\,(A_k/2)\cot 2\pi qk/p \end{aligned}$$

is real. We have

$$B_k = -\,i(A_k/2)((1-\eta^k)^{-1}-(1-\eta^{-k})^{-1}),$$

$$(14.3)\quad \begin{aligned} G(\xi\eta)-G(\xi) &= \sum B_k\xi^k(\eta^k-1) = (i/2)\left(\sum_1^{(p-1)/2}\xi^kA_k(1+\eta^k)\right)\\ &= (i/2)(F(\xi)+F(\xi\eta)+2), \end{aligned}$$

since $A_0=-1$. Also

$$G(1)=\sum_1^{(p-1)/2}B_k,$$

whence, by (13.8) and (14.3),

$$(14.4)\quad G(\eta^n) = in + \sum_1^{(p-1)/2}B_k,\ \text{for } n = -\,(p-3)/4,\cdots,(p-3)/4,$$

and

$$(14.5)\quad G(\eta^{(p+1)/4}) = i(p+1)/4 + (i/2)F(\eta^{(p+1)/4}) + \sum B_k.$$

Thus the B_k are real numbers for which, with δ_{jk} from (10.13),

$$(14.6)\qquad \sum B_k\delta_{jk} = in_j + \sum B_k, \qquad\qquad \text{if (10.9) with } j\neq q,$$

and

$$\sum B_k\delta_{qk} = i(p+1)/4 + (i/2)F(\eta^{(p+1)/4}) + \sum B_k.$$

Set

$$(14.7)\qquad \begin{aligned} \pi\Omega_{-1}(z) &= W_{-1}(z) - 2\pi\sum_1^{(p-1)/2}W_k^*(z)B_k\\ &\quad + W_0(z)\left(2\pi\sum B_k - \int_1^\infty e^tt^{-1}dt\right). \end{aligned}$$

By (14.7), (14.6), (11.6), (11.10) we have the limit relation

$$\begin{aligned} \lim_{x\to\infty}\Bigg[-\,\pi\Omega_{-1}(x) &+ 2\pi i\sum_j{}'c_jn_je^{\epsilon_jx} - 2\pi\sum_j{}'\sum_{k=1}^{(p-1)/2}B_k\delta_{jk}c_je^{\epsilon_jx}\\ &+ \sum_j{}'2\pi c_j\sum B_ke^{\epsilon_jx}\Bigg] = 0, \end{aligned}$$

where the sum $\sum'$ extends over all j with (10.9). This immediately yields

$$\lim_{x\to\infty}\left[-\pi\Omega_{-1}(x)-\pi i c_q F(\eta^{(p+1)/4})e^{-ix}\right]=0. \tag{14.8}$$

Compare (14.8) with the limit formula (13.12) for $\Omega_0(x)$. Clearly the nodes of $\Omega_{-1}(x)$ lie asymptotically midway between those of $\Omega_0(x)$. In a similar manner it is seen that $\Omega_{-1}(z)\to 0$ on the bottom of S_β.

For $z\to 0$, $\Omega_{-1}(z)$ has a logarithmic singularity while $\Omega_0(z)$ remains finite. Furthermore, $\Omega_{-1}(z)$ satisfies the differential equations on surface and bottom since it is a linear combination with real coefficients of functions having this property. Thus $R\Omega_{-1}(z)$ is the velocity potential of a second standing wave and

$$\cos tR\Omega_0(z)-\sin tR\Omega_{-1}(z)$$

is the time-depending velocity potential of a progressive wave, advancing toward the shore $z=0$, which at large distances from shore is a simple progressive wave of wave length 2π. We observe that it has a logarithmic singularity at $z=0$.

15. **Same for $p\equiv 1$ (mod 4).** Set, as before

$$G(\xi)=\sum_1^{(p-1)/2}B_k\xi^k,$$

with

$$\begin{aligned}B_k&=-(i/2)A_k((1-\eta^{2k})^{-1}-(1-\eta^{-2k})^{-1})\\&=-(A_k/2)\cot(2\pi kq/p),\end{aligned}$$

where now

$$\eta=e^{2\pi iq/p}.$$

We find

$$G(\xi\eta^2)-G(\xi)=(i/2)(F(\xi)+F(\xi\eta^2)+2), \tag{15.1}$$

since $A_0=-1$. Also

$$G(\eta^{2n+1})=G(\eta)+in,\quad n=-(p-1)/4,\cdots,+(p-5)/4, \tag{15.2}$$

$$G(\eta^{(p+1)/2})=G(\eta)+i(p-1)/4+(i/2)F(\eta^{(p+1)/2}). \tag{15.3}$$

Now, by (15.1),

$$G(\eta)-G(\eta^{-1})=i,$$

since $F(\eta)=F(\eta^{-1})=0$. Hence, because B_k is real,

$$G(\eta) = i/2 + \text{real.}$$

Thus (15.2), (15.3) become

$$G(\eta^{2n+1}) = i(n + 1/2) + RG(\eta),$$
$$n = -(p-1)/4, \cdots, (p-5)/4,$$
$$G(\eta^{(p+1)/2}) = i(p+1)/4 + (i/2)F(\eta^{(p+1)/2}) + RG(\eta).$$

Accordingly

$$\sum B_k\delta_{jk} = i(n_j + 1/2) + RG(\eta),$$

if (10.9′) holds with $j \neq q$,

$$\sum B_k\delta_{qk} = i(p+1)/4 + (i/2)F(\eta^{(p+1)/2}) + RG(\eta).$$

Set

$$\pi\Omega_{-1}(z) = W_{-1}(z) - 2\pi\sum W_k^*(z)B_k + W_0(z)\left(2\pi RG(\eta) - \int_1^{i\infty} e^{it}t^{-1}dt\right).$$

Then

$$0 = \lim_{x\to\infty}\Big[-\pi\Omega_{-1}(x) + 2\pi i\sum_j{}' c_j(2n_j+1)e^{\eta^j x} - 2\pi\sum_j{}' c_je^{\eta^j x}\sum_k B_k\delta_{jk} + \left(2\pi RG(\eta) - \int_1^{i\infty} e^{it}t^{-1}dt\right)\sum{}' c_je^{\eta^j x}\Big].$$

Consequently

$$0 = \lim_{x\to\infty}\left[-\pi\Omega_{-1}(x) - \pi ic_qF(e^{\pi i(p+1)q/p})e^{-ix}\right].$$

The same conclusions about the behavior of $\Omega_{-1}(z)$ on the bottom and near $z=0$, and about the asymptotic distribution of the nodes of $R\Omega_0(x)$ and $R\Omega_{-1}(x)$, hold as did for $p\equiv 3 \pmod 4$. Again

$$\cos tR\Omega_0(z) - \sin tR\Omega_{-1}(z)$$

is the velocity potential of a progressive wave advancing toward shore which at large distance from shore becomes a simple wave of length 2π and which has a logarithmic singularity at the shore.

16. **About the continuity of the standing wave as function of the angle $p\pi/2q$ and the connection with the quadratic reciprocity law of number theory.** Let us inspect the asymptotic location of the surface nodes of the finite standing wave $\Omega_0(z)$ as $z=x\to\infty$. On entering the

value of c_q from (6.18) in (13.13) we find that $\Omega_0(x)$ differs infinitely little from

$$(-i)^{(q-1)/2+(p-1)/2}(-1)^{N(2q/p)}e^{\pi i(p^2-1)q/4p}e^{-ix}p^{1/2}, \qquad \text{if } q \text{ odd,}$$
$$(-i)^{(p-1)/2}ie^{-\pi i(p+q)/4+\pi i(p^2-1)q/4p}(-1)^{N(2q,p)}e^{-ix}p^{1/2}, \qquad \text{if } q \text{ even.}$$

Thus the nodes of $R\Omega_0(x)$ come to lie asymptotically at the points $x=\pi/2+m\pi+qp\pi/4-q\pi/4p-\pi(q+p-2)/4$, with m integral for q odd, $x=m\pi+qp\pi/4-q\pi/4p-\pi(1+q)/4$, with m integral for q even. If the oddness of p is utilized, it is seen that the nodes lie asymptotically at

$$x = m\pi - \pi/4 - q\pi/4p, \tag{16.1}$$

no matter whether q is odd or even; for if q is odd,

$$qp - q - p \equiv -1 \pmod 4,$$

and if q is even

$$qp - 1 - q \equiv -1 \pmod 4.$$

The result (16.1) is the generalization of a property announced at the end of §7 for the case $p=1$.

Next let us study the ratio of $\lim_{x\to\infty}(e^{ix}\Omega_0(x))$ and $\Omega_0(0)$. By (13.1), (10.6)

$$\Omega_0(0) = W_0(0) = \sum_1^q c_j,$$

as given by (6.17). Accordingly, by elementary simplification of the exponents, one obtains, for example, for odd p and q the formula

$$\lim_{x\to\infty}(e^{ix}\Omega_0(x))/\Omega_0(0)$$
$$= (-1)^{N(2q,p)+N(p,q)+(qp-p-q+1)/4}e^{\pi i(1-q/p)/4}(p/q)^{1/2}.$$

Now suppose that the left-hand side depends continuously on the angle $p\pi/2q$ between surface and bottom. Then the same applies to the right-hand side, whence in particular the expression

$$(-1)^{N(2q,p)+N(p,q)+(qp-p-q+1)/4}$$

must be continuous in p/q for p and q odd. But since it equals ± 1, this expression must have a constant value, which is determined as $+1$ by taking $p=q=1$ or $p=1$, $q=3$. Furthermore it is easily seen from the definition of $N(p, q)$ that, for odd p and q, $N(p, q)=N(2p, q)$. Thus the assumption of a continuous dependence of the finite standing wave on p/q implies

$$(-1)^{N(2q,p)+N(2p,q)} = (-1)^{(p-1)(q-1)/4}.$$

This is, of course, the quadratic reciprocity law of number theory.

We shall not investigate here the above continuity question, nor shall we go here into a discussion of those problems which are immediately suggested by the above embedding of a number theoretical fact concerning two integers p and q into a continuous law concerning the quantity p/q.

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VECTOR FIELDS AND RICCI CURVATURE

S. BOCHNER

We shall prove theorems on nonexistence of certain types of vector fields on a compact manifold with a positive definite Riemannian metric whose Ricci curvature[1] is either everywhere positive or everywhere negative. Actually we shall have some relaxations of the requirements both as to curvature and as to compactness. We shall deal with real spaces with a customary metric and with complex analytic spaces with an Hermitian metric. In the latter case we shall impose on the metric a certain restriction, first explicitly stated by E. Kaehler,[2] which will be quite indispensable to our argument. In order to elucidate the rôle of this restriction we shall include a systematic introduction to the theory of Hermitian metric.

For positive curvature we shall have the theorem that on a compact space there exists no vector field for which the *divergence* and *curl* both vanish. In the complex case there exists no vector field whatsoever whose covariant components are analytic functions in the complex parameters. If we only assume that the curvature is nonnegative, then there are some "exceptional" vector fields in directions of spatial flatness. A principal result will be the following theorem on meromorphic functions. If a complex space with positive curvature is covered by a finite number of neighborhoods, if a meromorphic functional element is defined in each neighborhood, and if the difference of any two meromorphic elements is holomorphic wherever the elements overlap, then there exists *one* meromorphic function on the space which differs by a holomorphic function from each meromorphic element given. In a previous paper[3] this conclusion was drawn in the

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[1] Also called mean curvature; the definition will be restated later in the text. Interesting facts relating Ricci curvature to Riemann curvature have been given by T. Y. Thomas, *On the variation of curvature in Riemann spaces of constant mean curvature*, Annali di Matematica Pura ed Applicata (4) vol. 13 (1935) pp. 227–238, and *New theorems on Riemann-Einstein spaces*, Rec. Math. (Mat. Sbornik) N.S. vol. 3 (1938) pp. 331–340. Also, for the application of the Laplacean on compact Riemannian spaces see T. Y. Thomas, *Some applications of Green's theorem for compact Riemann spaces*, Tôhoku Math. J. vol. 46 (1940) pp. 261–266.

[2] *Ueber eine bemerkenswerte Hermitische Metrik*, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1933) pp. 173–186; see also S. Chern, *Characteristic classes of Hermitian manifolds*, Ann. of Math. vol. 47 (1946) pp. 85–121, especially pp. 109–112.

[3] S. Bochner, *Analytic and meromorphic continuation by means of Green's formula*, Ann. of Math. vol. 44 (1943) pp. 652–673, especially p. 672, Theorem 15.

case of a multi-torus; however, in that case, due to the flatness of the space, some necessary and sufficient conditions on the meromorphic elements had to be added. These conditions correspond to the "exceptional" vector fields mentioned before, and they do not occur if the curvature is strictly positive.

For negative curvature our theorem will in the main be a generalization of a theorem of Schwarz and M. Noether[4] from one to several complex variables. It states that on an algebraic curve of genus $p>1$ there is no one-parametric group of rational transformations. In other words, on such a compact two-dimensional surface in one complex variable there exists no continuous group of analytic homeomorphisms. Now, the universal covering surface of such a surface can be identified with the unit circle $|z|<1$ and thus endowed with a hyperbolic metric of (constant) negative curvature. Thus we can express the theorem of Schwarz and Noether in the following second version. If a "real" compact two-dimensional space with a Riemannian metric has (constant) negative curvature then there exists no continuous group of conformal (that is, angle-preserving) transformations into itself. Corresponding to these two versions we shall have two generalizations to several variables. The complex version will be that if a compact complex space in n variables with a suitable Hermitian metric has negative Ricci curvature (not necessarily constant), then there exists no continuous group of complex homeomorphisms. The "real" version will be less satisfactory but rather illuminating. We shall first show that for negative curvature in n variables there are no groups of motions. A motion preserves not only angles but also the lengths. In order to obtain a theorem on transformations which distort the length we shall have to add an assumption on the nature of the distortion. However, this assumption will be automatically fulfilled for $n=2$, and thus technically a proper generalization will be obtained. We shall also have "exceptional" groups in case the curvature is only nonpositive, and we shall have a further statement in case the given compact space is a minimal variety in some higher-dimensional space of nonpositive curvature.

Finally we shall have some converse theorems, but only locally. A converse theorem states that if there are sufficiently many vector fields of one kind or another then the curvature is negative or positive respectively. Locally, a gradient field is a scalar function from which it is derived, and several gradient fields is therefore a mapping into

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[4] H. A. Schwarz, *Gesammelte Abhandlungen*, vol. 2, pp. 285–291; M. Noether, Math. Ann. vol. 20 (1882) pp. 59–62 and vol. 21 (1883) pp. 138–140. See also A. Hurwitz, *Mathematische Werke*, vol. 1, pp. 241–259.

Euclidean space. Correspondingly we shall have the following theorems on *complex* space. Any piece of m-dimensional surface in n-dimensional Euclidean space has negative curvature, but the presence of sufficiently many contravariant vector fields in commutation gives rise to a fundamental tensor whose curvature is positive.

The most interesting problem in this connection is the problem of constructing a negative (hyperbolic) curvature from the existence of a noncommutative transitive group of analytic homeomorphisms on a piece of Euclidean or other space. This problem will not be approached at all in the present context.

I. REAL SPACES

1. **Lemmas.** Let S be an n-dimensional coordinate space with a fixed positive definite Riemannian metric

$$ds^2 = g_{ij}dx_i dx_j. \tag{1}$$

The space is of differentiability class C^4, and all scalars, vectors and tensors belong to class C^3. For any scalar ϕ we can form the Laplacean

$$\Delta\phi \equiv g^{ij}\phi_{,i,j} = \frac{1}{g^{1/2}}\frac{\partial}{\partial x_i}\left(g^{1/2}g^{ij}\frac{\partial\phi}{\partial x_j}\right) \tag{2}$$

and we shall apply it so the square length

$$\phi = \xi^i\xi_i \equiv g^{ij}\xi_i\xi_j = g_{ij}\xi^i\xi^j, \tag{3}$$

where $\{\xi_i\}$ is an arbitrary vector field on all of S. We shall next have a definition.

DEFINITION. A vector ξ_i on S is called a ***restrained*** vector, if either

$$\Delta\phi(P_0) < 0$$

at some point P_0 in S, or

$$\phi(P) \equiv \text{constant} \tag{4}$$

throughout S.

This definition will be justified by the following theorem.

LEMMA 1. *On a compact space S every vector is restrained relative to every metric.*

On a general S a vector is restrained if its length in some metric attains a (relative) maximum in S, and thus in particular if the length is less than ϵ outside some compact subset depending on ϵ.

PROOF. The compact case can be dealt with by a "global" argument, namely Stokes' formula

$$\int (\phi\cdot\Delta\phi + g^{ij}\phi_{,i}\phi_{,j})dv = 0. \tag{5}$$

In fact, if $\phi \geqq 0$, $\Delta\phi \geqq 0$, then (5) can hold only if $\phi_{,i} \equiv 0$, that is, $\phi \equiv$ constant, as claimed. But there is also a "local" argument which implies the lemma in its entirety. This is the following theorem of E. Hopf.[5] If $\Delta\phi \geqq 0$ in a neighborhood T, and if there exists a point P_0 in T such that $\phi(P) \leqq \phi(P_0)$ for all P in T, then $\phi(P) \equiv \phi(P_0)$ in T.

We now start from the formula

$$\xi_{i,j,k} - \xi_{i,k,j} - \xi_l R^{l}_{ijk} = 0 \tag{6}$$

where R^l_{ijk} is the curvature tensor. The Ricci tensor arises by contraction,

$$R_{ij} = R^{k}_{ijk} = g^{lk}R_{lijk} = R^{l}_{ijl}, \tag{7}$$

and it has the following interpretation.[6] If for a given point we consider n orthogonal unit vectors $\eta^i_{(m)}$, $m = 1, \cdots, n$, which are co-oriented with $x_1, \cdots, x_n$, and if we denote by κ_{pq} the curvature of the geodesic surface element which is determined by $\eta^i_{(p)}$ and $\eta^i_{(q)}$, then the sum

$$\kappa_{11} + \kappa_{12} + \cdots + \kappa_{1n}$$

is given by

$$-R_{ij}\eta^i_{(1)}\eta^j_{(1)},$$

thus being dependent only on $\eta^i_{(1)}$ itself. We shall call it the Ricci curvature in the direction $\eta^i_{(1)}$. The curvature is called positive if it is strictly positive in all directions.

LEMMA 2. *The quantity*

$$2^{-1}\Delta\phi = 2^{-1}g^{rs}(g^{ab}\xi_a\xi_b)_{,r,s} = g^{rs}g^{ab}\xi_{a,r}\xi_{b,s} + g^{rs}g^{ab}\xi_{a,r,s}\xi_b \tag{8}$$

has the value

$$g^{rs}g^{ab}\xi_{a,r}\xi_{b,s} - R^{lm}\xi_l\xi_m + A \tag{9}$$

where the remainder term is

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[5] *Elementare Bemerkungen ueber die Loesungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Preuss. Akad. Wiss. Sitzungsber. vol. 19 (1927) pp. 147–152.

[6] See L. P. Eisenhart, *Riemannian geometry*, p. 113. We have adopted Eisenhart's normalization throughout, and this explains why positive Ricci curvature corresponds to a negative Ricci quadratic form.

$$A = g^{rs}g^{ab}(\xi_{a,r,s} - \xi_{r,a,s} + \xi_{r,s,a})\xi_b; \tag{10}$$

it also has the value

$$g^{rs}g^{ab}\xi_{a,r}\xi_{b,s} + R^{lm}\xi_l\xi_m + B \tag{11}$$

where

$$B = g^{rs}g^{ab}(\xi_{a,r,s} + \xi_{r,a,s} - \xi_{r,s,a})\xi_b. \tag{12}$$

In fact, from (6) we obtain

$$g^{rs}g^{ab}(\xi_{r,s,a} - \xi_{r,a,s})\xi_b - g^{ab}R^l_a\xi_l\xi_b = 0.$$

If we add this to (8) we obtain (9), and if we subtract it from (8) we obtain (11).

2. Positive curvature. Since the term

$$g^{rs}g^{ab}\xi_{a,r}\xi_{b,s} \tag{13}$$

is positive for arbitrary quantities $\xi_{i,j}$ except in the case where the latter are all 0, we first obtain the following conclusion.

THEOREM 1. *The remainder term A is obviously* 0 *whenever*

$$\xi_{i,j} = \xi_{j,i} \quad \textit{and} \quad g^{ij}\xi_{i,j} = 0, \tag{14}$$

that is,

$$\text{curl } \xi = 0 \quad \textit{and} \quad \text{div } \xi = 0, \tag{15}$$

or more generally, whenever

$$(\xi_{i,j} - \xi_{j,i})_{,k} = 0 \quad \textit{and} \quad (g^{ij}\xi_{i,j})_{,k} = 0. \tag{16}$$

Therefore, there exists no restrained vector field which satisfies relation (16) *and*

$$- R^{ij}\xi_i\xi_j \geqq 0 \tag{17}$$

unless we have

$$- R^{ij}\xi_i\xi_j = 0, \qquad \xi_{i,j} = 0. \tag{18}$$

Thus, the only exceptions are fields of parallel vectors in directions of Ricci-flatness, and they cannot arise, for instance, if the space as such has positive Ricci curvature throughout.

A vector field for which $\xi_{i,j} = \xi_{j,i}$ is a local gradient field and can be described in the following way. There exists a covering of the space by a system of neighborhoods $\{U^\nu\}$, which for a compact space is finite, and a scalar $f^{(\nu)}(x)$ on each U^ν, such that

$$f^{(p)}(x) - f^{(q)}(x) = \text{constant} \tag{19}$$

whenever two neighborhoods U^p, U^q intersect. And ξ_i is defined as

$$\xi_i = \frac{\partial}{\partial x_i} f^{(p)}(x) \tag{20}$$

on U^p. The second part of (14) amounts to assuming that each $f^{(p)}(x)$ shall in its neighborhood be a solution of the Laplacean

$$\Delta f = 0, \tag{21}$$

and our theorem states in particular that on a space of positive Ricci curvature every $f^{(p)}(x)$ is itself constant. It should be noted that this conclusion is trivial if S is simply connected, or if some simply connected covering space of S is compact, since in this case the system of equations (20) can be solved by one function $f(x)$ in S; and since its Laplacean vanishes, it is a constant throughout.

3. **Negative curvature.** We now turn to an analysis of the expression (11) and we shall first have a detailed lemma on the influence of the remainder term B.

LEMMA 3. *The remainder term B vanishes if*

$$\xi_{i,j} + \xi_{j,i} = 0, \tag{22}$$

or more generally if

$$(\xi_{i,j} + \xi_{j,i})_{,k} = 0. \tag{23}$$

Thus it vanishes if the contravariant components ξ^i are an infinitesimal generator of a one-parametric group of motions, that is, metric-preserving homeomorphisms, or, more generally, of affine collineations.

We have $B \geq 0$, if there exists a scalar function $\lambda(x)$ such that

$$\xi_{i,j} + \xi_{j,i} = \lambda g_{ij}, \tag{24}$$

that is, if ξ^i represents a one-parametric group of angle-preserving (conformal) homeomorphisms, provided we also have

$$(1 - n/2) g^{ij} \lambda_i \xi_j \geq 0, \tag{25}$$

the latter condition being automatically fulfilled for $n = 2$.

PROOF. If we multiply (23) by g^{ij}, then due to the symmetry of g^{ij} we obtain

$$g^{rs}\xi_{r,s,k} = 0,$$

and thus (23) implies $B = 0$. On any differentiable manifold, with or without a Riemannian metric, a vector ξ^i describes a local group of differentiable homeomorphisms. If a metric is given we can form the covariant components ξ_i, (22) is the necessary and sufficient condition for the transformations to be motions,[7] and (23) for affine collineations.

Condition (24) expresses preservation of angles. If we differentiate it we obtain

$$\xi_{i,j,k} + \xi_{j,i,k} = \lambda_{,k} g_{ij} \tag{26}$$

and hence

$$g^{rs}g^{ab}(\xi_{a,r,s} + \xi_{r,a,s})\xi_b = g^{sb}\lambda_{,s}\xi_b.$$

If, however, we multiply (26) by $g^{ij}g^{kb}\xi_b$ we obtain

$$-\ g^{rs}g^{ab}\xi_{r,s,a}\xi_b = -\ (n/2)g^{sb}\lambda_{,s}\xi_b$$

and thus (25) is equivalent with $B \geqq 0$.

THEOREM 2. *There exists no restrained group of motions* $\{\xi^i\}$ *for which*

$$-\ R_{ij}\xi^i\xi^j \leqq 0, \tag{27}$$

unless we have

$$\xi_{i,j} = 0, \qquad R_{ij}\xi^i\xi^j = 0, \tag{28}$$

that is, unless it is a group of translations along Ricci-flat geodesics.

For $n = 2$, *the same conclusion holds for conformal mappings* $\{\xi^i\}$, *and for* $n \geqq 3$ *it will hold under the additional assumption* (25).

4. **Almost periodic vector fields.** Take a compact space S and its universal covering space T. If S has negative curvature, T most likely will not be compact. Introduce on T the group Γ of automorphisms for which S is the set of co-sets, and denote by S_0 a fundamental domain in T. It can be chosen compact. Every scalar or tensor function on S gives rise to such a function on T which is "periodic," that is, invariant under Γ. In particular the tensor g_{ij} can be periodically extended. Now take a vector field ξ_i on T which, though not necessarily periodic, is together with its second derivatives almost periodic with respect to Γ. This means that if $\{\gamma_p\}$ is a sequence in Γ then there exists a sub-sequence $\{\gamma_q\}$ such that the sequence of vector fields $\xi_i(\gamma_q P)$ and their second derivatives is uniformly convergent in T.

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[7] Eisenhart, p. 234.

THEOREM 3. *Any almost periodic vector field ξ_i on the covering space T of S is restrained in T, and thus Theorems* 1 *and* 2, *and similar theorems, apply to such vector fields in T.*

PROOF. Due to its almost periodicity, the square length $\phi(P)$ of ξ_i is bounded and uniformly continuous, and there exists a sequence of points P_p such that

$$\phi(P_p) \to \sup_P \phi(P).$$

Each P_p lies in some image $\gamma_p(S_0)$ of S_0, and owing to the compactness of S_0 there exists a subsequence γ_q such that the sequence of translated functions $\phi(\gamma_q P)$ and their second derivatives will converge uniformly in T towards a function $\Phi(P)$ which assumes its maximum in an interior point of T. Now, due to the periodicity of g_{ij}, we have

$$\Delta\phi(P) = \Delta\phi(\gamma_q P) = \Delta\Phi(P),$$

and thus $\Delta\phi \geqq 0$ implies $\Delta\Phi \geqq 0$. By Lemma 1, we therefore have $\Phi = \text{const}$. However this implies $\phi = \text{const.}$, and thus ξ_i is restrained.

5. **Minimal varieties.** If V_n is a subspace of any space V_{n+1} whose Riemann-Christoffel tensor vanishes, then for the Ricci tensor of V_n we have $R_{jk} = g^{il}(b_{ik}b_{jl} - b_{ij}b_{lk})$ where b_{ij} are the coefficients of the second differential form.[8] If V_n is a minimal hypersurface of V_{n+1}, then $g^{ij}b_{ij} = 0$,[9] and therefore

$$R_{jk}\xi^j\xi^k = g^{il}\eta_i\eta_l \tag{29}$$

where

$$\eta_i = b_{ik}\xi^k. \tag{30}$$

Thus the vanishing of (29) implies the vanishing of (30), and the vanishing of the latter for a group $\{\xi^k\}$ means that the orbits are asymptotic lines on V_n.[10] However, any geodesic line of V_n which is asymptotic on V_n is a geodesic of V_{n+1} itself. The same reasoning applies to a minimal variety in a space V_m of any higher dimension $m \geqq n+1$, since we have in general

$$R_{jk}\xi^j\xi^k = \sum_{r=1}^{m-n} g^{ij} b_{ik}^{(r)}\xi^k b_{jl}^{(r)}\xi^l.$$

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[8] Eisenhart, p. 190.

[9] Eisenhart, p. 178.

[10] Eisenhart, p. 167.

Thus we obtain the following theorem.

THEOREM 4. *If the space V_n is a minimal variety in a space V_m, $m \geqq n+1$, of vanishing Riemann curvature, and if there exists a one-parametric group of motions on V_n, then the orbits are geodesics in the enveloping V_m which are also asymptotic lines on V_n.*

II. HERMITIAN METRIC

6. **Power series.** We take independent *complex* variables

$$t_1, \cdots, t_m \tag{31}$$

and we consider power series in these variables with complex coefficients, each series converging in some (not fixed) neighborhood of the origin. Every "function," that is scalar, scalar density or any component of a tensor shall be such a series.

A "transformation"

$$t_i' = \phi_i(t_1, \cdots, t_m), \qquad i = 1, \cdots, m, \tag{32}$$

shall be given by functions ϕ_i which vanish at the origin, in order to allow unrestricted formation of the product of two transformations, and the Jacobian

$$\frac{\partial(t_1', \cdots, t_m')}{\partial(t_1, \cdots, t_m)}$$

shall be not equal to 0 at the origin. The totality of such transformations forms a group, and we consider an *arbitrary* fixed subgroup Γ of the group. The elements of Γ shall be termed allowable transformations and the resulting quantities t_i' allowable variables. We can now define as usual vectors and tensors

$$\xi_i, \xi^i, \xi_i{}^k{}_{lm\cdots},$$

with upper and lower indices; also, contraction of an upper and lower index is again permitted. A symmetric affine connection Γ_{jk} is again a system of functions with the customary rules of transformation. It gives rise to a mixed tensor

$$R^l_{ijk} = \frac{\partial}{\partial t_j}\Gamma^l_{ik} - \frac{\partial}{\partial t_k}\Gamma^l_{ij} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk} \tag{33}$$

with which we can form the Ricci tensor

$$R_{ij} = R^k_{ijk}; \tag{34}$$

it also gives rise to covariant differentiation of arbitrary tensors and tensor densities as usual. For instance

$$(35) \qquad \xi_{i,j} = \frac{\partial \xi_i}{\partial t_j} - \xi_k \Gamma^k_{ij},$$

$$(36) \qquad \xi^i_{,j} = \frac{\partial \xi^i}{\partial t_j} + \xi^k \Gamma^i_{kj}.$$

Also derivatives of sums, products and contractions are to be formed as usual.

We next introduce a fundamental tensor g_{ij}. For the present we only assume that it is symmetric

$$(37) \qquad g_{ij} = g_{ji},$$

and that the determinant

$$g = \|g_{ij}\|_{i,j=1,\cdots,m},$$

is not equal to 0 at the origin. We emphasize that each component g_{ij} is a complex power series in the complex variables (31) and that the symmetry condition (37) is not at all an "Hermitian" property. There exist functions g^{ij} which are solutions of $g^{ij}g_{jk} = \delta^i_k$, and they are a symmetric tensor. The expressions

$$(38) \qquad \Gamma^i_{jk} = 2^{-1} g^{il}\left(\frac{\partial g_{lk}}{\partial t_j} + \frac{\partial g_{jl}}{\partial t_k} - \frac{\partial g_{jk}}{\partial t_l}\right)$$

are an affine connection, and the tensors g_{ij}, g^{ij} can be used for pulling indices up and down in a manner compatible with covariant differentiation. Also

$$(39) \qquad g_{ij,k} = 0, \qquad g^{ij}_{,k} = 0,$$

and[11]

$$(40) \quad R_{ij} = 2^{-1} \frac{\partial^2}{\partial t_i \partial t_j} \log \frac{g}{C_0} - \frac{\partial}{\partial t_k} \Gamma^k_{ij} + \Gamma^m_{ik}\Gamma^k_{mj} - 2^{-1}\Gamma^m_{ij} \frac{\partial}{\partial t_m} \log \frac{g}{C_0},$$

where C_0 is the value of the determinant g at the origin. Also

$$(41) \qquad R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$$

and

$$(42) \qquad R_{ijkl} = -R_{jikl} = -R_{klij}.$$

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[11] Eisenhart, p. 21.

7. **Self-adjointness.** We now put $m=2n$ and we replace the variables (31) by the symbols

$$z_1, \cdots, z_n; \bar{z}_1, \cdots, \bar{z}_n. \tag{43}$$

Allowable transformations shall be those having the special form

$$z_\alpha' = \phi_\alpha(z_1, \cdots, z_n), \qquad \bar{z}_\alpha' = \psi_\alpha(\bar{z}_1, \cdots, \bar{z}_n), \tag{44}$$

where for each $\alpha=1, \cdots, n$ the power series $\psi_\alpha(t_1, \cdots, t_n)$ arises from the power series $\phi_\alpha(t_1, \cdots, t_n)$ by replacing each coefficient by its conjugate complex value. Therefore, if each $\bar{z}_\alpha$ is conjugate to z_α, then so is $\bar{z}_\alpha'$ to z_α', and therefore it will be consistent to assume henceforth that z_α are complex parameters and $\bar{z}_\alpha$ their conjugate values. However a function in general will continue to be a power series in all $2n$ variables (43), or rather it will be a power series with complex coefficients in the $2n$ real components x_α, y_α; $z_\alpha=x_\alpha+iy_\alpha$. Special scalar or tensor functions whose power series depend only on z_α will be termed "complex analytic" or "analytic" for short.

From now on *italic* indices i, j, k, $\cdots$ will run from 1 to $2n$; Greek indices α, β, $\cdots$ from 1 to n, and starred Greek indices will be n units more, $\alpha^*=\alpha+n$. We shall also star italic indices to indicate a change by n, thus $i^*=i\pm n$, $j^*=j\pm n$.

Let ζ_i be a covariant vector. For a transformation (44) we have

$$\zeta_\alpha' = \zeta_\beta \frac{\partial z_\beta'}{\partial z_\alpha}, \qquad \zeta_{\alpha^*}' = \zeta_{\beta^*} \frac{\partial \bar{z}_\beta'}{\partial \bar{z}_\alpha} \cdot$$

Thus the $2n$ components of ζ_i separate completely into the blocks ζ_α and ζ_{α^*}. For all tensors this separation applies to each upper and lower index independently of the others. For instance, g_{ij} separates into the blocks $g_{\alpha\beta}$, $g_{\beta^*\alpha^*}$, $g_{\alpha\beta^*}$ $g_{\alpha^*\beta}$, of n^2 components each.

We call a tensor *self-adjoint* if starring all indices simultaneously changes the value of a component into its conjugate:

$$\zeta^{j^*}_{i^*k^*l^*\cdots} = \overline{\zeta^{j}_{ikl\cdots}}\,.$$

For a vector it means

$$\zeta_{\alpha^*} = \bar{\zeta}_\alpha, \qquad \zeta^{\alpha^*} = \overline{\zeta^\alpha},$$

and for a *symmetric* tensor g_{ij} it means

$$g_{\alpha\beta} = g_{\beta\alpha} = \overline{g_{\alpha^*\beta^*}} = \overline{g_{\beta^*\alpha^*}}, \tag{45}$$

$$g_{\alpha\beta^*} = g_{\beta^*\alpha} = \overline{g_{\alpha^*\beta}} = \overline{g_{\beta\alpha^*}}. \tag{46}$$

A scalar shall be self-adjoint if it is real-valued. Self-adjointness is preserved under addition, multiplication and contraction. If g_{ij} is self-adjoint, then the inverse g^{ij}, and Γ^i_{jk} based on g_{ij}, are self-adjoint, and so are the Riemann and Ricci tensors. Furthermore it is preserved under covariant differentiation.

From now on all functions are assumed to be self-adjoint.

8. Restrictions on the fundamental tensor. The first substantial restriction on the fundamental tensor shall be

$$g_{\alpha\beta} = 0 = g_{\alpha^*\beta^*}. \tag{47}$$

Thus only the "interacting" components $g_{\alpha\beta^*} = g_{\beta^*\alpha}$ can be not equal to 0, and $g_{\alpha\beta^*} = \overline{g_{\alpha^*\beta}}$. We automatically also have

$$g^{\alpha\beta} = 0$$

and (38) gives

$$\Gamma^{\alpha}_{\beta\gamma} = 2^{-1} g^{\alpha\epsilon^*}\left(\frac{\partial g_{\epsilon^*\gamma}}{\partial z_\beta} + \frac{\partial g_{\beta\epsilon^*}}{\partial z_\gamma}\right), \tag{48}$$

$$\Gamma^{\alpha}_{\beta\gamma^*} = 2^{-1} g^{\alpha\epsilon^*}\left(\frac{\partial g_{\beta\epsilon^*}}{\partial \bar{z}_\gamma} - \frac{\partial g_{\beta\gamma^*}}{\partial \bar{z}_\epsilon}\right), \tag{49}$$

$$\Gamma^{\alpha}_{\beta^*\gamma^*} = 0, \tag{50}$$

and 5 other relations obtained by symmetry and self-adjointness. From (35) we now have

$$\zeta_{\alpha,\beta} = \frac{\partial \zeta_\alpha}{\partial z_\beta} - \zeta_\epsilon \Gamma^{\epsilon}_{\alpha\beta}, \tag{51}$$

$$\zeta_{\alpha^*,\beta} = \frac{\partial \zeta_\alpha}{\partial \bar{z}_\beta} - \zeta_\epsilon \Gamma^{\epsilon}_{\alpha\beta^*}. \tag{52}$$

However we also want to have $\Gamma^{\alpha}{}_{\beta\gamma^*} = 0$, and this will be obtained by Kaehler's restriction, which is:

$$\frac{\partial g_{\alpha\gamma^*}}{\partial z_\beta} = \frac{\partial g_{\beta\gamma^*}}{\partial z_\alpha}, \tag{53}$$

and by adjointness also implies

$$\frac{\partial g_{\gamma\alpha^*}}{\partial \bar{z}_\beta} = \frac{\partial g_{\gamma\beta^*}}{\partial \bar{z}_\alpha}. \tag{54}$$

This is equivalent with assuming that there exists (locally) a scalar function $\phi(z, \bar{z})$ such that

(55)
$$g_{\alpha\beta^*} = \frac{\partial^2\Phi}{\partial z_\alpha \partial \bar z_\beta}\,.$$

Self-adjointness demands that Φ shall be real-valued, and all these assumptions are indeed invariant under transformations of the type (43). We shall now list consequences of our restrictions.

LEMMA 4. *We have*

$$\zeta_{\alpha,\beta^*} = \frac{\partial \zeta_\alpha}{\partial \bar z_\beta}, \qquad \zeta^\alpha{}_{,\beta^*} = \frac{\partial \zeta^\alpha}{\partial \bar z_\beta}\,.$$

In particular if the component ζ_α *is complex analytic, we have* $\zeta_{\alpha,\beta^*} = 0$, *and if* ζ^α *is complex analytic we have* $\zeta^\alpha{}_{,\beta^*} = 0$.

More generally, if the indices $\alpha_1, \cdots, \alpha_p$ *are unstarred, we have*

$$\zeta_{\alpha_1,\cdots,\alpha_p,\beta^*} = \frac{\partial \zeta_{\alpha_1,\cdots,\alpha_p}}{\partial \bar z_\beta}\,; \qquad \zeta^{\alpha_1,\cdots,\alpha_p}{}_{,\beta^*} = \frac{\partial \zeta^{\alpha_1,\cdots,\alpha_p}}{\partial \bar z_\beta}\,.$$

For the appreciation of the lemma it should be noted that if the n components ζ_α are complex analytic the contravariant components

$$\zeta^\alpha = g^{\alpha\beta^*}\bar\zeta_\beta, \qquad \zeta^{\alpha^*} = g^{\alpha^*\beta}\zeta_\beta$$

will depend on all $2n$ variables (42) since the components $g^{\alpha\beta^*}$ so depend.

LEMMA 5. *For the Riemann tensor we have*

(56)
$$R_{\alpha\beta jk} = 0.$$

More generally, due to (42); *only those components of* R_{lijk} *can be different from* 0 *for which, in both pairs of indices* (l, i) *and* (j, k) *simultaneously, one element in the pair is a Greek unstarred and the other a Greek starred index.*

PROOF. It suffices to prove that $R^{\alpha^*}_{\beta jk}$ vanishes. From (33) we obtain for this component the value

$$\frac{\partial}{\partial t_j}\Gamma^{\alpha^*}_{\beta k} - \frac{\partial}{\partial t_k}\Gamma^{\alpha^*}_{\beta j} + \Gamma^{m}_{\beta k}\Gamma^{\alpha^*}_{mj} - \Gamma^{m}_{\beta j}\Gamma^{\alpha^*}_{mk},$$

and this vanishes since only the pure components $\Gamma^{\alpha}_{\beta\gamma}$ and $\overline{\Gamma^{\alpha}_{\beta\gamma}}$ can be not equal to 0.

LEMMA 6. *We also have*

$$R^{\alpha}_{\beta\gamma^* i} = -R^{\alpha}_{i\beta\gamma^*}.$$

PROOF. In the Bianchi identity

$$R^{\alpha}_{\beta\gamma^*\delta} + R^{\alpha}_{\gamma^*\beta\delta} + R^{\alpha}_{\delta\beta\gamma^*} = 0$$

the middle term is 0 by Lemma 5.

LEMMA 7. *For the Ricci tensor we have*

$$R_{\alpha\beta} = 0, \tag{57}$$

$$R_{\alpha\beta^*} = \frac{\partial^2 \log |G|}{\partial z_\alpha \partial \bar{z}_\beta} \tag{58}$$

where G is the determinant

$$G = \|g_{\alpha\beta^*}\|_{\alpha,\beta=1,\cdots,n}.$$

PROOF. (57) follows from

$$R_{\alpha\beta^*} = g^{lk}R_{l\alpha\beta k}$$

by Lemma 5, and (58) follows from (40).

9. **Positive fundamental tensor.** Finally, we make the assumption that the matrix $g_{\alpha\beta^*}$ is positive-definite. Consequently, for any vector ζ_i the "square length"

$$\phi = g^{ij}\zeta_i\zeta_j = 2g^{\alpha\beta^*}\zeta_\alpha\overline{\zeta_\beta} \tag{59}$$

is non-negative. For the Laplacean we have

$$\Delta\phi = g^{ij}\phi_{,i,j} = g^{\alpha\beta^*}\phi_{,\alpha,\beta^*} + g^{\alpha^*\beta}\phi_{,\alpha^*,\beta}, \tag{60}$$

and by Lemma 4 this is

$$2g^{\alpha\beta^*}\frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

Now, if we herein put $z_\alpha = x_\alpha + iy_\alpha$, then the resulting expression is a type of Laplacean in the variables (x_α, y_α) to which Lemma 1 will apply.

If Φ is the function occurring in (55), the positiveness of the tensor is expressed by the relation

$$\frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}\zeta^\alpha\overline{\zeta^\beta} \geqq 0, \tag{61}$$

which is supposed to hold for every vector ζ^α. Also, we shall say that the Ricci curvature is nonpositive if $R_{\alpha\beta^*}\zeta^\alpha\zeta^{\beta^*} \geqq 0$, and on account of (58) this is

$$\frac{\partial^2 \log G}{\partial z_\alpha \partial \bar{z}_\beta}\zeta^\alpha\overline{\zeta^\beta} \geqq 0, \tag{62}$$

which is the same type of relation as (61) is. It is not hard to see that relations (61) and (62) must hold for arbitrary complex numbers $\zeta^1, \cdots, \zeta^n$. In particular, if we introduce a complex curve $z_\alpha = z_\alpha(\omega)$, $\omega = u + iv$, and $\zeta^\alpha = dz_\alpha/d\omega$, then we must have $\partial^2\phi/\partial\omega\partial\bar{\omega} \geqq 0$, that is, $\partial^2\phi/\partial u^2 + \partial\phi^2/\partial v^2 \geqq 0$. In other words the function Φ must be a subharmonic function on any analytic curve. Similarly in the case (62), the function $\log G$ must be subharmonic in the manner stated.

III. COMPLEX SPACES

10. **Vector fields.** A complex space is a coordinate space in $2n$ real variables. Every basic neighborhood is the topological image of a neighborhood of the Euclidean space in n complex parameters. If two basic neighborhoods intersect, then the two images of the intersection are mapped into each other by analytic functions

$$z'_\alpha = \phi_\alpha(z_1, \cdots, z_n), \qquad \alpha = 1, \cdots, n. \tag{63}$$

It is easily seen that all concepts and results of Chapter II can be extended to a complex space in the large. In particular we assume the existence of a tensor g_{ij} with all properties and restrictions as enumerated before. We define a *restrained* vector field as before, and on a compact space every vector field is again restrained.

LEMMA 8. *If the covariant components ζ_α are complex analytic then for the scalar*

$$\phi = 2g^{\alpha\beta^*}\zeta_\alpha\zeta_{\beta^*} \tag{64}$$

we have

$$4^{-1}\Delta\phi = g^{\rho\sigma^*}g^{\alpha\beta^*}\zeta_{\alpha,\rho}\zeta_{\beta^*,\sigma^*} - R^{\alpha\beta^*}\zeta_\alpha\zeta_{\beta^*}.$$

If however the contravariant components ζ^α are analytic then we have

$$4^{-1}\Delta\phi = g^{\rho\sigma^*}g_{\alpha\beta^*}\zeta^\alpha{}_{,\rho}\zeta^{\beta^*}{}_{,\sigma^*} + R_{\alpha\beta^*}\zeta^\alpha\zeta^{\beta^*}.$$

PROOF. In the first case we have on basis of Lemmas 4 and 6

$$4^{-1}\Delta\phi = g^{\rho\sigma^*}(g^{\alpha\beta^*}\zeta_\alpha\zeta_{\beta^*})_{,\rho,\sigma^*} = g^{\rho\sigma^*}g^{\alpha\beta^*}\zeta_{\alpha,\rho}\zeta_{\beta^*,\sigma^*} + g^{\rho\sigma^*}g^{\alpha\beta^*}\zeta_{\alpha,\rho,\sigma^*}\zeta_{\beta^*}.$$

Now

$$\zeta_{\alpha,\rho,\sigma^*} = \zeta_{\alpha,\rho,\sigma^*} - \zeta_{\alpha,\sigma^*,\rho} = -\zeta_\lambda R^{\lambda}_{\alpha,\sigma^*,\rho},$$

and by Lemma 6 this is

$$+\zeta_\lambda R^{\lambda}_{\rho\alpha\sigma^*} = -\zeta_\lambda R^{\lambda}_{\rho\sigma^*\alpha}.$$

Therefore

$$g^{\rho\sigma^*}g^{\alpha\beta^*}\zeta_{\alpha,\rho,\sigma^*}\zeta_{\beta} = -\zeta_\lambda g^{\alpha\beta^*}R^{\lambda}_{\alpha}\zeta_{\beta^*} = -\zeta_\lambda\zeta_{\beta^*}R^{\lambda\beta^*}.$$

In the second case we have, as claimed,

$$\begin{aligned} g^{\rho\sigma^*}g_{\alpha\beta^*}\zeta^{\alpha}{}_{,\rho,\sigma^*}\zeta^{\beta^*} &= -g^{\rho\sigma^*}g_{\alpha\beta^*}\zeta^{\lambda}R^{\alpha}_{\lambda\rho\sigma^*}\zeta^{\beta^*} = -g^{\rho\sigma^*}R_{\beta^*\lambda\rho\sigma^*}\zeta^{\beta^*}\zeta^{\lambda} \\ &= -g^{\rho\sigma^*}R_{\beta^*\sigma^*\lambda\rho}\zeta^{\beta^*}\zeta^{\lambda} = R_{\beta\lambda}\zeta^{\beta^*}\zeta^{\lambda}. \end{aligned}$$

From Lemma 8 we obtain directly the following theorem, the first half of which is an unqualified extension of the theorem of Schwarz and Noether from one to several complex variables.

THEOREM 5. *On a compact space of negative Ricci curvature there exists no one-parametric group of analytic homeomorphisms whatsoever. More specifically, if on any space in a restricted vector field the contravariant components* ζ^α *are complex analytic and* $-R_{\alpha\beta^*}\zeta^\alpha\zeta^{\beta^*}\leq 0$ *holds, then we of necessity have* $\zeta^{\alpha}{}_{,\beta}=0$ *and* $R_{\alpha\beta^*}\zeta^\alpha\zeta^{\beta^*}=0$.

If however the covariant components ζ_α *are analytic then the inequality* $-R^{\alpha\beta^*}\zeta_\alpha\zeta_{\beta^*}\geq 0$ *must be a strict equality and it implies* $\zeta_{\alpha,\beta}=0$.

It should be noted that the second half of the theorem is *not* restricted to local gradient fields as was Theorem 1.

It is very easy to extend Theorem 5 from vectors to tensors.

THEOREM 6. *If a compact space has positive Ricci curvature then there exists no analytic covariant tensor* $\zeta_{\alpha_1\ldots\alpha_p}$ *whatsoever, and for negative curvature there exists no analytic contravariant tensor whatsoever.*

PROOF. For instance, in the case $p=2$, if we put

$$\phi = 4g^{\alpha_1\beta^*_1}g^{\alpha_2\beta^*_2}\zeta_{\alpha_1\alpha_2}\zeta_{\beta^*_1\beta^*_2}$$

we obtain

$$8^{-1}\Delta\phi = g^{\rho\sigma^*}g^{\alpha_1\beta_2}g^{\alpha_2\beta_1}\zeta_{\alpha_1\alpha_2,\rho}\zeta_{\beta^*_1\beta^*_2,\sigma^*} + A$$

where

$$A = g^{\rho\sigma^*}g^{\alpha_1\beta_1}g^{\alpha_2\beta_2}\zeta_{\alpha_1\alpha_2,\rho,\sigma^*}\zeta_{\beta^*_1\beta^*_2}.$$

However,

$$\begin{aligned} \zeta_{\alpha_1\alpha_2,\rho,\sigma^*} &= \zeta_{\alpha_1\alpha_2,\rho,\sigma^*} - \zeta_{\alpha_1\alpha_2,\sigma^*,\rho} = -\zeta_{\lambda\alpha_2}R^{\lambda}{}_{\alpha_1\sigma^*\rho} - \zeta_{\alpha_1\lambda}R^{\lambda}{}_{\alpha_2\sigma^*\rho} \\ &= -\zeta_{\lambda\alpha_2}R^{\lambda}{}_{\rho\alpha_1\sigma^*} - \zeta_{\alpha_1\lambda}R^{\lambda}_{\rho}{}_{\alpha_2\sigma^*}. \end{aligned}$$

Therefore

$$A = -\zeta_{\lambda\alpha_2}\zeta_{\beta^*_1\beta^*_2}R^{\lambda\beta^*_1}g^{\alpha_2\beta^*_2} - \zeta_{\alpha_1\lambda}\zeta_{\beta^*_1\beta^*_2}R^{\lambda\beta^*_2}g^{\alpha_1\beta^*_1},$$

and the conclusion follows.

11. Meromorphic functions. We consider a (self-adjoint) vector field which in the neighborhood of every point can be represented in the form

$$\zeta_\alpha = \frac{\partial f}{\partial z_\alpha}, \qquad \zeta_{\alpha^*} = \overline{\zeta_\alpha} = \frac{\partial \bar f}{\partial \bar z_\alpha}. \tag{65}$$

It is *not* a local gradient field in the proper sense unless $f = \bar f$; that is, if f is real-valued, and in our application it will definitely not be so. We introduce the associate vector

$$\eta_\alpha = \frac{\partial \bar f}{\partial z_\alpha}, \qquad \eta_{\alpha^*} = \frac{\partial f}{\partial \bar z_\alpha}.$$

These two vectors are not subject to Lemma 4; however they have the following alternate properties

$$\begin{aligned} \zeta_{\alpha,\beta} &= \zeta_{\beta,\alpha}, & \zeta_{\alpha^*,\beta^*} &= \zeta_{\beta^*,\alpha^*}, \\ \eta_{\alpha,\beta} &= \eta_{\beta,\alpha}, & \eta_{\alpha^*,\beta^*} &= \eta_{\beta^*,\alpha^*}, \\ \zeta_{\alpha,\beta^*} &= \eta_{\beta^*,\alpha}, & \zeta_{\alpha,\beta} &= \eta_{\beta,\alpha^*}. \end{aligned}$$

For $\phi = 2g^{\alpha\beta^*}\zeta_\alpha\zeta_{\beta^*}$ we obtain

$$8^{-1}\Delta\phi = A + B + C$$

where

$$\begin{aligned} A &= g^{\rho\sigma^*}g^{\alpha\beta^*}(\zeta_{\alpha,\rho}\zeta_{\beta^*,\sigma^*} + \zeta_{\alpha,\sigma^*}\zeta_{\beta^*,\rho}), \\ B &= g^{\rho\sigma^*}g^{\alpha\beta^*}\zeta_{\alpha,\rho,\sigma^*}\zeta_{\beta^*}, \\ C &= g^{\rho\sigma^*}g^{\alpha\beta^*}\zeta_\alpha\zeta_{\beta^*,\rho,\sigma^*}. \end{aligned}$$

If we substitute $\zeta_{\alpha,\rho,\sigma^*} = \zeta_{\rho,\alpha,\sigma^*} = (\zeta_{\rho,\alpha,\sigma^*} - \zeta_{\rho,\sigma^*,\alpha}) + \zeta_{\rho,\sigma^*,\alpha} = \zeta_{\rho,\sigma^*,\alpha} - \zeta_\lambda R^\lambda{}_{\rho\alpha\sigma^*}$ we obtain

$$B = g^{\alpha\beta^*}(g^{\rho\sigma^*}\zeta_{\rho,\sigma^*})_{,\alpha}\zeta_{\beta^*} - R^{\lambda\beta^*}\zeta_\lambda\zeta_{\beta^*},$$

and if we put

$$\zeta_{\beta^*,\rho,\sigma^*} = \eta_{\rho,\beta^*,\sigma^*} = \eta_{\rho,\sigma^*,\beta^*} + (\eta_{\rho,\beta^*,\sigma^*} - \eta_{\rho,\sigma^*,\beta^*}) = \eta_{\rho,\sigma^*,\beta} + \zeta_\lambda R^\lambda{}_{\rho\beta^*\sigma^*}$$

we obtain by Lemma 5

$$C = g^{\alpha\beta^*}(g^{\rho\sigma^*}\eta_{\rho,\sigma^*})_{,\beta^*}\zeta_\alpha.$$

Finally we introduce the assumption

$$g^{\rho\sigma^*}\zeta_{\rho,\sigma^*} = 0, \tag{66}$$

that is,

$$\Delta f = 0. \tag{67}$$

This will also imply $g^{\rho\sigma^*}\eta_{\rho,\sigma^*} = 0$, and if for the sake of a later application we interchange the variables (z_α) and $(\bar{z}_\alpha)$ we obtain the following lemma.

Lemma 9. *If on a compact space with positive Ricci curvature a (self-adjoint) vector field* ζ_i *has the property that in the neighborhood of every point the components* ζ_{α^*} *can be represented in the form*

$$\zeta_{\alpha^*} = \frac{\partial f}{\partial \bar{z}_\alpha} \quad \text{with} \quad \Delta f = 0 \tag{68}$$

then $\zeta_{\alpha^*} = 0$, *that is,* f *is complex analytic.*

If the curvature is only non-negative then $\zeta_{\alpha^*,i} = 0$, *that is, the derivatives* $\partial f/\partial \bar{z}_\alpha$ *are not necessarily* 0 *but have covariant derivative* 0.

Finally we shall have the following theorem.

Theorem 6. *If a compact space with positive Ricci curvature is subdivided into simplices* $S_1, \cdots, S_t$, *if for each* r, $1 \leq r \leq t$, *there is defined a meromorphic functional element* $\phi_r(z_1, \cdots, z_n)$ *in some neighborhood* U_r *of the closure of* S_r, *and if each difference* $\phi_p - \phi_q$ *is holomorphic in the intersection of* U_p *and* U_q *(whenever not empty), then there exists one meromorphic function* $\Phi(z_1 \cdots z_n)$ *in* S *such that, for each* U_r, *the difference* $\Phi - \phi_r$ *is holomorphic.*

Proof. The proof proceeds as follows. We replace each ϕ_r in U_r by another function $f_{(r)}$ which has no singularities but which on the other hand may also depend on the conjugate values $\bar{z}_\alpha$. In other words, $\partial f_{(r)}/\partial \bar{z}_\alpha$ need not be not equal to 0. However, each $f_{(r)}$ shall be a solution of the Laplacean $\Delta f = 0$, and the difference $f_{(p)} - f_{(q)}$ shall be equal to $\phi_p - \phi_q$ for all (p, q). If such functions $f_{(r)}$ can be found then the components

$$\zeta_{\alpha^*} = \frac{\partial f_{(r)}}{\partial \bar{z}_\alpha}, \qquad \alpha = 1, \cdots, n,$$

are uniquely defined in S, and Lemma 9 applies immediately. Thus, for positive curvature, the components all vanish. In other words, the functions $f_{(r)}$ are automatically analytic, and the expression

$$\Phi = \phi_p - f_{(p)} = \phi_q - f_{(q)}$$

will define a meromorphic function in S. A similar proof holds for non-negative curvature. Therefore the only remaining task is the construction of the functions $f_{(r)}$. This is a problem in potential theory and has a solution in either real or Hermitian space. We shall only indicate its solution since an elaboration of details would lead us far afield.

Suppose we are dealing with a compact m-dimensional space with a real positive fundamental form. We set up the Laplacean, and it is known[12] that there always exists a Green's function $G(x;\xi)$. It has the proper type of singularity for $x=\xi$, and it satisfies the relation

$$\Delta_x G(x;\xi) = -1.$$

Let S_r and U_r have the same meaning as in the theorem and let B_{pq} denote the $(n-1)$-dimensional face separating S_p and S_q, if such a face exists. In the integral to follow each such face will appear twice with opposite orientations. Now, for each r, let $\phi_{(r)}$ be an "arbitrary" function in U_r with "singularities." By this we mean that it need be defined and differentiable only in some subset of U_r. However, the differences

$$\phi_{pq} = \phi_p - \phi_q \tag{69}$$

shall be each defined and differentiable in some neighborhood of the closure of B_{pq}. We now set up the Cauchy-Green integral

$$\frac{1}{\omega}\sum_{p,q=1}^{t}\int_{B_{pq}}\sum_{\nu=1}^{m}\left(\phi_{pq}(\xi)\,\frac{\partial G(x,\xi)}{\partial \xi_\nu} - G(x,\xi)\,\frac{\partial \phi_{pq}(\xi)}{\partial \xi_\nu}\right)\sigma^\nu.$$

The symbol σ^ν denotes a certain invariant $(m-1)$-dimensional external differential form, which in the Euclidean case is

$$(-1)^\nu d\xi_1\cdots d\xi_{\nu-1}d\xi_{\nu+1}\cdots d\xi_m,$$

whereas ω is a numerical constant. For variable x the integral as it stands defines a function in the interior of each S_r. Denoting this function by $f_{(r)}(x)$, we can show that it can be continued beyond S_r through the interior of each $(m-1)$-dimensional face, that $f_{(p)}(x) - f_{(q)}(x)$ has the value $\phi_{pq}(x)$ on B_{pq}, and that it is a solution of the Laplacean.[13] All this would be true for arbitrary functions $\phi_{pq}(\xi)$ on B_{pq}, but then $f_{(r)}(x)$ need not be continuable through the interior of the boundary faces of dimension not greater than $m-2$. If however the prescribed saltus function has the special form (69) then $f_{(r)}(x)$ can be continued into an entire neighborhood of the closure of S_r, as needed.

In our complex space with a Laplacean of Hermitian type the existence of the Green's function could be proven by an adaptation of the proof for the real case. However the duplication can be avoided, and the complex case can be directly reduced to the real case, if we make the additional assumption on S that it is basically a real space which in some of its allowable coordinates is also Hermitian. By this we mean that there exists a $2n$-dimensional Riemannian space S_0 which is in topological correspondence with S. Each allowable coordinate system in S is in correspondence with some coordinate system of S_0, and the Laplacean on S is a transcription into complex parameters of the real Laplacean as defined on S_0.

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[12] Compare, for instance, S. Bochner, *Analytic mapping of compact Riemann spaces into Euclidean space*, Duke Math. J. vol. 3 (1937) pp. 339–354.

[13] See the paper quoted in footnote 3, p. 657.

12. **Subspaces of Euclidean space.** The following statements are of a local character. Suppose we are given several analytic covariant vector fields in a neighborhood of n complex parameters. We denote them by $\zeta_{k/\alpha}$, where the subscript k identifies the vector, and we attempt to define a fundamental tensor by the formula

$$g_{\alpha\beta^*} = \sum_{k=1}^{l} \zeta_{k/\alpha}\zeta_{k/\beta^*}.$$

Our first concern must be to secure restriction (53). The obvious way of securing it is to assume that we have

$$\zeta_{k/\alpha} = \frac{\partial \phi_k}{\partial z_\alpha} \tag{70}$$

where $\phi_k(z)$ is an analytic scalar, because in this case

$$g_{\alpha\beta^*} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}\left(\sum_k |\phi_k|^2 \right).$$

This is a crude way of securing (53) and there probably is some more elaborate condition; we shall deal with it nevertheless, since geometrically it corresponds to a mapping of our z-neighborhood into a W-neighborhood by the functions $W_k = \phi_k(f)$.

THEOREM 7. *If*

$$W_k = \phi_k(z_1 \cdots z_n), \qquad k = 1, \cdots, l,$$

is a parametrisation of a subspace of Euclidean space, then its Hermitian tensor

$$g_{\alpha\beta^*} = \sum_{k=1}^{l} \frac{\partial \phi_k}{\partial z_\alpha}\frac{\partial \bar{\phi}_k}{\partial \bar{z}_{\alpha^*}}$$

generates a negative Ricci curvature.

PROOF. If we evaluate the determinant G of $g_{\alpha\beta^*}$ then we find for its value the expression

$$\sum_{\nu=1}^{\sigma_{l,n}} D_\nu \bar{D}_\nu, \tag{71}$$

where each D_ν is the value of some n-dimensional determinant of the n by l matrix

$$\left|\frac{\partial \phi_k}{\partial z_\alpha}\right|, \qquad \alpha = 1, \cdots, n; \; k = 1, \cdots, l. \tag{72}$$

We now put

$$R_{\alpha\beta^*} = \frac{\partial^2 \log G}{\partial z_\alpha \partial \bar{z}_\beta}$$

and form the sum

$$- R_{\alpha\beta^*}\lambda^\alpha\bar{\lambda}^\beta \tag{73}$$

for an arbitrary complex number λ^α. Now, if we carry out the differentiations and if we put

$$\eta^p = \frac{\partial D^p}{\partial z_\alpha}\lambda^\alpha,$$

then it turns out that the sum (71) has the value

$$G^{-2}\sum_{p,q} \left| \eta^p D^q - \eta^q D^p \right|^2$$

and this is non-negative. Also, if the matrix (71) has rank n, this must be strictly positive for some λ^α, as claimed.

For the positiveness of (71) it is not necessary to assume that our vectors have the special form (70); however in general the matrix (73) is not the Ricci tensor and has no immediate geometrical interpretation.

13. **Commutative analytic groups.** We now assume the existence of contravariant analytic vectors ζ_k^α and we set up the corresponding tensor

$$g^{\alpha\beta^*} = \sum_{k=1}^{l} \zeta_k^\alpha \bar{\zeta}_k^\beta.$$

If we denote its determinant by H, then by our last remark the matrix

$$-\frac{\partial^2 \log H}{\partial z_\alpha \partial \bar{z}_\beta}$$

is positive definite. If we introduce the inverse tensor $g_{\alpha\beta^*}$, then $G=H^{-1}$, and this would prove that the Ricci curvature is positive, provided $g_{\alpha\beta^*}$ satisfies restriction (53). Now the identity

$$g_{\beta\epsilon^*}g^{\epsilon^*\alpha} = \delta_\beta^\alpha$$

implies

$$\frac{\partial g_{\beta\epsilon^*}}{\partial z_\gamma} g^{\epsilon^*\alpha} + g_{\beta\epsilon^*}\frac{\partial g^{\epsilon^*\alpha}}{\partial z_\gamma} = 0$$

or

$$\frac{\partial g_{\beta\rho^*}}{\partial z_\gamma} = -g_{\beta\epsilon^*}\frac{\partial g^{\epsilon^*\alpha}}{\partial z_\gamma} g_{\alpha\rho^*}.$$

In applying (53) we hence obtain

$$\frac{\partial g^{\epsilon^*\alpha}}{\partial z_\beta} g^{\beta\rho^*} = \frac{\partial g^{\rho^*\alpha}}{\partial z_\beta} g^{\beta\epsilon^*}$$

which is equivalent with

$$\sum_{p,q}\frac{\partial \zeta_p^\alpha}{\partial z_\beta}\overline{\zeta_p^\epsilon}\zeta_q^\beta\overline{\zeta_q^\rho} = \sum_{pq}\frac{\partial \zeta_q^\alpha}{\partial z_\beta}\overline{\zeta_q^\rho}\zeta_p^\beta\overline{\zeta_p^\epsilon}.$$

The only obvious way of securing this relation is to assume that

$$\frac{\partial \zeta_p^\alpha}{\partial z_\beta}\zeta_q^\beta - \frac{\partial \zeta_q^\alpha}{\partial z_\beta}\zeta_p^\beta = 0$$

for all α, p, q. But this is the classical condition for commutative groups. Thus if the rank of the matrix $|\zeta_p^\alpha|$ is n, and the group operators are commutative, they give rise to an Hermitian metric with positive curvature.

PRINCETON UNIVERSITY

BOOK REVIEW

Vorlesungen über Differential- und Integralrechnung. Vol. 1. *Funktionen einer Variablen.* By A. Ostrowski. Basel, Birkhäuser, 1945. 12+373 pp. 47.50 s.fr.

According to the author's statement, this volume is intended to serve as the basis of a one semester course introducing the student to the fundamental concepts and techniques of analysis. The book consists of seven chapters. Chapter I is devoted to a treatment of the real number system, not as something built up in stages from the integers, but as a system described axiomatically. Mathematical induction, ordered fields, and Dedekind cuts are discussed carefully and at length. In Chapter II the limit concept for sequences and functions is introduced, first for convergence to zero, and then in general. Chapters III and IV introduce the two fundamental concepts of calculus. The definition of the definite integral precedes that of the derivative. A proof, employing Dedekind cuts, is given of the existence of the integral of a continuous function, but the author suggests that the student may very well pass over the proof lightly at first. The development of technique in differentiation and integration is attended to in Chapters V and VI. These chapters also contain discussions of monotone functions and their inverses, and of logarithmic and exponential functions. Curiously, the general development in Chapter III is interrupted by placing a section on the trigonometric functions between the treatment of continuous functions and the definition of the definite integral. The book concludes with a chapter containing the following topics: uses of the first and second derivatives in studying the properties of functions and in tracing curves; indeterminate forms; series expansions of the logarithm and the inverse tangent; Taylor's formula with remainder; Euler's formulas.

The emphasis throughout is upon hypotheses, definitions, and fundamental concepts. The book begins with a brief discussion of the characteristics of mathematics as a discipline. There are some good observations about the importance of clear definitions and about the necessity for proof even of the intuitively obvious. The proofs of certain theorems (the properties of functions continuous on a closed interval, and the sufficiency of the Cauchy convergence criterion) are omitted. It is interesting to note, in comparison with American practice in introductions to calculus, that nearly one fourth of the book is taken up with the subject matter of the first two chapters alone.

Roughly one fourth of the book is devoted to exercises. Virtually all of these pertain strictly to analysis; there are no applications to physics or the other sciences, nor to the calculation of volumes and arc lengths. Analytic geometry is used sparingly. There are relatively few (48) figures; these are uneven in quality.

Several interesting and unusual features deserve brief mention. There is a short discussion of the possibility of defining the inverse trigonometric functions by definite integrals, and then studying the trigonometric functions as the inverses of the functions so defined. The history of elliptic integrals and the theory of elliptic functions is cited in this connection (pp. 206–207). There is a beautifully simple discussion of the law of natural growth (pp. 262–263). There is a more than usually full discussion of the computing of logarithms by series (pp. 337–340).

ANGUS E. TAYLOR

NOTES

Professor E. S. Pearson of University College, London, has received the honor of being named Commander of the Order of the British Empire.

The following have been elected honorary members of the Royal Society of Edinburgh: Professor P. A. M. Dirac of the University of Cambridge, Professor Emeritus Jacques Hadamard of the Collège de France and l'École Polytechnique, and Professor Emeritus G. H. Hardy of the University of Cambridge.

Among the new members recently elected to the American Academy of Arts and Sciences were the following members of the American Mathematical Society: Professor J. L. Barnes of Tufts College, Professor Subrahmanyan Chandrasekhar of Yerkes Observatory, University of Chicago, and Professor O. E. Neugebauer of Brown University.

Professor Theodor von Kármán of the Guggenheim Aeronautics Laboratory, California Institute of Technology, has been elected to membership in the Royal Society.

Professor Emeritus L. L. Dines of Carnegie Institute of Technology has been awarded the honorary degree of doctor of laws by the University of Saskatchewan.

Professor Norbert Wiener of Massachusetts Institute of Technology has received an honorary degree from Tufts College.

Associate Professor R. F. Rinehart of the Case School of Applied Science has received the Medal for Merit from the United States Navy and the Medal of Freedom from the United States Army.

Professor B. L. Remick of Kansas State College of Agriculture and Applied Science has retired with the title emeritus.

Professor D. H. Richert of Bethel College, North Newton, Kansas, has retired with the title emeritus.

Dr. D. S. Miller and Dr. W. C. Strodt have received fellowship appointments for 1946–1947 from the National Research Council.

Assistant Professor T. A. Bancroft of Iowa State College of Agriculture and Mechanical Arts has been appointed to an associate professorship at the University of Georgia.

Dr. Valentine Bargmann of the Institute for Advanced Study has been appointed visiting lecturer at Princeton University.

Dr. W. D. Berg has been appointed to a visiting assistant professorship at Kenyon College.

Associate Professor R. A. Beth of Michigan State College has been appointed to a professorship of physics at Western Reserve University.

Associate Professor M. T. Bird of Utah State Agricultural College has been appointed to an assistant professorship at Allegheny College.

Assistant Professor D. H. Blackwell of Howard University has been promoted to an associate professorship.

Dr. Archie Blake has been appointed senior statistician in the Office of the Army Surgeon General, Washington, D. C.

Assistant Professor W. E. Bleick of the Postgraduate School, United States Naval Academy, has been promoted to an associate professorship.

Associate Professor Salomon Bochner of Princeton University has been promoted to a professorship.

Dr. T. A. Botts has been appointed to an assistant professorship at the University of Delaware.

Associate Professor Richard Brauer of the University of Toronto has been promoted to a professorship.

Dr. H. K. Brown of Lehigh University has been promoted to an assistant professorship of mechanics.

Assistant Professor L. H. Bunyan of Rutgers University has been promoted to an associate professorship.

Dr. R. S. Burington has been appointed chief mathematician in the Bureau of Ordnance.

Dr. A. W. Burks of Swarthmore College has been appointed to an assistant professorship in philosophy at the University of Michigan.

Mr. W. R. Callahan of Yale University has been appointed to an assistant professorship of mechanical engineering at Northeastern University, Boston, Massachusetts.

Dr. J. D. Campbell of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. C. R. Cassity of the New Mexico School of Mines has been promoted to an associate professorship.

Assistant Professor Harold Chatland of Montana State University has been promoted to an associate professorship.

Associate Professor Claude Chevalley of Princeton University has been promoted to a professorship.

Dr. D. E. Christie of Bowdoin College has been promoted to an assistant professorship.

Dr. Paul Civin of the University of Buffalo has been appointed to an assistant professorship at the University of Oregon.

Mr. C. A. Cope has been appointed to an assistant professorship at the University of Georgia.

Dr. Robert Cortell of the College of the City of New York has been promoted to an assistant professorship.

Dr. V. F. Cowling of Ohio State University has been appointed to an assistant professorship at Lehigh University.

Associate Professor J. T. Culbertson of Southwestern University, Georgetown, Texas, has been appointed to an assistant professorship of Philosophy at Michigan State College.

Mr. W. J. A. Culmer has been appointed to an assistant professorship at Hamline University, St. Paul, Minnesota.

Dr. L. E. Cunningham has been appointed to an assistant professorship of astronomy at the University of California.

Assistant Professor John DeCicco of the Illinois Institute of Technology is on leave and will be a visiting associate professor at Columbia University.

Associate Professor C. H. Denbow of Ohio University, Athens, Ohio, has been appointed to an associate professorship at the Postgraduate School, United States Naval Academy.

Associate Professor W. J. Dixon of the University of Oklahoma has been appointed to an associate professorship at the University of Oregon.

Dr. C. L. Dolph has been appointed a member of the technical staff of Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. D. M. Dribin has been appointed research analyst with the Army Security Agency, Washington, D. C.

Dr. Roy Dubisch of Montana State University has been appointed to an assistant professorship at Syracuse University.

Dr. W. D. Duthie of the University of Michigan has been appointed to a professorship in aerological engineering at the Postgraduate School, United States Naval Academy.

Dr. J. E. Eaton of Queens College, Flushing, New York, has been appointed mathematician at the Naval Research Laboratory.

Mr. W. R. Eikelberger has been appointed to an assistant professorship at the University of Denver.

Mr. D. H. Erkiletian of the University of Missouri has been promoted to an assistant professorship.

Dr. A. B. Farnell of the United States Military Academy has been appointed to an assistant professorship at the University of Colorado.

Dr. Herbert Federer of Brown University has been promoted to an assistant professorship.

Assistant Professor D. G. Fulton of the University of New Hampshire has been appointed to an associate professorship at Tufts College.

Dr. B. H. Gere of the Postgraduate School, United States Naval Academy, has been promoted to an assistant professorship.

Dr. David Gilbarg of the Naval Ordnance Laboratory has been appointed to an assistant professorship at Indiana University.

Assistant Professor B. E. Gillam of Drake University has been promoted to an associate professorship.

Dr. L. J. Green of Case School of Applied Science has been promoted to an assistant professorship.

Mr. Edison Greer of the University of Kansas has been appointed to an associate professorship at Kansas State College.

Assistant Professor P. R. Halmos of Syracuse University has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor H. J. Hamilton of Pomona College has been promoted to an associate professorship.

Assistant Professor R. W. Hamming of the University of Louisville has been appointed a member of the technical staff at Bell Telephone Laboratories, Murray Hill, New Jersey

Dr E. A. Hedberg has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor A. E. Heins of Purdue University has been appointed to an associate professorship at the Carnegie Institute of Technology.

Dr. Olaf Helmer has been appointed research mathematician at the Naval Ordnance Test Station, Inyokern, California.

Dr. R. G. Helsel of Ohio State University has been promoted to an assistant professorship.

Dr. Erik Hemmingsen has been appointed to an assistant professorship at the University of Georgia.

Dr. J. G. Herriot of Stanford University has been promoted to an assistant professorship.

Dr. Edwin Hewitt of the Institute for Advanced Study has been appointed to an assistant professorship at Bryn Mawr College.

Assistant Professor F. E. Hohn of Guilford College, Guilford College, North Carolina, has been appointed to an assistant professorship at the University of Maine.

Dr. P. L. Hsu has been appointed to an associate professorship at the University of North Carolina.

Associate Professor G. B. Huff of the Southern Methodist University has been appointed to an associate professorship at the University of Georgia.

Mr. W. N. Huff of the University of Rochester has been appointed to an assistant professorship at the University of Oklahoma.

Dr. H. D. Huskey of the University of Pennsylvania has been appointed to an assistant professorship at the University of Oklahoma.

Assistant Professor S. B. Jackson of the University of Maryland has been promoted to an associate professorship.

Dr. L. (G.) Jones has been appointed research engineer at General Precision Laboratory, Pleasantville, New York.

Associate Professor E. C. Kennedy of the Texas College of Arts and Industries has accepted a position as research engineer with the Consolidated Vultee Aircraft Corporation, Daingerfield, Texas.

Dr. D. E. Kibbey of the United States Military Academy has been appointed to an assistant professorship at Syracuse University.

Associate Professor B. O. Koopman of Columbia University has been promoted to a professorship.

Dr. D. M. Krabill of West Virginia University has been appointed to an associate professorship at Bowling Green State University, Bowling Green, Ohio.

Dr. Mary E. Ladue of Barnard College, Columbia University, has been promoted to an assistant professorship.

Dr. J. A. Larrivee of the Applied Physics Laboratory, Johns Hopkins University, has been appointed to an assistant professorship at the University of Vermont.

Dr. J. P. LaSalle has been appointed to an assistant professorship at the University of Notre Dame.

Miss Mary A. Lee has been appointed to an assistant professorship at Sweet Briar College.

Mr. (T.) H. Lee of Louisiana State University has been appointed to an adjunct professorship at the University of South Carolina.

Assistant Professor B. A. Lengyel of the University of Rochester has accepted a position as physicist with the Naval Research Laboratory, Washington, D. C.

Associate Professor D. C. Lewis of the University of New Hampshire has been appointed to a professorship at the University of Maryland.

Dr. J. V. Lewis has been appointed to an assistant professorship at the University of Nevada.

Assistant Professor C. C. Lin of Brown University has been promoted to an associate professorship.

Dr. J. F. Locke of the United States Naval Academy has been appointed to a professorship at Birmingham Southern College

Assistant Professor W. L. Massey of the University of Chattanooga has been promoted to an associate professorship.

Professor C. E. Miller of the University of New Brunswick has been appointed to a professorship at the University of Saskatchewan.

Mr. R. A. Miller has been appointed to an assistant professorship at the University of Mississippi.

Dr. W. M. Miller has been appointed to a professorship at Robert College, Istanbul, Turkey.

Mr. W. B. Morgan has been appointed to an assistant professorship at Bluffton College, Bluffton, Ohio.

Dr. H. T. Muhly of the United States Naval Academy has been promoted to an assistant professorship.

Associate Professor W. R. Murray of Franklin and Marshall College, Lancaster, Pennsylvania, has been promoted to a professorship.

Assistant Professor E. N. Nilson of Mount Holyoke College has been appointed research mathematician at Pratt and Whitney Aircraft, East Hartford, Connecticut.

Assistant Professor J. W. Odle of Pennsylvania State College has been appointed acting head of the mathematics section, Naval Ordnance Test Station, Inyokern, California.

Mr. E. R. Ott has been appointed to an associate professorship at Rutgers University.

Dr. O. G. Owens of the University of California has been appointed to an assistant professorship at the University of Nevada.

Associate Professor Gordon Pall of McGill University has been appointed to a professorship at Illinois Institute of Technology.

Dr. A. M. Peiser has been appointed to an assistant professorship at Rutgers University.

Dr. C. R. Phelps has been appointed to an assistant professorship at Rutgers University.

Dr. R. S. Phillips of the Massachusetts Institute of Technology has been appointed to an assistant professorship at New York University.

Dr. Harry Pollard of Yale University has been appointed to an assistant professorship at Cornell University.

Associate Professor R. G. Putnam of New York University has been promoted to a professorship.

Miss Margaret Ramsey of Linfield College, McMinnville, Oregon, has been promoted to an associate professorship.

Associate Professor G. E. Raynor of Lehigh University has been promoted to a professorship.

Mr. E. K. Ritter has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Dr. H. E. Robbins of New York University has been appointed to an associate professorship at the University of North Carolina.

Assistant Professor M. S. Robertson of Rutgers University has been promoted to an associate professorship.

Dr. D. H. Rock has been appointed to an assistant professorship at Iowa State College.

Assistant Professor Arthur Rosenthal of the University of New Mexico has been promoted to an associate professorship.

Dr. Robert Schatten of the Institute for Advanced Study has been appointed to an associate professorship at the University of Kansas.

Associate Professor C. H. W. Sedgewick of the University of Connecticut has been promoted to a professorship.

Dr. J. K. Senior of the University of Chicago has been promoted to an associate professorship.

Dr. C. B. Smith has been appointed to an associate professorship at the University of Florida.

Professor I. S. Sokolnikoff of the University of Wisconsin has been appointed to a professorship at the University of California at Los Angeles.

Mr. E. L. Stanley of Clemson Agricultural College, Clemson, South Carolina, has been promoted to an assistant professorship.

Dr. A. C. Sugar of Brown University has been appointed to an associate professorship at Oklahoma Agricultural and Mechanical College.

Dr. Feodor Theilheimer of Trinity College, Hartford, Connecticut, has been promoted to an assistant professorship.

Mr. J. S. Thompson has been elected president of The Mutual Benefit Life Insurance Company, Newark, New Jersey.

Mr. C. W. Topp of Fenn College, Cleveland, Ohio, has been promoted to an assistant professorship.

Associate Professor A. W. Tucker of Princeton University has been promoted to a professorship.

Assistant Professor C. B. Tucker of Kansas State Teachers College, Emporia, Kansas, has been promoted to an associate professorship.

Assistant Professor H. L. Turrittin of the University of Minnesota has been promoted to an associate professorship.

Assistant Professor R. J. Walker of Cornell University has been promoted to an associate professorship.

Professor H. S. Wall of the Illinois Institute of Technology has been appointed to a professorship at the University of Texas.

Dr. W. R. Wasow of New York University has been appointed to an assistant professorship at Swarthmore College.

Associate Professor M. A. Zorn of the University of California at Los Angeles has been appointed to a professorship at Indiana University.

The following appointments to instructorships are announced: Brown University: Mr. Winfield Keck; Columbia University: Dr. T. W. Anderson, Mr. J. L. Bogdanoff; Dartmouth College: Dr. William H. Durfee, Dr. W. C. G. Fraser; Fenn College, Cleveland, Ohio: Mr. F. H. McGar; University of Florida: Mr. R. G. Blake, Mr. R. E. Lee, Mr. W. L. Pickard, Mr. J. W. Young; University of Georgia: Mr. Eugene Park; Harvard University: Dr. Lowell Schoenfeld; Hofstra College, Hempstead, New York: Miss Ethel M. Hove; University of Illinois: Mr. B. E. Meserve; Johns Hopkins University: Dr. Murray Mannos; Madison College, Harrisonburg, Virginia: Miss Mary F. Suter; Massachusetts Institute of Technology: Dr. F. E. Bothwell, Dr. A. V. Martin; University of Michigan: Dr. W. F. Eberlein, Dr. Leonard Tornheim; Montana State University: Mr. Walter Hook; Mount Holyoke College: Miss Grace E. Bates; Paul Smith's College of Arts and Sciences, Paul Smiths, New York: Miss Dorothy I. Coyne; Princeton University: Mr. Richard Bellman; Rensselaer Polytechnic Institute: Mr. A. F. Bartholomay; Syracuse University: Dr. Ruth O. Goodman, Miss Kathryn A. Morgan; Temple University: Miss Irma R. Moses; United States Naval Academy: Mr. K. F. McLaughlin; University of Virginia: Mr. V. L. Klee; Washington University: Mr. R. E. Roberson; University of Wisconsin: Dr. R. E. Fullerton; Wright Junior College: Miss Alice M. Christiansen.

Assistant Professor A. S. McMaster of the University of Colorado died May 24, 1946. He had been a member of the Society since 1929.

Dr. J. A. Miller of Wallingford, Pennsylvania, died June 15, 1946, at the age of eighty-seven years. He had been a member of the Society since 1899.

Associate Professor A. E. Staniland of the University of Pittsburgh died June 28, 1946, at the age of forty-eight years.

The editors wish to correct an error which appeared in the Notes department of the July issue of the Bulletin, page 610. Dr. Shiffman's appointment at New York University was at the rank of associate professor.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

262. P. T. Bateman: *On the representations of a number as the sum of three squares.*

In determining $r_s(n)$, the number of representations of a positive integer n as the sum of s squares, it is known that the singular series $\rho_s(n)$ constructed by Hardy (Trans. Amer. Math. Soc. vol. 21 (1920) pp. 255–284) gives exact results for $3 \leqq s \leqq 8$. For $5 \leqq s \leqq 8$ Hardy proved this by showing that the function $\Psi_s(\tau) = 1 + \sum_{n=1}^{\infty} \rho_s(n) e^{\pi i \tau n}$, $\vartheta(\tau) > 0$, has exactly the same behavior under the modular subgroup Γ_2 as the function $\vartheta_3(0|\tau)^s = (\sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2})^s = 1 + \sum_{n=1}^{\infty} r_s(n) e^{\pi i \tau n}$. For $s = 3, 4$ the double series of partial fractions for $\Psi_s(\tau)$ which Hardy used to establish the modular properties of $\Psi_s(\tau)$ is no longer absolutely convergent, even though the proof is correct formally. For $s = 4$ absolute convergence is easily restored by grouping terms, but for $s = 3$ this is not possible. In this paper the case $s = 3$ is successfully treated by supplementing the Hardy method with a limit process of the kind used by Hecke in defining his generalized Eisenstein series. There are some analytical intricacies in applying the limit process, but no formal difficulties. A particular result is that for n square free, $r_3(n) = C\pi^{-1} n^{1/2} \sum_{k=1}^{\infty} (-n/k) k^{-1}$, where $C = 0$ if $n \equiv 7 \pmod 8$, $C = 16$ if $n \equiv 3 \pmod 8$, and $C = 24$ if $n \equiv 1, 2, 5, 6 \pmod 8$. (Received July 13, 1946.)

263. Garrett Birkhoff and P. M. Whitman: *Representation theory for certain non-associative algebras.*

It is known that the ways of embedding a Lie algebra L in a linear associative enveloping algebra are all obtained from a "universal" enveloping algebra $A_u(L)$ with infinite basis, by setting a suitable ideal of L equal to zero. It is shown that a corresponding theorem holds for any Jordan algebra J, but that $A_u(J)$ has a finite basis if J does. Particular examples are worked out for Lie and Jordan algebras. If L or J is "solvable," then the finite basis theorem is valid in $A_u(L)$ and $A_u(J)$. (Received July 15, 1946.)

264. A. T. Brauer: *Limits for the characteristic roots of a matrix.*

Let $A = (a_{\kappa\lambda})$ be a matrix with real or complex elements. Set $\sum_{\lambda=1}^{n} |a_{\kappa\lambda}| = R_\kappa$, $\sum_{\kappa=1}^{n} |a_{\kappa\lambda}| = T_\lambda$; $\max_{\kappa=1,2,\ldots,n}\{R_\kappa\} = R$, $\max_{\lambda=1,2,\ldots,n}\{T_\lambda\} = T$; $R_\kappa - |a_{\kappa\kappa}| = P_\kappa$, $T_\lambda - |a_{\lambda\lambda}| = Q_\lambda$. It is proved that each characteristic root ω_ν of A lies in at least one of the circles $|z - a_{\kappa\kappa}| \leqq P_\kappa$ and in at least one of the circles $|z - a_{\lambda\lambda}| \leqq Q_\lambda$. It follows that $|\omega_\nu| \leqq \min(R, T)$. This generalizes a result of Frobenius for matrices with positive elements and improves results of E. T. Browne, W. V. Parker, and A. B. Farnell. Also $\max|\omega_\nu| = \min (R, T)$ if and only if $R_1 = R_2 = \cdots = R_n$ or $T_1 = T_2 = \cdots = T_n$ and if $\arg (a_{\kappa\lambda}) = \phi + \phi_\kappa - \phi_\lambda$. If these conditions are satisfied, then $\omega = e^{i\phi} \min (R, T)$ is one of the characteristic roots. Suppose that A has real elements. If for a given m and for each λ one has $|a_{mm} - a_{\lambda\lambda}| > P_m + P_\lambda$, then the circle $|z - a_{mm}| \leqq P_m$ contains one and only one characteristic root; this root is real. If this holds for each m, then all the characteristic roots are real. (Received July 13, 1946.)

265. Ellen F. Buck and R. C. Buck: *A note on finitely-additive measures.*

In the space I of integers, the class $\mathfrak{D}_s$ of sets which are essentially finite unions of arithmetic progressions forms a finitely additive measure class. If S is a countable space and Ω is a countable family of subsets of S which are measurable under a finitely additive measure function, satisfying certain simple restrictions, there exist one-to-one point transformations of S into I which preserve measure in the Carathéodory closure of Ω. This is in a sense a stronger form of a theorem stated by Ulam. The topology in I determined by the class $\mathfrak{D}_s$ is also discussed. (Received July 12, 1946.)

266. R. C. Buck: *The measure-theoretic approach to density.*

Ordinary density, $D(A) = \lim n/a_n$, has some of the properties of a finitely additive measure on the countable space I composed of the positive integers. This paper is devoted to an analysis of a special measure μ obtained by applying the Carathéodory extension to a simple basic measure-density. The measurability and nonmeasurability of certain sets is proved using methods of elementary number theory; in particular, if α is irrational, the set $\{[\alpha n + \beta]\}$ intersects every set of positive measure in an infinite set, and is nonmeasurable. Sequences of sets are used to show that the range of values of μ is the closed unit interval. Generalized limit densities are discussed, using properties of regular summability transforms; it is shown that if A is any set having a positive limit density, determined by a regular transform, and if A does not contain a pair of consecutive integers, then there is a set B, also having a generalized density, such that $A \cup B$ and $A \cap B$ do not. (Received July 12, 1946.)

267. C. J. Everett and H. J. Ryser: *Rational vector spaces.*

Infinite-dimensional rational vectors spaces $V(R)$ are isomorphic if their cardinals are equal. Every cardinal is the cardinal of some $V(R)$. There is no universal $V(R)$. Spaces $V(R)$ with *rational* inner product possess analogues of Schwarz, Bessel, triangle, and Hadamard inequalities, and Schmidt process. Necessary and sufficient conditions that a norm be an inner product norm are given. Inner products in finite-dimensional $V(R)$ are completely determined via the Hasse invariants. Infinite inner product spaces are studied in the countable, and in the separable case. Legendre polynomials arise naturally in the former. Normal algebraic number fields $R(\rho)$ with normal maximal real subfield admit trace $T(\bar{\alpha}\beta)$ as inner product (α, β). A field with basis $\delta_1, \cdots, \delta_n$ so that $T(\bar{\delta}_i\delta_j) = \delta_{ij}$ is called an I-field. The only quadratic I-field is $R(i)$; every normal cubic is; the cyclotomic (p^α) field is if and only if $p = 2$. Dirichlet I-fields are determined. The function $\|\alpha\| = (T(\bar{\alpha}\alpha))^{1/2}$ is a pseudo-valuation of $R(\rho)$, the $\|\ \|$-completion of which is a direct sum of n real or $n/2$ complex fields. (Received June 17, 1946.)

268. J. S. Frame: *On the reduction of the conjugating representation of a finite group.*

Let γ be a column vector whose entries are the g elements γ_i of a finite group G, so arranged that the k_σ elements of a class of conjugate elements C_σ ($\sigma=1, \cdots, r$) come together. The $g\times g$ permutation matrices $R(\gamma_i)$ and $L(\gamma_i)$ of the right and left regular representations, respectively, are defined by the equations $\gamma\gamma_i=R(\gamma_i)\gamma$, $\gamma_i\gamma=L(\gamma_i)\gamma$. Write $\gamma_i^{-1}\gamma\gamma_i=P(\gamma_i)\gamma$, where $P(\gamma_i)=L(\gamma_i^{-1})R(\gamma_i)$, and call the permutation representation P of degree g, whose matrices are $P(\gamma_i)$, the *conjugating representation* of G. Because of the arrangement of group elements, P is a direct sum of transitive permutation matrix representations P_σ of degrees k_σ, $\sigma=1, \cdots, r$, each of which is the same as a group of permutations on the right cosets of the normalizer N_σ of an element of C_σ. A $g\times g$ "entry" matrix $Z=\|z_{ij}\|$ with unitary orthogonal columns is defined by assigning to the ith row in a specified order the $\sum n_\rho^2=g$ linearly independent entries (or coefficients) for the group element γ_i in a complete set of r nonequivalent irreducible representations Γ_ρ. The principal theorem states that the representation $Z^{-1}PZ$ is the direct sum of the representations $\Gamma_\rho\times\bar{\Gamma}_\rho$. Some illustrations and consequences of this are discussed. (Received July 11, 1946.)

269. S. A. Jennings: *The group ring of a class of infinite nilpotent groups.*

Let G be a nilpotent group with lower central series $G=H_1\supset H_2\supset\cdots\supset H_c\supset H_{c+1}=\{1\}$ such that H_i/H_{i+1}, $i=1, 2, \cdots, c$, is a direct product of a finite number of infinite cyclic groups, and let Γ be the group ring of G over any field of characteristic 0. The structure of Γ is studied, and in particular it is shown that Γ contains a two-sided ideal Δ such that Γ/Δ is isomorphic to the coefficient field and $\bigcap\Delta^k=0$. By taking the powers of the ideal Δ as a system of neighborhoods of 0 a natural topology may be introduced in Γ. If Γ^*, Δ^* are the completions of Γ, Δ in this topology then Δ^* is the Jacobson radical of Γ^*. The methods of this paper are similar to those of an earlier investigation of the group ring of a p-group over a field of characteristic p (Trans. Amer. Math. Soc. vol. 50 (1941) pp. 175–185). (Received July 15, 1946.)

270. B. E. Meserve: *Inequalities of higher degree.*

A system composed of a finite number of inequalities of the form $0<f$ where f is a real-valued function of real variables $x_1, x_2, \cdots, x_p$ has as *content* all sets of x's (or points) for which $0<f$ is satisfied. Bounds are determined for the number k of segments, $\alpha<x<\beta$, in the content of a single inequality $0<f$ where f is a polynomial and for the number K of segments in the content of a finite system $0<g_i(x)$. The content of a single polynomial inequality in two indeterminates is discussed. A system $0<b_iE_i$ ($i=1, 2, \cdots, m$) where E_i is the elementary symmetric polynomial of degree i in $x_1, x_2, \cdots, x_m$ has a solution if and only if $0\neq b_1b_2\cdots b_m$. General solutions are given for several particular systems $0<b_iE_i$. (Received July 3, 1946.)

271. C. N. Moore: *On the infinitude of prime pairs.*

In his address entitled *Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion*, delivered at the Fifth International Congress of Mathematicians (Cambridge, 1912), Edmund Landau mentioned four problems which he regarded as "unangreifbar beim gegenwartigen Stande der Wissenschaften." One of these problems was the question of the infinitude of prime pairs.

It therefore seems desirable to place on record any attack on this problem that gives promise of leading to a solution. In the present communication the problem is solved under the assumption of an unproved property of the functions corresponding to certain types of Dirichlet's series. The Dirichlet's series in question are similar to those that correspond to the functions $(L'/L)(s, \chi)$, arising in the discussion of the number of primes in an arithmetic progression, and the property assumed for the related functions is one possessed by the functions $(L'/L)(s, \chi)$. Accordingly the assumption, which concerns the asymptotic behavior of the related functions on certain curves asymptotic to the axis of imaginaries, appears at least equally reasonable with the classical Riemann hypothesis regarding the roots of the zeta-function. (Received July 2, 1946.)

272. Irma R. Moses: *On the representation, in the ring of p-adic integers, of a quadratic form in n variables by one in m variables.*

The following theorem is proved: Let S and T be symmetric, nonsingular, rationally integral matrices, of dimensions m by m and n by n respectively. If there exist a real transformation and transformations in all p-adic rings, each taking S into T, then there exists a rational transformation without essential denominator, taking S into T. This theorem was proved by C. L. Siegel (Amer. J. Math. vol. 63 (1941) pp. 678–680) in case $m=n$. A considerable portion of his proof must be modified in case $m\neq n$. The modification consists principally in establishing the hypothesis of another theorem of Siegel (Ann. of Math. vol. 36 (1935) p. 536), which gives a relation between p-adic transformations taking S into T; or, more specifically, it consists in showing that a certain determinant is nonvanishing. In doing this, the author uses canonical forms for T in the p-adic rings. In the case of the 2-adic ring, canonical forms were obtained from some results of B. W. Jones (Duke Math. J. vol. 9 (1942) pp. 726–727) and Gordon Pall (Quart. J. Math. Oxford Ser. vol. 6 (1935) pp. 35–38). (Received July 16, 1946.)

273. C. E. Rickert: *The singular elements of a Banach algebra.*

An element of a B-algebra $\mathcal{A}$ is said to be *regular* provided it possesses a two-sided inverse, otherwise it is said to be *singular*. If there exists $\{x_n\} \subset \mathcal{A}$ such that $\|x_n\|=1$ and either $xx_n \to 0$ or $x_n x \to 0$, then x is called a *generalized null-divisor* (g.n.d.) (Silov, C.R. (Doklady) Acad. Sci. URSS. vol. 22 (1939) pp. 7–10). If x is a g.n.d., then it is necessarily singular in every B-algebra which contains $\mathcal{A}$. If every singular element of $\mathcal{A}$ is a g.n.d., then $\mathcal{A}$ is said to be *proper*. The class P of all elements of $\mathcal{A}$, not g.n.d., is an open set and the component of P which contains the identity element is a group under multiplication. This generalizes a result of M. Nagumo (Jap. J. Math. vol. 13 (1936) pp. 61–80). It follows that, if P is connected, $\mathcal{A}$ is proper. $\mathcal{A}$ will also be proper if it possesses a certain general adjoint operation. If $\mathcal{A}$ is commutative and its norm satisfies the condition $\|x^2\|=\|x\|^2$, then it can be embedded in a B-algebra $\mathcal{A}'$ which is proper and in which every element of $\mathcal{A}$, not a g.n.d., is regular. (Received July 13, 1946.)

274. R. M. Robinson: *Primitive recursive functions.*

Recursion with one parameter has the form $F(u, 0)=A(u)$, $F(u, x+1)=B(u, x, F(u, x))$. If $B(u, x, y)$ does not depend on the parameter u, the scheme is called iteration; if it does not depend on x, pure recursion; and if it depends only on y, pure iteration. It is shown that all primitive recursive functions, that is, all arithmetical functions which can be obtained from the identity, zero, and successor functions, by repeated substitutions and recursions with any number of parameters, can be defined using recursion only in the special form of iteration with one parameter. It is shown further that still more special schemes are sufficient if certain adjunctions are made to the initial functions: pure recursion with one parameter, if the predecessor function is adjoined; pure iteration with one parameter, if the characteristic function of squares is adjoined; recursion with no parameter, by adjoining sum and the characteristic function of squares, or absolute difference; pure recursion with no parameter, by adjoining sum and excess over the preceding square, or absolute difference and the characteristic function of squares. In the last case, sum and the characteristic function of squares would not be sufficient to adjoin. (Received July 11, 1946.)

275. R. D. Schafer: *Concerning automorphisms of non-associative algebras.*

N. Jacobson pointed out (*A note on non-associative algebras*, Duke Math. J. vol. 3 (1937) pp. 544–548) that an automorphism S of a non-associative algebra $\mathfrak{A}$ determines in a natural way an automorphism Σ of the (associative) transformation algebra $T(\mathfrak{A})$. Let $\mathfrak{A}$ have unity element 1. With a suitable multiplication defined, the set of residue classes of $T(\mathfrak{A})$ modulo $\mathfrak{R}$, the right ideal of transformations $N \in T(\mathfrak{A})$ annihilating 1, forms an algebra equivalent to $\mathfrak{A}$. Then $S \rightarrow \Sigma$ defines a natural isomorphism between the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$ and the group $\mathfrak{H}$ of all automorphisms Σ of $T(\mathfrak{A})$ under which $\mathfrak{R}$ and the right multiplication space $R(\mathfrak{A})$ are their own images. The study of $\mathfrak{G}$ reduces to that of a well-defined subgroup $\mathfrak{H}$ of the automorphism group of the associative algebra $T(\mathfrak{A})$. (Received July 15, 1946.)

276. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* III.

The quasi-determinant $\Delta(\lambda; \alpha) = |\lambda_{ij} \ \ \alpha_i\alpha_j|$ $(i, j = 1, 2, \cdots, n+1; n = 1, 2, \cdots)$ is "modular" if the configurational sets of dyads A_{ij} in its expansion are subject to equivalence relations; $\Delta(\lambda; \alpha)$ is then reduced relatively to the latter. This reduction may be effected (1) corresponding to the reduction (by equivalence relations) of the symmetric group G of degree $n+1$ to any subgroup H of G (2) by assuming that two A_{ij} are equivalent if they occur in a subgroup H of G or in the same co-set of G relatively to H. Then if $G = H + \sum H \ \ s_i = H + \sum s_i' \cdot H$ and if I, $s_i(I, s_i')$ are elements of the cyclic group of order $n+1$, numerical expressions result from replacing configurational sets corresponding to I, $s_i(I, s_i')$ by roots of the equation $s^{n+1} = 1$ with $\alpha_1\alpha_1 \cdot \alpha_2\alpha_2 \cdots \alpha_{n+1}\alpha_{n+1} = 1$. For $n = 1$, $\Delta(\lambda; \alpha)$ then reduces to a determinant. Another type of modular quasi-determinants is obtained by subjecting the dyads $\alpha_i\alpha_j$ in $\Delta(\lambda; \alpha)$ to equivalence relations. For example, by assuming (1) $\alpha_i\alpha_j = \alpha_k\alpha_l$ if and only if $\alpha_i\alpha_j$ and $\alpha_k\alpha_l$ occur in the same A_{ij} of a regular group, and (2) $\lambda_{ij} = \lambda_{kl}$ if and only if $\alpha_i\alpha_j = \alpha_k\alpha_l$, a modular "group quasi-determinant" is obtained analogous to the group determinant discussed by Frobenius and others. (Received July 11, 1946.)

277. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* IV.

The quasi-determinant $\Delta(\lambda; \alpha)$ in expanded form is interpreted as a special case of a hypercomplex number with group elements as units in a group algebra. More generally, a representation of a group algebra is based on the addition and multiplication of linear functions of ordered dyads $X_{m+1}(\alpha)=\sum x_j\alpha_1\alpha_j$ $(j=1, 2, \cdots, m+1)$ with coefficients x_j in a suitable domain D (field F) relatively to any given abstract group G of order $m+1$ $(m=1, 2, \cdots)$ represented as a regular group of configurational sets of dyads on $m+1$ elements. Two dyads $\alpha_i\alpha_j$ and $\alpha_k\alpha_l$ are then equivalent if and only if they occur in the same configurational set of dyads. Multiplication is determined by $\alpha_i\alpha_j\times\alpha_j\alpha_k=\alpha_i\alpha_k$ and by the preceding equivalences. Other instances of dyadic representation of linear algebras are given by two examples: 1. $X_2(\alpha)=x_1\alpha_1\alpha_1+x_2\alpha_1\alpha_2$ with equivalences $\alpha_1\alpha_1=\alpha_2\alpha_2$, $\alpha_1\alpha_2=-\alpha_2\alpha_1$. 2. $X_4(\alpha)=\sum x_j\alpha_1\alpha_j$ $(j=1, 2, 3, 4)$ with equivalences corresponding to those given by the author (Math. Ann. vol. 69 p. 584). In both examples multiplication is determined by $\alpha_i\alpha_j\times\alpha_j\alpha_k=\alpha_i\alpha_k$ and by the associated equivalences. (Received July 11, 1946.)

278. M. C. Sholander: *On the existence of the inverse operation in certain spaces.*

In a set S of elements $x, y, \cdots$ which admits a binary operation—here denoted by multiplication—an element a will be called regular if both (i) $ax=ay$ implies $x=y$ and (ii) $xa=ya$ implies $x=y$. An element a will be called proper if for each element b in S there exist unique solutions x and y in S for the equations $ax=b$ and $ya=b$. It is well known that if the multiplication is commutative and associative S can be imbedded in a space S' of the same type in such a way that all elements regular in S are proper in S'. In this paper it is shown the imbedding process can also be carried out in case the multiplication is one satisfying the alternation law $(ab)(cd)=(ac)(bd)$ and in case the regular elements of S are closed under multiplication. Thus if all elements of S are regular, S' is a quasi-group of a type studied, for example, by D. C. Murdoch (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 392–409) and R. H. Bruck (Trans. Amer. Math. Soc. vol. 55 (1944) pp. 19–52). Various conditions which insure the necessary closure property in S are given in the paper. (Received July 26, 1946.)

279. J. M. Thomas: *Eliminants.*

If $R(a, b)$ denotes the resultant to two polynomials $x(t)$, $y(t)$ whose constant terms are a, b, the polynomial $R(a-x, b-y)$ in the two indeterminates x, y is the *eliminant* $E(x, y)$ of $x(t)$, $y(t)$. This paper (i) proves $E(x, y)=f^k$, where f is an irreducible polynomial and k is a positive integer; (ii) proves $E(x, y)$ is reducible $(1<k)$ if and only if $x(t)$, $y(t)$ are also polynomials in a second parameter which is itself a polynomial of degree at least two in t; (iii) expresses in terms of $E(x, y)$ algebraic conditions that a single polynomial $y(t)$ be a polynomial in $x(t)$ of degree k, where $1<k<\deg y$ (these last polynomials have been called by Ritt composite polynomials, Trans. Amer. Math. Soc. vol. 23 (1922) pp. 51–66). (Received July 27, 1946.)

280. J. H. M. Wedderburn: *Note on Goldbach's theorem.*

It is shown in this note that, if p and q are primes and $r=(p+q)/2$ is a prime, then $p-q$ is a multiple of 12 unless r has the form $2p-3$ or, when unity is reckoned as a prime, also $r=2p-1$. The proof is elementary and depends on reducing modulo 12. Similar statements apply if q is replaced by $-q$. (Received July 30, 1946.)

ANALYSIS

281. R. P. Agnew: *Methods of summability which evaluate sequences of zeros and ones summable C_1.*

It is shown that if A is a method of summability belonging to a familiar general class, and if each divergent sequence of zeros and ones summable C_1 is summable A to the same value, then A must be regular. (Received July 13, 1946.)

282. R. P. Agnew: *Subseries of series which are not absolutely convergent.*

If $a(1)+a(2)+\cdots$ is a series of real or complex terms which fails to converge absolutely, then there is an increasing sequence $1\leqq n_1<n_2<n_3<\cdots$ of integers such that $n_{p+1}-n_p\rightarrow\infty$ as $p\rightarrow\infty$ and the subseries $a(n_1)+a(n_2)+\cdots$ is divergent. (Received July 13, 1946.)

283. H. P. Atkins: *On fractional derivatives of univalent functions.*

The following theorem is proved: If $f(z)$ is analytic and univalent in the unit circle with $f(0)=0$ and $f'(0)=1$, and if a is any real number and $n+1$ the smallest non-negative integer greater than a, then (without restriction if $a\leqq 3$, but assuming the Bieberbach conjecture if $a>3$) $\left|{}_0D_z^a f(z)-\sum_{j=1}^{n}{}_0D_z^{a-j}f^{(j)}(0)\right|\leqq(1-|z|)^{-a-2}\Gamma(a+1)\cdot\{aI(n+1-a,\, a+2)+|z|I(n-a,\, a+2)\}$ for $z\neq 0$. Equality is attained for real positive z by the function $f(z)=z(1-z)^{-2}$. The sum from 1 to n is to be interpreted as 0 if $n<1$, and ${}_0D_z^{a-j}f^{(j)}(0)$ means the derivative of order $a-j$ of the constant $f^{(j)}(0)$. The function $I(p,q)$ as used above represents the ratio of the incomplete beta function from 0 to $|z|$ to the corresponding complete beta function. A correspondingly weaker relation may be obtained using the Littlewood inequality instead of the Bieberbach conjecture. The theorem is a generalization of a relation for $a=0$ and $a=1$ due to G. Pick and proved for the remaining positive integers by F. Marty (*Sur les dérivées d'une fonction univalente*, C. R. Acad. Sci. Paris vol. 194 (1932) p. 1308). (Received June 19, 1945.)

284. J. D. Bankier: *Modified regular continued fractions.* Preliminary report.

If the algorithm for the expansion of a real number, y_0, into a regular continued fraction (r.c.f.), is modified by requiring that b_n $(n=0, 1, 2, \cdots)$ be either the integer nearest to y_n, or, (1) if $y_n=I+1/2$, where I is an integer, I or $I+1$ according as y_n is positive or negative, the resulting expansion is called a *modified* regular continued fraction (m.r.c.f.). The m.r.c.f. expansion terminates, if y_0 is rational, and does not have more elements than the corresponding r.c.f. If y_0 is irrational, its m.r.c.f. expansion converges to the value y_0 at least as rapidly as the corresponding r.c.f. Necessary and sufficient conditions that a continued fraction be a m.r.c.f. are that its elements be integers such that (2) $|b_n|\geqq 2$, (3) if $|b_{n-1}|=2$, $b_{n-1}b_n>0$ $(n=1, 2, 3, \cdots)$ and (1) holds if the continued fraction terminates. A m.r.c.f. is periodic if and only if it converges to a quadratic irrational. A method is given for obtaining the m.r.c.f. expansion of a number directly from the corresponding r.c.f. (Received July 2, 1946.)

285. Stefan Bergman: *The "extended class" in the theory of functions of two complex variables.*

Let $\mathfrak{M}^4$ be a domain with a distinguished boundary surface $\mathfrak{F}^2$. Let $H(z_1, z_2)$ be a (real) continuous function defined on $\mathfrak{F}^2$. As is well known, there does not exist in general a biharmonic function (real part of an analytic function of two complex variables) which is regular in $\mathfrak{M}^4$ and equals $H(z_1, z_2)$ on $\mathfrak{F}^2$. Let $\mathcal{U}=\mathcal{U}(\mathfrak{M}^4, H)$ be the totality of functions which are biharmonic in open $\mathfrak{M}^4$, continuous in closed $\mathfrak{M}^4$, and such that for every $B \in \mathcal{U}$, $B = H$ on $\mathfrak{F}^3$. Let $U(z_1, z_2) = \lim\inf B(z_1, z_2)$, $B \in \mathcal{U}$, $(z_1, z_2) \in \mathfrak{M}^4$. Similarly, let $\mathcal{L}$ denote the totality of biharmonic functions for which $B = H$ on $\mathfrak{F}^3$ and $L(z_1, z_2) = \lim\sup B(z_1, z_2)$, $B \in \mathcal{L}$. Under certain assumptions on H, it is possible to show that $U = L$ on the boundary $\mathfrak{m}^3$ of $\mathfrak{M}^4$. A harmonic function of four real variables which assumes on $\mathfrak{m}^3$ these values is denoted as a function of M-extended class belonging to H. Various applications to the theory of value distribution of functions of two complex variables and to the theory of pseudo-conformal mapping are made. (Received July 15, 1946.)

286. J. P. Brown: *Cyclic characteristic curves of a system of differential equations.*

Geometrical properties of the *cyclic characteristic curves* (*c.c.c.*), and in particular of the limit cycles of Poincaré, corresponding to the periodic solutions of $\{dx/dt = X(x, y);\ dy/dt = Y(x, y)\}$ are studied. Hurewicz's results (*Ordinary differential equations in the real domain*, Brown University, 1943) on the topology and stability of c.c.c. and of other limit sets of characteristic curves are restated. A *curve without contact* (*c.w.c.*) is defined to be a simple closed curve with continuously turning tangent, passing through no singularity of the vector field (X, Y) and nowhere tangent to (X, Y). There are two types of c.w.c., according as the vector (X, Y) points inside or outside the c.w.c. A c.c.c. is defined to be *persistent* if small arbitrary changes of X and Y change its position arbitrarily little. A c.c.c. is found to be persistent if and only if there exist c.w.c. of both kinds arbitrarily close to it. This result is interpreted in terms of the existence of limit cycles arbitrarily close to the c.c.c. Analogous definitions are set up and results obtained for the *stable persistence* of a c.c.c. (Received July 19, 1946.)

287. C. C. Camp: *Integral equations with kernels having discontinuities along two diagonals.*

If the kernel of $u(x) = \lambda\int_0^1 K(x, y)u(y)dy$ is the smallest of the quantities x, y, $1-x$, $1-y$, its partial derivatives have discontinuities along $y = x$ and $y = 1-x$. For this simple case $u(x) = C \sin (2k+1)\pi x$. An equivalent integro-differential system has been used to extend the theory to kernels which are more general. For example, if the kernel is normalized, has symmetry with respect to the line $x = 1/2$, satisfies the condition of a constant jump in K_x across the secondary diagonal, and if $K_{xx}(x, y)$ multiplied by a certain constant possesses a reciprocal kernel, then it may be shown that the usual expansion theorem holds for a function $f(x)$ such that $f(x) = f(1-x)$ (Received July 15, 1946.)

288. J. A. Clarkson: *A characterisation of C-spaces.*

The set $C(T)$ of all bounded continuous real-valued functions $x(t)$ defined over a topological space T is a real Banach space. The following conditions are shown to be necessary and sufficient for a given real Banach space B to be equivalent to a space of this kind (that is, a "C-space"). (Space B is not assumed to be a ring, or partially ordered.) (1) S, the surface of the unit sphere in B, contains an element v such that if x is any element of S, one of the two segments $[x, v]$, $[x, -v]$ lies in S. (2) Let $\mathcal{C}$ be the set of all points x such that x lies on a ray terminating at v and passing through another point of S. Then $\mathcal{C}$ has the property that the intersection of two of its translates is again a translate. The proof utilizes results of Kakutani concerning partially ordered Banach spaces (*Concrete representation of abstract* (*M*)-*spaces*, Ann. of Math. vol. 42 (1941) pp. 994–1024). (Received July 6, 1946.)

289. H. V. Craig: *On the structure of intrinsic derivatives.*

The primary purpose of the present paper is to express the Mth order iterated intrinsic derivative of a higher order absolute tensor as a contraction. A method is developed for constructing extensors from higher order tensors by differentiating with respect to the curve parameter. Contraction of these derived extensors with certain generalized components of connection yields the Mth order iterated intrinsic derivative of the original tensor. The iterated intrinsic derivatives considered are the tensors obtained by repeatedly applying the classical intrinsic differentiation process of tensor analysis. (Received July 15, 1946.)

290. Bernard Epstein: *Integral representations of solutions of certain classes of partial differential equations.*

The method of integral operators developed by Bergman (see especially Trans. Amer. Math. Soc. vol. 53 and recent volumes of Duke Math. J.) is employed to study singularities of solutions of certain classes of elliptic equations in three variables, in particular the existence of solutions having singularities only along a sequence of prescribed closed curves. As a preliminary step, a theorem for harmonic functions of three variables similar to the Mittag-Leffler theorem of function theory is established. Then the use of the integral operator method leads directly to an analogous theorem for solutions of the above-mentioned more general types of equations. Finally, a similar theorem is obtained for solutions of a certain class of nonlinear equations. (Received July 8, 1946.)

291. N. J. Fine: *On the Walsh functions.* Preliminary report.

In 1923 Walsh completed the Rademacher system and studied some of its properties, pointing out the great similarity between the completed system and the trigonometric system. It is shown in the present paper that this similarity is not accidental, but is based on the fact that each system may be regarded as the character group of a certain compact commutative group. Some of the differences are shown to stem directly from the topological and algebraic structures of the underlying groups. The behavior of the Fourier expansion of a given function in terms of the Walsh functions is studied in more detail, and certain conditions sufficient for convergence are obtained. Questions of $(C, 1)$ and Abel summability are taken up for certain classes of functions. A discussion of the Lebesgue constants is given, covering their recursive and explicit representations, their generating function, and their maximal and average orders. The Riemann theory for general Walsh series is developed, with some important differences in method. It is proved that a Walsh series which converges everywhere to an integrable function is the Fourier series of its sum. (Received July 24, 1946.)

292. A. W. Goodman: *On some determinants related to p-valent functions.* Preliminary report.

The elementary inequality $|\sin n\theta/\sin \theta| \leqq n$ is generalized to give exact upper bounds for the ratios of certain pth order determinants involving $\sin n\theta_i$,

$i=1, 2, \cdots, p$. Exact upper bounds are also obtained for analogous expressions involving $\cos n\theta_i$ which are generalizations of $|\cos n\theta| \leqq 1$. A similar result is obtained for the ratios of certain modified Vandermonde determinants. Bieberbach's conjecture that $|b_n| \leqq n|b_1|$ for univalent functions is generalized to the set of functions p-valent in the unit circle. It is shown that the first set of trigonometric inequalities has a relation to this generalized conjecture. (Received July 11 1946.)

293. S. H. Gould: *Lagrange multipliers and functional determinants.*

A geometric interpretation is given of the Lagrange multiplier rule, which is shown to be identical with the Jacobian rule for functional dependence with substitution of the words "stationary at a point" for "constant in a neighborhood." (Received July 12, 1946.)

294. R. N. Haskell: *Mean values and biharmonic polynomials.*

A class of functions is studied which at the centers of regular polygons are equal to certain linear combinations of the areal and peripheral means of the functions on the polygons. Let $P_n(x_0, y_0; r; \psi)$, $|P_n(x_0, y_0; r; \psi)|$ and $|p_n(x_0, y_0; r; \psi)|$, or for brevity P, $|P|$ and $|p|$, be the closed finite region, its area and the length of the boundary respectively of the regular n-gon $p_n(x_0, y_0; r; \psi)$ whose center is at (x_0, y_0). The letters r and ψ denote respectively the radius of the inscribed circle and the smallest positive angle made by any diagonal of the polygon with the positive x-axis. A typical result of the paper is the following. Given a function $f(x, y)$ summable superficially in the interior of a finite simply connected domain D, n a fixed integer, $n \geqq 3$, and ψ fixed, then a necessary and sufficient condition that $f(x_0, y_0) = 2|P|^{-1}\int\int_P f d\xi d\eta - |p|^{-1}\int_p f ds$ hold at each point (x_0, y_0) of D, and for all $p_n(x_0, y_0; r; \psi)$ in D for which $\int_p f ds$ exists, is that $f(x, y)$ shall be a biharmonic polynomial of degree at most $n+2$. (Received July 15, 1946.)

295. M. H. Heins: *On the Phragmén-Lindelöf principle.*

The problem has been proposed by Ahlfors (Trans. Amer. Math. Soc. vol. 41) to determine under the hypotheses of the classical Phragmén-Lindelöf principle for a half-plane whether or not $\lim_{r\to+\infty}(\log M(r))/r$ exists in all cases, the notation being that conventionally used (cf. Nevanlinna, *Eindeutige analytische Funktionen*, p. 43). This question is answered affirmatively. The limit exists and is never finite and strictly negative. The proof is based upon the following lemma: Let $f(z)$ denote a function which is analytic and of modulus not exceeding one for $R(z) > 0$, let ϵ denote a given positive number, and let $E(r, \epsilon)$ denote for given r the set of θ for which $\log|f(re^{i\theta})| < -\epsilon r$. If $\limsup_{r\to+\infty}$ meas. $E(r, \epsilon) > 0$, then there exists a strictly positive constant κ such that $|f(z)| \leqq \exp[-\kappa R(z)]$ for $R(z) > 0$. This lemma admits generalization in many directions and taken with its generalizations serves as a useful instrument for investigating the properties of the class of functions which are admitted by the hypotheses of the Phragmén-Lindelöf principle and in addition satisfy $\lim_{r\to+\infty}(\log M(r))/r < +\infty$. (Received June 10, 1946.)

296. J. G. Herriot: *On the restricted summability of multiple Fourier series.* Preliminary report.

Consider the Fourier series of an integrable function of k variables, periodic of period 2π in each variable. In a previous paper (Duke Math. J. vol. 11 (1944) pp.

735–754) the author showed that if $k \geqq 3$, restricted Abel summability and many restricted multiple Nörlund methods including restricted $(C, \alpha, \beta, \cdots, \kappa)$, $\alpha, \beta, \cdots, \kappa \geqq 0$, applied to such Fourier series do not possess the localization property. Bochner (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207) showed that the same result is true for restricted Riesz summability of type $\lambda_n = n$ and any positive order. It is now shown that restricted Riesz summability of type $\lambda_n = n^2$ and non-negative integral order δ applied to such Fourier series does not possess the localization property for $\delta \leqq k-2$. The condition $\delta \leqq k-2$ is necessary at least for $k \leqq 5$ since it is shown that restricted Riesz summability of type $\lambda_n = n^2$ and non-negative integral order δ applied to these Fourier series does possess the localization property for $\delta \geqq k-1$ provided $\delta \leqq 4$. This contradicts a remark of Bochner that these methods possess the localization property if $\delta > k$ but not if $\delta < k$. It is conjectured that the restriction $\delta+4$ is not essential. (Received July 10, 1946.)

297. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of types of generalised Bessel functions.*

Methods already developed by W. B. Ford, Newsom, and Hughes are employed to obtain general theorems giving the asymptotic behavior in the neighborhood of $s = \infty$ of the entire functions $f_1(s) = \sum_{n=0}^{\infty} s^n/\Gamma(kn+q)\Gamma(ln+p)$ $(l>0; l+k>0)$ and $f_2(s) = \sum_{n=0}^{\infty} s^n\Gamma(kn+q)/\Gamma(ln+p)$ $(l>k>0)$, where p and q are any suitable real or complex constants. From these theorems follow the asymptotic expansions of several important functions of Bessel types. One of these functions is the "generalized Bessel function" $j(s) = \sum_{n=0}^{\infty} s^n/n!\Gamma(kn+q)$, $k > -1$. (Received July 13, 1946.)

298. D. C. Lewis: *On polynomial interpolation and extrapolation by least squares with arbitrary weight function.*

Let $G_n^{(m)}(x, t)$ be defined as that polynomial in x of degree not greater than n which best fits (in the sense of least squares with distribution, $\alpha(x)$, having at least $n+1$ points of increase, on the interval $a \leqq x \leqq b$) the function $g^{(m)}(x, t)$ defined as follows: $g^{(m)}(x, t) = (x-t)^m$ for $a \leqq x \leqq t$, and $g^{(m)}(x, t) = 0$ for $t < x \leqq b$. Let $f(x)$ be any continuous function (defined for $A \leqq x \leqq B$, with $A \leqq a < b \leqq B$) with continuous derivatives up to and including the one of order $m-1$, and let the mth derivative $f^{(m)}(x)$ exist almost everywhere (including the points $x=a$ and $x=b$, where it is also continuous) and let $f^{(m)}(x)$ be of bounded variation. Let $P_n(x)$ be the polynomial of degree not greater than n which best fits $f(x)$ on the interval $a \leqq x \leqq b$ with respect to the same distribution $\alpha(x)$. It is proved in this paper that the following formula holds at every point on the interval $A \leqq x \leqq B$ with $1 \leqq m \leqq n$: $f(x) = P_n(x) + (1/m!)\int_b^x (x-t)^m df^{(m)}(t) + (1/m!)\int_a^b G_n^{(m)}(x, t) df^{(m)}(t)$. Certain generalizations of this theorem are also proved. (Received July 8, 1946.)

299. Szolem Mandelbrojt and G. R. MacLane: *On functions holomorphic in a strip region and an extension of Watson's problem.*

Let Δ be a region in the $s = \sigma + it$ plane given by $\sigma > a$ $(a \geqq -\infty)$, $|t| < g(\sigma)$, where $g(\sigma)$ is of bounded variation in (a, ∞) and $\lim_{\sigma\to\infty} g(\sigma) = \pi/2$. Let $N(\sigma)$ be an increasing function, and let $S(\sigma) = \pi/2\int_c^\sigma (1/g(x))dx$, $c > a$. A necessary and sufficient condition that there exist a function $F(s)$, not identically zero, holomorphic in Δ, continuous in the closure of Δ, and satisfying the inequality $\log|F(\sigma + ig(\sigma))| \leqq \log M(\sigma) \leqq -N(\sigma)$, where $M(\sigma) = \mathrm{Max}_{|t| \leqq g(\sigma)} |F(\sigma+it)|$, is that (1) $\int_c^\infty N(\sigma)e^{-S(\sigma)}d\sigma < \infty$. If moreover $g'(\sigma)$

exists with $|g'(\sigma)| < M$, $g'(\sigma_2) - g'(\sigma_1) > A(\sigma_2 - \sigma_1)$ $(A < 0,\ \sigma_2 > \sigma_1)$ then if $F(s) \not\equiv 0$, $\int_0^\infty \log |F(\sigma + ig(\sigma))| e^{-S(\sigma)} d\sigma > -\infty$. If $M_n > 0$ $(n \geqq 1)$ and if $\lim M_n^{1/n} = \infty$, then a necessary and sufficient condition (since $F(s)$ is holomorphic in Δ, continuous in the closure of Δ, and satisfies there the inequalities $|F(s)| < M_n e^{-n\sigma}$ $(n \geqq 1)$ implying that $F(s) \equiv 0$) is that (2) $\sum \exp\left[-S(\log(M_{n+1}^c/M_n^c))\right] = \infty$, where $\{M_n^c\}$ is the convex regularized sequence of $\{M_n\}$, regularized by logarithms. These theorems generalize classical theorems known for the case in which Δ is the strip $|t| < \pi/2$, $S(\sigma) = \sigma - c$, and (2) becomes $\sum M_n^c/M_{n+1}^c = \infty$. (Received June 11, 1946.)

300. P. R. Masani: *Laurent factorization of analytic functions in normed rings.*

Let D_1 and D_2 be regions in the complex plane, $D = D_1 \cap D_2 \neq 0$, and f be an analytic function on D which possesses an inverse, with values in a normed ring X. This paper shows that under suitable assumptions as to D_1 and D_2, f can be factored on D: $f(z) = f_1(z) \cdot f_2(z)$, where f_1 and f_2 are analytic and possess inverses on D_1 and D_2 respectively. This question, which is deep when X is not commutative, has been studied for matrix-valued functions by Plemelj, G. D. Birkhoff, and H. Cartan. In this paper their results are generalized and proved by product integration. It is first shown that every function f analytic inside a closed contour C and continuous on C admits the Cauchy product integral representation $\exp(2\pi i f(z)) = \oint [1 + D_w\{\exp \int (\zeta - z)^{-1} f(\zeta) d\zeta\} \cdot dw]$. Here z is a point inside C, 1 is the unit of X, D_w denotes product differentiation, $\int$ stands for the additive integral along C from a fixed point to w, and $\oint$ stands for the product contour integral along C. From here on this proof is practically parallel to the classical proof of Laurent's theorem. (Received July 9, 1946.)

301. P. R. Masani: *Multiplicative Riemann integration in normed rings.*

The author studies Riemann product-integration of functions from a real interval with values in a normed ring X. Volterra, Schlesinger, and Rasch have treated continuous matrix-valued functions. Garrett Birkhoff has studied continuous functions in an infinite-dimensional nonlinear space. The present theory is completely general in that it includes discontinuous functions. This extension is not trivial for even functions discontinuous everywhere may be R-integrable; and the product of such functions need not be R-integrable. The following results are obtained: properly product integrable functions are bounded; additive and product integrability are equivalent; the class of R-integrable functions is not in general a ring, when X is infinite-dimensional; the Peano-series development is valid, if the integrand is R-integrable; the indefinite product integral and product primitive agree up to a constant factor, whenever both exist; the rule for product integration by parts holds if the functions involved are R-integrable; the familiar method of integration after change of variable is generally valid. (Received July 9, 1946.)

302. Josephine M. Mitchell: *Value distribution of a function of two complex variables in a domain with distinguished boundary surface.*

The author considers meromorphic functions $f(z_1, z_2)$ of two complex variables $z_k = x_k + iy_k$, $k = 1, 2$, in a certain class of domains with distinguished boundary surface F^2, namely in $M^4 = \sum_{|z_2| \leq 1} B^2(z_2)$ where $B^2(z_2^{(0)})$ is a domain in the plane $z_2 = z_2^{(0)}$ bounded by a simple closed curve $b^1(z_2^{(0)}) = E[z_1 = h(z_2^{(0)}, \lambda),\ 0 \leqq \lambda \leqq 2\pi]$, $h(z_2^{(0)}, 0)$

$=h(z_2^{(0)}, 2\pi)$. Domains with distinguished boundary surfaces were introduced by S. Bergman (Math. Zeit. vol. 39 (1934) pp. 76–94, 605–608). As an example cases where $h(z_2, \lambda)=\sum_{k=-\infty}^{\infty} A_k(z_2)e^{ik\lambda}$, $A_k(z_2)$ analytic in $|z_2| \leqq 1$, are discussed. The notion of density function $D(a, z_1, z_2)$ in M^4 with respect to $F^3=\sum_{|z_2|\leqq 1} b^1(z_2)$ of the surfaces $f(z_1, z_2)=a$ is introduced and bounds are derived for $D(a, z_1, z_2)$. Let K^3 be a manifold which lies in F^3: $K^3=\sum_{0\leqq\lambda\leqq a} C^1(\lambda)$, $0<a\leqq 2\pi$, where $C^1(\lambda)=E[z_1=h(z_2, \lambda), z_2=z_2(t, \lambda), 0\leqq t\leqq 1]$, $z_2(0)=0$, $|z_2(1)|=1$. By means of the Milloux theorem and a method indicated by Bergman (Travaux de l'Institut Mathématique de Tbilissi vol. 1 (1937) pp. 187–204) the author obtains theorems concerning the value distribution of f in K^3 in terms of $D(0, z_1, z_2)$, $D(\infty, z_1, z_2)$ and the maximum of $|f|$ on F^3. (Received July 10, 1946.)

303. George Piranian: *Infinite iteration of totally regular transformations*. Preliminary report.

The *core* $C(s_n)$ of a sequence s_n in the complex plane is the intersection of all convex sets that contain all finite and infinite limit points of the sequence. A transformation A is *totally regular* provided $C(s_n)\supset C(As_n)$ for every sequence s_n. For a totally regular transformation A the *central A core* $C(A^{\infty}s_n)$ of s_n is the intersection of the cores $C(A^k s_n)$ $(k=1, 2, \cdots)$. THEOREM. *If A and B are totally regular and* $AB=BA$, *the central cores* $C(A^{\infty}s_n)$ *and* $C(B^{\infty}s_n)$ *have a common point, for each sequence* s_n. The *essential central A core* of s_n is the intersection of all central X cores of s_n, where X ranges over the set of totally regular transformations for which $AX=XA$. (Received July 17, 1946.)

304. Harry Pollard: *The inversion of the transforms with reiterated Stieltjes kernels.*

It is shown that the transforms $\int_{0+}^{\infty} H_n(x, y)d\alpha(y)$, where $H_n(x, y)$ is the nth iterate of the Stieltjes kernel $(x+y)^{-1}$, can be inverted by a linear differential operator of infinite order. (Received July 12, 1946.)

305. Harry Pollard: *The representation of $e^{-x^{\lambda}}$ as a Laplace integral.*

The inverse Laplace transform of $\exp(-x^{\lambda})$, $0<\lambda<1$, is computed explicitly, using Post's theory of the Laplace transform (Trans. Amer. Math. Soc. vol. 32 (1930) pp. 723–781). (Received July 12, 1946.)

306. W. T. Reid: *Solutions of linear differential equations.*

This paper is concerned with criteria for a function $y(x)$ to be a "weak solution" in the sense of Bochner (Ann. of Math. vol. 47 (1946) pp. 202–212) of a differential equation $\mathcal{L}[y]+g(x)=0$, where $\mathcal{L}[y]$ is a homogeneous linear differential operator involving ordinary or partial derivatives whose coefficients satisfy suitable differentiability conditions, and $g(x)$ is a given integrable function. One criterion presented is an extension of the condition on weak convergence used by Bochner for equations $\mathcal{L}[y]=0$ with constant coefficients, while another involving difference quotients is closely related to previous results of the author (Duke Math. J. vol. 12 (1945) pp. 685–694) for ordinary linear differential equations. (Received July 12, 1946.)

307. I. M. Sheffer: *On k-periodic systems of linear equations.*

Let (1) $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=c_n$ $(n=0, 1, 2, \cdots)$ be a system of linear equations, and let (2) $A_s(t)=\sum_0^\infty a_{sj}t^j$. The system (1) is k-periodic, if (3) $A_{j+sk}(t)=A_j(t)$ ($j=0, 1, \cdots, k-1$; $s=0, 1, 2, \cdots$). The following results are obtained for suitable conditions on $\{a_{sj}\}$ and $\{c_s\}$: (i) A k-periodic system can be transformed into a 1-periodic system (but the process is not reversible). (ii) A k-periodic system can be factored into a product of k-periodic systems each of simple type. (iii) If (suitably restricted) perturbation terms are added to a k-periodic system, a factorization into factors of simple type is again possible. (iv) The number of independent solutions of a k-periodic system is determined. (Received July 8, 1946.)

308. Abraham Spitzbart: *The mininum of a certain integral.*

Let $f(s)$ be analytic within the unit circle C: $|s|=1$, continuous within and on C, and with the value $f'(\alpha)=A$ prescribed at a point $s=\alpha$ within C. It is shown that for any such function the minimum of the integral $\int_C|f(s)|^p|ds|$, $p\geqq 1$, is given by $2\pi|A|^p(1-|\alpha|^2)^{p+1}\{2(1+|\alpha|^2)\}^{1-p}[(p-1)|\alpha|+(|\alpha|^2-p^2+2p)^{1/2}]^{p-2}[p+|\alpha|^2-|\alpha|(|\alpha|^2-p^2+2p)^{1/2}]$ if $p\leqq 1+|\alpha|$, and by $2\pi|A|^p2^{1-p}p^p(1-|\alpha|^2)(1-|\alpha|)^p$ if $p\geqq 1+|\alpha|$. If $p=1$, the first form applies and reduces to a result of Macintyre and Rogosinski (Mathematical Notes, Edinburgh Math. Soc., vol. 35 (1945) pp. 1–3). (Received July 12, 1946.)

309. W. J. Trjitzinsky: *Singular integral equations of the first kind and those related to permutability and iteration.*

This is a systematic study of the topics indicated in the title when the kernels involved are L_2 separately in each variable, a fact which renders these problems singular. Essentially two methods are used, of which one involves suitable types of spectral theory. (Received July 12, 1946.)

310. Y. W. Tschen: *Branch points and flat points of minimal surfaces in R^3.*

The object of this paper is to study the behavior of minimal surfaces (in R^3) in the neighborhood of a branch point and to find an expression for the Anzahl of the branch points and of the flat points, respectively, for minimal surfaces with boundaries and given topological structure. Let the surface be denoted by the real parts of $\phi_i(\omega)$, $i=1, 2, 3$, with branch points defined by $\phi_i'(\omega)=0$, $i=1, 2, 3$. To each branch point B belong two integers $k>1$ and $l\geqq 1$: in the neighborhood of B the surface has k sheets and the normal vectors (with components ξ_i, $i=1, 2, 3$) turn l times around the normal at B. Self-intersection of the surface at B is discussed. Branch points and flat points together are the points where the analytic function (of ω) $\xi_1\phi_1''+\xi_2\phi_2''+\xi_3\phi_3''=0$. From this is derived an analytic function which vanishes only at the branch points. Regularity of the functions on the boundaries is assumed in order to form the Anzahl by integration. Specific results are derived for polygon boundaries. (Received July 11, 1946.)

311. František Wolf: *On the continuation of analytic functions.*

T. Carleman has proved that if $x\in(a, b)\to\lim_{y\to 0}[f_1(x+iy)-f_2(x-iy)]=0$ uniformly, then $f_1(z), f_2(z)$ are parts of the same analytic function $f(z)$ which is analytic also on the open segment (a, b). If one drops the uniformity the same is true, except that $f(z)$ is analytic only in an everywhere dense set of intervals in (a, b). If the additional condition $f_1(x+iy), f_2(x-iy)=o(1/y^n)$ is introduced, then $f(z)$ may have on (a, b) only poles and limit points of them of an even order less than n. The set of poles forms a set, of which no component is dense in itself. (Received July 29, 1946.)

Applied Mathematics

312. Hillel Poritsky and D. W. Dudley: *Conjugate action of involute helical gears with inclined axes.*

By an involute helical surface is meant a surface consisting of helices of a proper screw motion about an axis and such that a section by a plane perpendicular to the axis is an involute of a proper "base" circle. It is shown that two involute helical surfaces with inclined axes mate correctly in the sense that a uniform rotation of the one surface about its axis A is transformed into a uniform rotation of the other surface about its axis A', provided that certain inequalities in the distance and angle between A, A' are satisfied; the ratio of the two rotational velocities ω/ω' is independent of the distance between the two axes. The point of contact of the mating surfaces always moves along a fixed straight line in space with uniform velocity, this line being normal to each surface. Certain industrial applications are briefly discussed. (Received July 23, 1946.)

313. William Prager: *On the variational principles of plasticity.*

In earlier papers the author outlined a new mathematical theory of plasticity (Prikladnaia Matematika i Mekhanika N.S. vol. 5 (1941) pp. 419–430) and discussed certain variational principles associated with this theory (Duke Math. J. vol. 9 (1942) pp. 228–233). These variational principles were established under the assumption that the velocities or rates of stressing prescribed at the surface produce "loading" throughout the body. In the present paper this restriction is dropped and the general variational principles associated with the new theory of plasticity are established. (Received July 15, 1946.)

314. S. S. Shü: *On Taylor and Maccoll's equation of a cone moving in the air with supersonic speed.*

When an infinite cone is moving in the air with supersonic speed, the shock phenomena occur. Taylor and Maccoll (Proc. Roy. Soc. London, Ser. A. vol. 139 (1933) pp. 278–311) considered a conical flow behind an oblique shock wave and deduced a nonlinear ordinary differential equation which was integrated numerically. The purpose of the present note is first to transpose the equation to the differential-integral form $-\lambda'(\theta)=S_w \exp \int_\theta^{\theta_w}((2+\lambda'^2)/\lambda')d\theta/\rho \sin \theta$ (where S_w is a constant, σ^λ is the ratio of the radial component of the velocity of the flow to the speed of the gas if allowed to be discharged into a vacuum and $\rho^{\gamma-1}=1-\sigma^{2\lambda}(1+\lambda'^2)$) for which the author assumes the position and the intensity of the shock wave known and for which successive approximations are applied. In some cases, the sequence generated by the successive approximations is proved to be monotone and equi-continuous and therefore it converges uniformly to a solution of the problem in the large. A method is suggested for the practical calculation of the angle of the solid cone. The first approximation in which only the rational forms of elementary functions are involved gives a fairly good coincidence with Taylor and Maccoll's calculations. (Received July 20, 1946.)

315. J. L. Synge: *Approximations in elasticity based on the concept of function space.*

A state of an elastic body is defined by a set of six stress components, given as functions of position throughout the body. Such a state defines a point or vector S in function space, without any implication that the equations of equilibrium or compatibility or the boundary conditions are satisfied. A metric in the function space is defined by means of the strain-energy function. If S is the solution to a problem in which surface stress is given, S' an arbitrary state satisfying the equations of equilibrium and the boundary conditions (but not the equations of compatibility), and S'' another arbitrary state satisfying the equations of compatibility (but not the equations of equilibrium or the boundary conditions), then S is situated on the intersection of a hypersphere determined by S' and a hyperplane determined by S''. The center C of this hypercircle may be regarded as the "best" approximation. Its energy-error is given through the radius R of the hypercircle, which may be calculated from the formula $R=2^{-1}\left|S'-S''(S'\cdot S'')/S''^2\right|$. The method can be extended by using a sequence of states S_1'', S_2'', $\cdots$, and may also be used when the surface displacement is given instead of the surface stress. (Received July 22, 1946.)

316. C. A. Truesdell: *On Behrbohm and Pinl's linearization of the equation of two-dimensional steady flow of a compressible adiabatic fluid.*

In a recent note Behrbohm and Pinl (Zeitschrift für angewandte Mathematik und Mechanik vol. 21 (1941) pp. 193–203) have achieved a new linearization of the potential equation of two-dimensional steady adiabatic compressible flow in generalization of the Minkowski linearization of the equation for minimal surfaces. The author shows that Behrbohm and Pinl's result is equivalent by a simple change of variable to the ordinary linearization by Legendre's transformation, that Behrbohm and Pinl's subsidiary condition on the variables is superfluous, and that hence two of their variables may be interpreted physically as components of the velocity vector. He shows that Behrbohm and Pinl's equation suggests immediately the classical solutions of Tschaplygin. He discusses the possibility of other separations of the variables, and concludes that it is unlikely that any exist. (Received July 12, 1946.)

317. Alexander Weinstein: *On the method of sources and sinks.*

This paper contains an extension of the method of sources and sinks. New types of flows are obtained by taking sources distributed on circumferences, disks and cylinders. The procedure requires a modification of several formulae given by Beltrami who failed to recognize that Stokes' stream function of a circumference is a many-valued function. (Received July 6, 1946.)

GEOMETRY

318. L. A. Dye: *A Cremona transformation in S_3 defined by a pencil of quartic surfaces.*

An involutorial Cremona transformation in S_3 is defined by means of a pencil of quartic surfaces $|F_4|$ with γ_{11} of genus 14 and γ_5 of genus 2, $[\gamma_{11}, \gamma_5]=18$ as a base. The γ_5 lies on a quadric H which intersects $|F_4|$ in γ_5 and a system of twisted cubics $|C_3|$. The residual intersections of the bisecants of a C_3 cut from the associated F_4 are pairs of conjugate points in the involution I. The γ_5 is invariant, not fundamental, and each C_3 is parasitic. After five factors H are removed from the equation of a general homoloid there remains an S_{18}: $\gamma_{11}^{8}+35l_4+4\gamma_2^{2}$. The 35 quadrisecants of γ_{11} and the 4 conics which meet it in 8 points are fundamental curves of the second species. (Received July 11, 1946.)

319. Michael Goldberg: *Tubular linkages.*

Special cases of movable polyhedral linkages are derived. They are composed of two or more prismatic linkages. In particular, if two quadrilateral prismatic linkages are joined to form a movable linkage, the quadrilateral joint may be a plane parallelogram, a skew deltoid or a skew isogram (a skew quadrilateral in which the opposite sides are equal). Some of these are derivable from the Bricard deformable octahedron. (See G. T. Bennett, Proc. London Math. Soc. (2) vol. 10 (1911–1912) pp. 309–343.) Combinations, including toroidal configurations, are exhibited. (Received July 2, 1946.)

320. V. G. Grove: *On congruences and conjugate nets.*

This paper is concerned with a metric study of congruences and conjugate nets on a surface in euclidian space of three dimensions. The formulas convenient to the study are written in terms of arbitrary parameters of the surface, and in terms of certain normalized components of vectors in the tangents to the curves of the net. An invariant is found which plays a role similar to that played by the geodesic curvature. Geometric constructions of all congruences conjugate to a surface and of all normal congruences are found. (Received June 27, 1946.)

321. C. C. Hsiung: *Projective theory of surfaces and conjugate nets in four-dimensional space.*

The purpose of this paper is to establish a theory of the projective differential geometry of surfaces sustaining a conjugate net in space S_4 of four dimensions. By purely geometric determinations a completely integrable system of linear homogeneous partial differential equations in canonical form is introduced, defining a conjugate net in S_4 except for a projective transformation. Some invariants of the parametric conjugate net N_x on an integral surface S of these equations are calculated. If one uses the local power series expansions for S, the curves of N_x and the Laplace transformed surfaces S_{-1}, S_1, the equations in local coordinates of some canonical configurations for S and hyperquadrics having contact of various orders with the surfaces and the curves are obtained. Further, some special classes of conjugate nets are also studied. One of their characterizations may be stated as follows: Conjugate nets with equal and nonzero Laplace-Darboux invariants in S_4 are characterized by the property that there exists a proper hyperquadric (and therefore ∞^3 such hyperquadrics) having second order contact with both S_{-1}, S_1 at the Laplace transformed points x_{-1}, x_1 respectively. (Received July 5, 1946.)

322. Edward Kasner and John DeCicco: *Geometry of harmonic transformations.*

The authors study the geometry of the infinite set (H) of harmonic transformations of the plane. Any such correspondence is defined by a pair of independent harmonic functions. This set (H) does not constitute a group. It is proved that the only real groups contained in (H) are the conformal group, the affine group, and the subgroups of these. The set (H) is characterized by means of the Kasner circle of the polygenic function in the following manner. A point transformation T is harmonic if and only if the associated center transformation is direct conformal. Another characterization is as follows. If, by a point transformation T, every parallel pencil of lines in the transformed plane corresponds to an isothermal family of curves in the original plane, then T is harmonic. Conformalities are the only real maps whereby every pencil of lines and every concentric set of circles correspond to isothermal families. Finally the transformation theory of the third order differential elements at a point, which is induced by (H), is studied. The harmonic set (H) induces a twelve-parameter set S_{12} of the total fifteen-parameter group G_{15} which is induced by the group of arbitrary point transformations. (Received July 19, 1946.)

323. Fred Supnick: *On the packing of spheres.* I.

Let a set of circles $c_1, \cdots, c_n$ with radii $r_1 \leqq r_2 \leqq \cdots \leqq r_n$, respectively, be packed (tangent, non-overlapping, connected) into a strip of width, A, $A/2 \geqq r_n$, bounded by the parallel lines l_1 and l_2, such that if c_1 is tangent to l_1, and c_2 and c_3 tangent to c_1 and l_2, then c_2 and c_3 have no interior point in common. Then the smallest rectangle of width A containing $c_1, \cdots, c_n$ is obtained by the packing $\{\cdots c_{2k} \cdots c_4 c_2 c_1 c_3 c_5 \cdots c_{2k-1} \cdots\}$ and the largest by $\{c_1 c_n c_3 c_{n-2} \cdots - \cdots c_{n-3} c_4 c_{n-1} c_2\}$ where any two adjacent c's are tangent to each other and to opposite sides of the strip. Analogous theorems are proved for the packing of spheres into containers, in particular, into cylinders of circular or regular polygonal cross section. (Received July 13, 1946.)

324. M. L. Vest: *An involutorial space transformation associated with a $Q_{1,n}$ congruence.*

The plane n-ic r, a line s meeting r at an $(n-1)$-point A, and a pencil of surfaces $|F_m| : s^{m-2} g_{4m-4}$ are given. Through a generic point $P(y)$ there passes a single F or $|F|$. The unique line t belonging to the congruence and passing through $P(y)$ meets F a second time in one residual point $Q(x)$, image of $P(y)$ under the transformation thus defined. The residual base curve g of $|F|$ is of order $4m-4$ and is considered noncomposite. It is shown that r, s and g are fundamental curves of the involution and that A is a fundamental point of the second kind. (Received June 28, 1946.)

325. G. W. Walker: *Games of the checkers family in line, plane, and space.*

Many variants from classical checkers can be made by varying the rules of play, particularly the typical move of the typical piece. If the move is in one dimension only, one has linear checkers. In two dimensions, the move may be across an edge of the cell, or through a corner, or both, or there may be a more complicated move like the knight's move in chess. Games can be based on various combinations. Interesting new features appear. In space, using a suitable frame, several excellent cube checker games can be defined, with many interesting new features. The field of play may be a network of white or black cells, or a looser network of cells holding together by their corners, or the entire frame. Places of local safety like the familiar double corner, and other strategical features, appear in new forms. There are many possible kinds of cube checker games, pure, combination, and hybrid games, multiple games, interfering games, cyclical games, and others. The best have already proved more interesting to play than the classical checker game. The checkers family can also be extended to spaces of higher dimensions. (Received July 12, 1946.)

STATISTICS AND PROBABILITY

326. H. W. Becker: *Stirling's numbers of the third kind.*

These are defined by ${}^{x}_{c\pm}S_{\pm r+1} = (x \pm c \pm r)\cdot {}^{x}_{c\pm}S_{\pm r} + {}^{x-1}_{c\pm}S_{\pm r}$, in which the weighting coefficient combines the characteristics of the recurrences for the Stirling's numbers of the first and second kinds. However, tables of the four varieties ± ± are not only matrix products of certain tables of the first two kinds, but also unexpectedly simple vector multiples of the table of binomial coefficients. Summed over c, these are expressible by polynomials which are Chrystal-Jordan factorials of Stirling's polynomials of the second kind, and have the remarkable property of a triple recurrence. For example, ${}^{x}_{+}S_{-r+1} = (1+(x))^{r+1} = ({}_{+}S+x)_{r+1} = {}^{x}_{+}S_{-r} + x\cdot {}^{x-1}_{+}S_{-r} = {}^{x-1}_{+}S_{-r+1} + (r+1)\cdot {}^{x-1}_{+}S_{-r} = {}^{x+1}_{+}S_{-r} + (x-r)\cdot {}^{x}_{+}S_{-r}$. The combinatorial interpretations are rather intricate in terms of either permutations or rhyme schemes. (Received July 15, 1946.)

327. David Blackwell: *Conditional expectation and unbiased sequential estimation.*

It is shown that $E[f(x_\alpha)E_\alpha y] = E(fy)$ whenever $E(fy)$ is finite, and that $\sigma^2(E_\alpha y) \leq \sigma^2(y)$, with equality holding only if $E_\alpha y = y$, where $E_\alpha y$ denotes the conditional expectation of y with respect to the family of chance variables x_α. These results imply that whenever there is a sufficient statistic u and an unbiased estimate t, not a function of u only, for a parameter p, the function $E_u t$, which is a function of u only, is an unbiased estimate for p with variance smaller than that of t. A sequential unbiased estimate for a parameter is obtained, such that when the sequential test terminates after i observations, the estimate is a function of a sufficient statistic for the parameter with respect to these observations. A special case of this estimate is that obtained by Girshick, Mosteller, and Savage (Ann. Math. Statist. vol. 17 (1946) pp. 13–23) for the parameter of a binomial distribution. (Received July 5, 1946.)

328. Paul Boschan: *The consolidated Doolittle technique.*

The quadratic matrix notation is interpreted as a segment in a sequence of matrices wherein each successor matrix is augmented by a bordering row and column. Extension theorems based on this idea date back into the last century. The step from the original concept to one of higher order is also fruitful in discussing inverse matrices, specifically the inverse of a symmetric matrix. The symmetry of the matrix of normal equations for a set of multiple regression-coefficients is restored by adding the transpose of the column on the right side of the equations, that is, the co-variances with the dependent variable and the variance of the dependent variable itself. The inverse of this matrix can be constructed as a partial sum over a series of matrices. Each individual element of this series is in itself meaningful. The solution for the set of multiple regression coefficients relating the kth variable to the preceding $(k-1)$ variables is a column matrix. The product of this matrix with its transpose expressed in terms of the residual variance forms the kth term in the matrix series. The summation of the first n products yields the inverse matrix. This characteristic of the inverse can be used to great advantage in the standardization of elementary computational steps. (Received July 17, 1946.)

329. P. G. Hoel: *The efficiency of the mean moving range.*

The statistic $w=\sum_1^{n-1}|x_{i+1}-x_i|\ \pi^{1/2}/2(n-1)$ is studied as an estimate of σ for a normal variable subject to trend effects. It is shown that the efficiency of w compares favorably with that of the mean square successive difference, δ^2. The proof that w, and also δ^2, is asymptotically normally distributed is made to depend upon a general result that can be derived from a theorem of S. Bernstein on dependent variables. (Received July 13, 1946.)

330. Leo Katz: *On the class of functions defined by the difference equation* $(x+1)f(x+1)=(a+bx)f(x)$.

The difference equation defines only three discrete functions: the binomial, the Poisson and the Pascal functions; the first and third have one parameter (N) slightly generalized. It is shown that the Pascal function with this generalization is identical with the Polya-Eggenburgher distribution, which is a very useful form of the compound Poisson law and has been used to explain probability situations involving contagion. Areas for all functions in the class are given in terms of existing tables of the incomplete- and B-functions. Observed distributions are fitted by two moments. As Carver (*Handbook of mathematical statistics*) pointed out, the advantages of fitting by difference equations are many; not the least is the fact that it is unnecessary to discriminate among the various functions in fitting an observed distribution. The problem of discrimination, posed by Frisch (Metron vol. 10) and others, may be resolved in terms of the sampling distribution of variances for the Poisson function, since the three functions correspond to situations where the variance is less than, equal to, or greater than the mean, respectively. (Received July 13, 1946.)

331. B. F. Kimball: *Some basic theorems for developing tests of fit for the case of the non-parametric probability distribution function.*

Given a universe with c.d.f. $P[X\leq x]=F(x)$, consider a random sample of n values x_i which have been ordered so that $x_i\leq x_{i+1}$. The successive differences of the true c.d.f. values at $X=x_i$ are denoted by u_i. Thus $u_1=F(x_1)$, $u_i=F(x_i)-F(x_{i-1})$, $2\leq i\leq n$, $u_{n+1}=1-F(x_n)$. Theorem 1: The product power moments $E(u_r^p u_s^q u_t^w\cdots)$ for any or all different indices from 1 to $n+1$, where the powers are real numbers greater than -1, are given by $E(u_r^p u_s^q u_t^w\cdots)=\Gamma(n+1)\Gamma(p+1)\Gamma(q+1)\Gamma(w+1)\cdots/\Gamma(n+1+p+q+w+\cdots)$. Theorem 2: Given a test function of u_i, $Y=\sum_m u_i^p$, where p is a real positive number, and the sum is for m indices chosen at random on the range 1 to $n+1$, let $\bar{Y}$ and σ^2 denote the mean and variance of this test function. Establish a convention for increasing the indices included in the above sum for increasing m as n increases, such that $[m/(n+1)]=\text{const.}$, to nearest multiple of $1/(n+1)$. It is proved that the asymptotic distribution of $(Y-\bar{Y})/\sigma$ for increasing n, subject to the above condition, is the normal distribution with zero mean and unit variance, except in the trivial case $m=n+1$, $p=1$. (Received July 16, 1946.)

332. Herman Rubin: *Asymptotic distribution of moments from a system of linear stochastic difference equations.*

Let $\sum_{\tau=0}^{n}B_\tau y'_{t-\tau}+\Gamma z'_t=u'_t$ $(t=1,2,\cdots)$ be a complete system of linear stochastic difference equations determining y_{it} (the coordinates of y_t), $t>0$, in terms of y_{it}, $t\leq 0$, and z_{ik} (the coordinates of z_t), which are assumed to be fixed variates, and the random variables u_{it} (the coordinates of u_t). Such a system is called stable if for every bounded set of fixed variates, and $E(u'_t u_t)$ uniformly bounded, $E(y'_t y_t)$ is uniformly bounded. This condition is shown to be equivalent to $\sum |k_{ij\tau}|$ finite, where $y'_t = \sum_{\tau=0}^{\infty} H_\tau(u'_{t-\tau} - \Gamma z'_{t-\tau}) + \sum_{\nu=1}^{\infty} J_{t,\nu} y'_{t-\nu}$ is the solution of the above difference equation. Let Q_t be an infinite quadratic form in $y_{t-\tau,i}$ and $z_{t-\nu,k}$ (τ, $\nu = 0, 1, \cdots$) with coefficients depending only on i, k, τ, and ν. Such a quadratic form is called convergent if the sum of the absolute values of the coefficients is finite. It is shown under fairly general conditions that the mean of a convergent quadratic form is asymptotically normally distributed with variance $O(1/T)$. (Received July 8, 1946.)

333. F. E. Satterthwaite: *Retention of decimal places in matrix calculations.*

The accumulation of errors in matrix calculations has been studied by the author and others for special types of matrices and for special methods of calculation. In the present paper, error formulae are developed for the standard Doolittle and Waugh-Dwyer compact routines. These formulae do not place any restrictions on the matrices involved and do not require any extra calculations or initial approximations. Simple rules are developed which give for each step in the calculations the number of decimal places which must be retained. These rules are efficient in the sense that the retention of fewer places will, except for good fortune in balancing of errors, lead to results less accurate than those specified. The rules also assist in choosing that arrangement of the calculations which will lead to the smallest average number of significant figures which must be retained for the calculation as a whole. (Received July 15, 1946.)

334. J. W. Tukey: *Sampling from contaminated distributions.* Preliminary report.

A contaminated distribution is a nearly normal distribution in which extreme observations are more frequent than in a normal distribution. By studying the bias and variability of several measures of dispersion when applied to samples from particular one-parameter families of contaminated distributions it is shown that: (i) for nearly normal distributions, the mean deviation is often better than the standard deviation; (ii) small changes in the underlying distribution may increase the sampling variance of the standard deviation by a factor of three. This suggests that, in a broad class of cases, the mean deviation is safer than the standard deviation when a single dispersion is estimated from a set of data. This conclusion need not apply in an analysis of variance situation. (Received July 13, 1946.)

335. Jacob Wolfowitz: *Confidence limits for the fraction of a normal population which lies between two given limits.*

Let $x_1, \cdots, x_N$ be N independent observations from a normal population with mean μ and variance σ^2, both unknown. Let $N\bar{x} = \sum x_i$ and $(N-1)s^2 = \sum (x_i - \bar{x})^2$ define $\bar{x}$ and s^2 Let L_1 and L_2 be given constants with $L_1 < L_2$, and let $\gamma = ((2\pi)^{1/2}\sigma)^{-1}\int_{L_1}^{L_2} \exp(-1/2)\{(y-\mu)/\sigma\}^2 dy$. By a lower confidence limit on γ with confidence coefficient α is meant a function $D(x_1, \cdots, x_N)$ such that the probability is α that $D \leqq \gamma$. Since $\bar{x}$ and s^2 are sufficient estimates of μ and σ^2 the restriction that D be a function of $\bar{x}$ and s only is imposed. It is assumed that there exist (a) a positive d such that $L_1 + d < \mu < L_2 - d$, (b) a positive C such that $\sigma < C$. From these it follows that there exists a lower bound $G = G(d, C)$ on γ. Let $\chi^2_{1-\alpha}$ be that number for which $P\{\chi^2 < \chi^2_{1-\alpha}\} = 1 - \alpha$, where χ^2 has $N-1$ degrees of freedom, and let

$w=(N-1)^{1/2}s/\chi_{1-\alpha}$. It is shown that if D be defined as follows: (1) if $L_1\leqq t\leqq L_2$, $D=(2\pi)^{-1/2}\int_{(L_1-t)/w}^{(L_2-t)/w}\exp\{-y^2/2\}dy$, (2) $D=G$ otherwise, then $|P\{D\leqq\gamma\}-\alpha|$ approaches zero as $N\to\infty$. Thus D is a large sample lower confidence limit. The extension to upper and two-sided limits presents no difficulty. (Received July 8, 1946.)

Topology

336. R. F. Arens: *Convex topological algebras.* Preliminary report.

A convex topological algebra A is a convex topological linear space in which a multiplication of elements is defined, which is, as is addition and scalar multiplication (for definiteness, take the case of real scalars), continuous simultaneously in both factors. This is a generalization of the concept of normed rings. However, the elements with inverses do not form an open set, nor is inversion continuous when possible. The author proves that if A is a division algebra, and is complete in some metric, then A is finite-dimensional, and hence its structure follows from Frobenius' theorem. This result is fundamental for the representation theory of convex topological algebras. (Received June 7, 1946.)

337. R. F. Arens: *Duality in topological linear spaces.*

Let L be any topological linear space with elements x. Let L^* be the set of continuous linear functionals f defined on L, and use in L^* the topology in which convergence of directed sets means uniform convergence on each compact subset of L (the k-topology). Let this construction be repeated, using L^* instead of L, and giving rise to L^{**}, with elements X. Then for each X there is an $x\in L$ such that $X(f)=f(x)$ for each $f\in L^*$. This natural mapping from L^{**} back into L is 1-1 and continuous if L is convex; if furthermore L is complete in some invariant metric (in particular, if L is a Banach space) then the natural mapping is bicontinuous. (Received July 10, 1946.)

338. R. H. Bing: *Skew sets.*

No plane set G contains a collection of five mutually separated sets such that the closure of the sum of any pair of these five sets is the closure of a connected subset of G which is open in G. (Received July 10, 1946.)

339. G. D. Birkhoff and D. C. Lewis: *Chromatic polynomials.*

The number of ways a map P_{n+3} of $n+3$ regions can be colored with λ colors is given by a polynomial $P_{n+3}(\lambda)$ of degree $n+3$. Certain new properties of these chromatic polynomials are established. For instance, if P_{n+3} is regular and if $(-1)^k a_k$ is the coefficient of $(\lambda-2)^{n-k}$ in the expansion of $P_{n+3}(\lambda)/\lambda(\lambda-1)(\lambda-2)$ in powers of $\lambda-2$, it is shown that binomial coefficient $C_k^n\leqq a_k\leqq C_k^{n+k-1}$. Similar results are obtained for expansions in powers of $\lambda-5$. Moreover, extensive numerical calculations indicate that both $P_{n+3}(\lambda)/\lambda(\lambda-1)(\lambda-2)-(\lambda-3)^n$ and $(\lambda-2)^n-P_{n+3}(\lambda)/\lambda(\lambda-1)(\lambda-2)$ are positively completely monotonic for $\lambda\geqq 4$. This conjecture is a very strong form of the usual four-color proposition that $P_n(4)>0$. In connection with reducibility, reduction formulas, and the analysis of rings, the theory of Kempe chains, which has been applied qualitatively with considerable success to the case $\lambda=4$, is generalized so as to yield quantitative results on chromatic polynomials for all values of λ. Typical results on reducible configurations, previously obtained only by use of Kempe chains, are also obtained inductively. The present paper therefore to some extent attempts to bridge the gap between two previously separated methods of attacking the four-color problem. (Received June 27, 1946.)

340. Herbert Federer: *Dimension and measure.*

According to Szpilrajn the dimension of a separable metric space X does not exceed p, $\dim X \leqq p$, if and only if X has a homeomorph $Y \subset E_{2p+1}$ such that the $(p+1)$-dimensional Hausdorff measure of Y equals zero, $H^{p+1}_{2p+1}(Y)=0$. (See Hurewicz and Wallman, *Dimension theory*, chap. 7.) In the present paper it is shown that the foregoing statement remains true if the Hausdorff measure H^{p+1}_{2p+1} is replaced by the $(p+1)$-dimensional integralgeometric Favard measure F^{p+1}_{2p+1}. (For its definition see Federer, *The* (ϕ, k) *rectifiable subsets of* n *space*, Bull. Amer. Math. Soc. Abstract 52-5-145.) Now suppose $A \subset E_n$, $H^k_n(A) < \infty$ and B is the set of all those points of A at which A does *not* have a k-dimensional approximate tangent plane. Then Szpilrajn's theorem implies that $\dim B \leqq \dim A \leqq k$. However it was proved by the writer (in the paper quoted above) that $F^k_n(B)=0$. Hence the new theorem implies that $\dim B \leqq k-1$. This inequality is the best possible. (Received July 10, 1946.)

341. Witold Hurewicz: *Algebraic and topological classification of mappings.*

Let X and Y be locally finite polytopes of finite dimensions. Given an abelian group G, denote by $H^m(X, G)$, $H^m(Y, G)$ the m-dimensional cohomology groups of X and Y, with coefficients from the group G. If f_1 and f_2 are continuous mappings of X into Y, f_1 *is homologous to* f_2 *in dimension* m, if for any abelian group G the mappings f_1 and f_2 induce identical homomorphisms of $H^m(Y, G)$ into $H^m(X, G)$; and f_1 *is homotopic to* f_2 *in dimension* m, if f_1 and f_2 are homotopic when restricted to the m-dimensional skeleton of X. For $m \geqq 2$, π_m is the m-dimensional homotopy group of Y, and H_m the m-dimensional homology group of Y with integer coefficients. The natural homomorphism of π_m into H_m induces a homomorphism of $H^m(X, \pi_m)$ into $H^m(X, H_m)$. The following assumptions are made: (1) Y is simple in dimension m (this means that a continuous image of the m-sphere in Y determines uniquely an element of π_m). (2) The natural homomorphism of $H^m(X, \pi_m)$ into $H^m(X, H_m)$ is an isomorphism (between $H^m(X, \pi_m)$ and a subgroup of $H^m(X, H_m)$). Under these conditions the theorem holds: *If* f_1 *is homotopic to* f_2 *in dimension* $m-1$ *and homologous to* f_2 *in dimension* m, *then* f_1 *is homotopic to* f_2 *in dimension* m. (Received July 19, 1946.)

342. Fred Supnick: *A theorem on rectilinear deformation.*

Let G_i, $i=1, \cdots, n$, be simple closed polygons with G_i in the interior of G_{i-1}. Let rectilinear suspended chains lying in the interior of the ring bounded by G_{i-1} and G_i join a vertex of G_{i-1} to a vertex of G_i such that no two chains have the same end points or intersect each other. The author proves that any such graph can be rectilinearly deformed so that the G_i become convex and the chains rectified. (Received July 13, 1946.)

343. Fred Supnick: *Topology of sphere clusters.* I.

The author defines a *regular* n*-sphere cluster* as a connected set of mutually external, equal Euclidean n-spheres, each tangent to exactly N others. N is called the *degree* of the cluster. The regular circle clusters fall into classes of degrees 0, 1, 2, and 3.

NEW PUBLICATIONS

Geary, A., Lowry, H. V., and Hayden, H. A. Advanced mathematics for technical students. Part 1. London, Longmans, Green, 1945. 8+419 pp. 12s. 6d.

Hayden, H. A. See Geary, A.

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McKay, H. The world of numbers. Cambridge University Press, 1946. 7+198 pp. 8s. 6d.

A manual of operation for the automatic sequence controlled calculator. Cambridge, Harvard University Press; London, Oxford University Press, 1946. 10+561 pp. $10.00.

Mathematical tables. Vol. I: Circular and hyperbolic functions, exponential and sine and cosine integrals, factorial function and allied functions, Hermitian probability functions. 2d ed. Cambridge University Press, 1946. 11+72 pp. $2.50.

Tables of the modified Hankel functions of order one-third and of their derivatives. Cambridge, Harvard University Press; London, Oxford University Press, 1945. 36+235 pp. $10.00.

The structural properties of these classes are studied. The regular 3-sphere clusters fall into classes of degrees 0, 1, · · · , 8. Certain sets of regular 3-sphere clusters of degrees 2, 3, 4, and 5 are constructed and their structural properties studied. The author has constructed an infinite set of regular 3-sphere clusters of degree 6. These have a complicated structure. Sphere clusters of higher dimension are also considered. (Received July 13, 1946.)

344. Fred Supnick: *Topology of sphere clusters. II. Analogue of Kuratowski's theorem.*

Let a sphere cluster of equal, mutually external, Euclidean 3-spheres be given. The linear graph whose vertices are the centers of the spheres and whose edges are the line segments joining the centers of two spheres if and only if they are tangent is called by the author the *structural graph* of the cluster. The author calls a sphere cluster planar if the structural graph of the cluster is planar. It is proved that (1) every sphere cluster of less than eight spheres must be planar and (2) eight spheres are sufficient to obtain each of the non-planar graphs appearing in Kuratowski's theorem. (Received July 13, 1946.)

345. Sister Petronia Van Straten: *Toroidal and non-toroidal graphs.*

A one-dimensional complex of 10 points 0, · · · , 9, and 30 arcs 01, 02, 03, 05, 06, 09, 12, 13, 16, 17, 18, 23, 24, 25, 28, 34, 37, 39, 45, 47, 48, 49, 56, 58, 59, 67, 68, 69, 78, 79 which are mutually disjoint except for terminal points is called a Desargues graph. The Desargues graph is irreducibly non-toroidal in the sense that it is not homeomorphic with a subset of the torus while the sum of every 29 of the 30 segments can be embedded into the torus. The Desargues graph can be embedded into the sphere with two handles. The complex of 9 points 1, · · · , 9 and 27 disjoint arcs 12, 23, 31, 45, 56, 64, 78, 89, 97, 14, 47, 71, 25, 58, 82, 36, 69, 93, 35, 57, 73, 24, 49, 92, 68, 81, 16 is called a Pappus graph. The Pappus graph can be topologically embedded into the torus. The names of the graphs are derived from the corresponding projective configurations. (Received June 28, 1946.)

346. J. W. T. Youngs: *The representation problem for Fréchet surfaces.* Preliminary report.

This paper contains a solution of the representation problem for Fréchet surfaces in which the base space is the closure of a region on the place bounded by a finite number of Jordan curves. The solution is a generalization of a result announced in Bull. Amer. Math. Soc. Abstract 52-5-221. (Received July 5, 1946.)

Since $\gamma_1, \gamma_2, \cdots$ are all on $0 \leqq t < p$, it follows that $c_1, c_2, \cdots$ are integers satisfying $0 \leqq c_\nu \leqq p-1$. Hence we have a correspondence

$$\text{(B)} \qquad \gamma_0 \to \{c_\nu\}$$

from all reals $\gamma_0 \geqq 0$, to sequences of integers as described.

3. **Termination in $p-1$.** We ask now whether, under the algorithm, sequences may appear terminating in $p-1, p-1, \cdots$. Such is the case if and only if, for the function $f(t)$:

(C) There exists a $\gamma_0 = p-1+f(\gamma_0)$, $p-1 < \gamma_0 < p$.

Obviously such a $\gamma_0 \to \{p-1, p-1, \cdots\}$ under (A). On the other hand, if a number δ_0 under (A) yields a sequence terminating in $p-1, p-1, \cdots$, this implies that some δ_ν itself yields $p-1, p-1, \cdots$. Suppose then that γ_0 $(=\delta_\nu)$ under (A) gives $\gamma_0 = p-1+f(\gamma_1)$, $\gamma_1 = p-1+f(\gamma_2), \cdots, \gamma_\nu = p-1+f(\gamma_{\nu+1})$, and so on. If (C) is false, it follows from continuity of $f(t)$ that:

$$\text{(D)} \qquad f(t) > t - (p-1), \quad \text{for all } t \text{ on } p-1 \leqq t < p.$$

We should then have $\gamma_1 - (p-1) < f(\gamma_1) = \gamma_0 - (p-1)$, $\gamma_2 - (p-1) < f(\gamma_2) = \gamma_1 - (p-1)$, and so on, with $p > \gamma_0 > \gamma_1 > \gamma_2 > \cdots > p-1$. Hence $t_0 = \lim \gamma_\nu$ exists, with $p-1 \leqq t_0 < p$. But from $\gamma_\nu = p-1+f(\gamma_{\nu+1})$, we have $t_0 = p-1+f(t_0)$, a contradiction.

Indeed, the condition "not C" is equivalent to (D), and (D) may in turn be rephrased as a slope condition

$$\text{(D}') \qquad (f(p) - f(t))/(p-t) < 1 \text{ on } p-1 \leqq t < p.$$

Moreover, if a γ_0 satisfying (C) exists, then not only γ_0 but every δ_0 on $\gamma_0 < \delta_0 < p$ will yield $\{p-1, p-1, \cdots\}$ under (A). For $\gamma_0 = p-1 + f(\gamma_0) < \delta_0 < p$ implies $\delta_0 = p-1+f(\delta_1)$, hence $\gamma_0 < \delta_1 < p$, and so on. Since our final object is to obtain a one-one correspondence (B), we assume from this point on the necessary condition (D$'$). *The correspondence* (B) *then maps all reals* $\gamma_0 \geqq 0$ *onto non-*$(p-1)$*-terminating sequences.*

4. **Upper and lower limits.** Let $\{c_\nu\}$ be an arbitrary sequence of integers with $0 \leqq c_0 \leqq \infty$; $0 \leqq c_\nu \leqq p-1$, $\nu = 1, 2, \cdots$, not $(p-1)$-terminating. We define $C_\nu^\lambda = c_\lambda + f(c_{\lambda+1} + \cdots + f(c_{\lambda+\nu})$ and $\Gamma_\nu^\lambda = c_\lambda + f(c_{\lambda+1} + \cdots + f(c_{\lambda+\nu}+1)$, where the last parentheses are ν-fold. Then, from monotonicity, one has $c_\lambda \leqq C_\nu^\lambda \leqq C_{\nu+1}^\lambda < \Gamma_{\nu+1}^\lambda \leqq \Gamma_\nu^\lambda \leqq c_\lambda + 1$, so that the limits $C^\lambda = \lim C_\nu^\lambda$, $\Gamma^\lambda = \lim \Gamma_\nu^\lambda$ exist and satisfy

$$\text{(E)} \qquad c_\lambda \leqq C^\lambda \leqq \Gamma^\lambda \leqq c_\lambda + 1.$$

Since $C_\nu^\lambda = c_\lambda + f(C_{\nu-1}^{\lambda+1})$ we have $C^\lambda = c_\lambda + f(C^{\lambda+1})$ and similarly

REPRESENTATIONS FOR REAL NUMBERS

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1. **Introduction.** In a recent paper[2] [1] B. H. Bissinger generalized continued fractions by iteration of more general decreasing functions than the $1/x$ of the classical case. We extend here the algorithm by which real numbers are represented as decimals of base p, to general continuous increasing functions on $(0, p)$, including the classical x/p as special case. This sets up a correspondence from real numbers to sequences of integers mod p. Weak sufficient conditions are given that the correspondence be one-one. In the one-one case, algebraic examples are noted. The limit involved in the inscribed polygon problem appears here in a natural way. In the many-one case, the algorithm defines a set L of limit numbers which is perfect and nowhere dense. These sets are closely related to the Cantor perfect set. Finally, the relation between the above theory and the topological transformations F_1 of the unit interval into itself is studied. The latter yield sequences $\{F_p\}$ of our functions, $p=2, 3, \cdots$, and their structure is reflected in the limit sets $L_2, L_3, \cdots$.

2. **The algorithm.** Let $p \geqq 2$ be a fixed integer and $f(t)$ a continuous, strictly increasing function on the interval $0 \leqq t \leqq p$, with $f(0)=0$ and $f(p)=1$ (cf. [4]).

Such a function may be used to associate with every real number $\gamma_0 \geqq 0$, a sequence $\{c_\nu\}$ of integers, with $0 \leqq c_0 < \infty$, $0 \leqq c_\nu \leqq p-1$, $\nu=1, 2, \cdots$, by way of the following algorithm. We write, for $\gamma_0 \geqq 0$,

$$\text{(A)} \qquad \begin{array}{ll} \gamma_0 = c_0 + f(\gamma_1), & c_0 \leqq \gamma_0 < c_0 + 1, \\ & 0 \leqq c_0 < \infty, 0 \leqq \gamma_1 < p, \\ \gamma_1 = c_1 + f(\gamma_2), & c_1 \leqq \gamma_1 < c_1 + 1, \\ & 0 \leqq c_1 \leqq p-1, 0 \leqq \gamma_2 < p, \end{array}$$

and so on.

Thus, at each step, c_ν is the greatest integer in γ_ν, and $\gamma_{\nu+1}$ is the uniquely defined real number on the interval $0 \leqq t < p$ such that $f(\gamma_{\nu+1}) = \gamma_\nu - c_\nu$, where $0 \leqq \gamma_\nu - c_\nu < 1$.

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[2] The numbers in brackets refer to the references cited at the end of the paper.

must have a limiting relative frequency of exactly β^{-1}. In a similar manner it can be shown that the limiting relative frequency of any combination of k digits is β^{-k}. Hence the theorem is proved.

We make the following conjectures. First let $f(x)$ be any polynomial. It is very likely that $0.f(1)f(2)\cdots$ is normal. Besicovitch[4] proved this for $f(x)=x^2$. In fact he proved that the squares of almost all integers are (ϵ, k) normal. This no doubt holds for polynomials.

Second let $\beta_1, \beta_2, \cdots, \beta_r$ be integers such that no β is a power of any other. Then for any $\eta>0$ and large enough r the number of integers $m\leqq n$ which are not (ϵ, k) normal for any of the bases β_i, $i\leqq r$, is less than n^η. We cannot prove this conjecture.

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[4] Ibid. p. 154.

$$\beta\left(\sum\nolimits_1 \beta^k + \sum\nolimits_2 \beta^k\right) < \beta(x+1)\left\{(\rho_1^{-\epsilon/2}\beta)^n + (\rho_2^{-\epsilon/2}\beta)^n\right\}$$
$$< \beta^{\delta(n-1)} \leqq N^\delta$$

and the lemma is established for $(\epsilon, 1)$ normality.

The extension to the case of (ϵ, k) normality is accomplished by a method similar to that used by Borel[3] and we shall only outline the proof. Consider the digits $b_0, b_1, \cdots$ of a number $m \leqq N$ grouped as follows:

$$b_0, b_1, \cdots, b_{k-1}; b_k, \cdots, b_{2k-1}; b_{2k}, \cdots, b_{3k-1}; \cdots.$$

Each of these groups represents a single digit of m when m is expressed in the base β^k. Hence there are at most N^δ integers $m \leqq N$ for which the frequency among these groups of a given combination of k digits falls outside the interval from $\beta^{-k}-\epsilon$ to $\beta^{-k}+\epsilon$.

The same holds for

$$b_1, b_2, \cdots, b_k; b_{k+1}, \cdots, b_{2k}; \cdots,$$

and so on. This gives our result.

To prove the theorem consider the numbers $a_1, a_2, \cdots$ of the increasing sequence up to the largest a less than or equal to N where $N=\beta^n$. At least $N^\theta - N^{(1-\epsilon)}$ of these numbers have at least $n(1-\epsilon)$ digits since by hypothesis there are at least N^θ of the numbers in this sequence and since at most $\beta^{n(1-\epsilon)} = N^{1-\epsilon}$ of them have fewer than $n(1-\epsilon)$ digits. Hence these numbers altogether have at least $n(1-\epsilon)(N^\theta - N^{1-\epsilon})$ digits. Let f_N be the relative frequency of the digit 0. It follows from the lemma that the number of a's for which the frequency of the digit 0 exceeds $\beta^{-1}+\epsilon$ is at most N^δ and hence

$$f_N < \beta^{-1} + \epsilon + \frac{nN^\delta}{n(1-\epsilon)(N^\theta - N^{1-\epsilon})}$$
$$= \beta^{-1} + \epsilon + \frac{N^{\delta-\theta}}{(1-\epsilon)(1-N^{1-\epsilon-\theta})}.$$

Since we are permitted to take θ greater than δ and greater than $1-\epsilon$ it follows that $\lim_{N\to\infty} f_N$ is at most $\beta^{-1}+\epsilon$ and hence at most β^{-1}. Of course we have allowed N to become infinite only through values of the form β^n but this restriction can readily be removed. A similar result holds for the digits $1, 2, \cdots, \beta-1$ and hence each of these digits

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[3] Ibid. p. 147.

We prove the following lemma.

LEMMA. *The number of integers up to N (N sufficiently large) which are not (ϵ, k) normal with respect to a given base β is less than N^{δ} where $\delta = \delta(\epsilon, k, \beta) < 1$.*

First we prove the lemma for $(\epsilon, 1)$ normality. Let x be such that $\beta^{x-1} \leqq N < \beta^{x}$. Then there are at most

$$\beta \sum\nolimits_1 \beta_k + \beta \sum\nolimits_2 \beta_k$$

numbers up to N among whose digits there are less than $x(1-\epsilon)/\beta$ 0's, 1's, and so on, or more than $x(1+\epsilon)/\beta$ 0's, 1's, and so on, where $\beta_k = (\beta-1)^{x-k}C_{x,k}$ and where the summations $\sum_1$ and $\sum_2$ are extended over those values of k for which $k < (1-\epsilon)x/\beta$ and $k > (1+\epsilon)x/\beta$, respectively. The remaining numbers must have between $x(1-\epsilon)$ and $x(1+\epsilon)$ digits and hence for these remaining numbers the relative frequencies of 0's, 1's, 2's, and so on, must lie between $(1-\epsilon)/\beta(1+\epsilon)$ and $(1+\epsilon)/\beta(1-\epsilon)$. We have to show that $\beta(\sum_1\beta_k + \sum_2\beta_k) < N^{\delta}$. The following inequalities result from the fact that the terms of the binomial expansion increase up to a maximum and then decrease.

$$\sum\nolimits_1 \beta_k < (x+1)\beta_{r_1}, \qquad \sum\nolimits_2 \beta_k < (x+1)\beta_{r_2}, \tag{1}$$

where

$$r_1 = [(1-\epsilon)x/\beta], \qquad r_2 = [(1+\epsilon)x/\beta] \tag{2}$$

and where $[(1-\epsilon)x/\beta]$ is the largest integer less than or equal to $(1-\epsilon)x/\beta$. Similarly for r_2. By repeated application of the relation

$$\beta_{k+1}/\beta_k = (x-k)/(k+1)(\beta-1) \tag{3}$$

we obtain

$$\beta_{r_1}\rho_1^{\epsilon x/2} < \beta_{r'_1} < \beta^{x}$$

where

$$r'_1 = [(1-\epsilon/2)/\beta], \qquad \rho_1 = (x-r_1)/(r_1+1)\beta - 1)$$

and where $\rho_1 > 1$ for x sufficiently large. It follows that

$$\beta_{r_1} < (\rho_1^{-\epsilon/2}\beta)^{x}$$

and similarly

$$\beta_{r_2} < (\rho_2^{-\epsilon/2}\beta)^{x}.$$

Hence

NOTE ON NORMAL NUMBERS

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D. G. Champernowne[1] proved that the infinite decimal

$$0.123456789101112 \cdots$$

was normal (in the sense of Borel) with respect to the base 10, a normal number being one whose digits exhibit a complete randomness. More precisely a number is normal provided each of the digits $0, 1, 2, \cdots, 9$ occurs with a limiting relative frequency of $1/10$ and each of the 10^k sequences of k digits occurs with the frequency 10^{-k}. Champernowne conjectured that if the sequence of all integers were replaced by the sequence of primes then the corresponding decimal

$$0.12357111317 \cdots$$

would be normal with respect to the base 10. We propose to show not only the truth of his conjecture but to obtain a somewhat more general result, namely:

THEOREM. *If $a_1, a_2, \cdots$ is an increasing sequence of integers such that for every $\theta < 1$ the number of a's up to N exceeds N^θ provided N is sufficiently large, then the infinite decimal*

$$0.a_1a_2a_3 \cdots$$

is normal with respect to the base β in which these integers are expressed.

On the basis of this theorem the conjecture of Champernowne follows from the fact that the number of primes up to N exceeds $cN/\log N$ for any $c<1$ provided N is sufficiently large. The corresponding result holds for the sequence of integers which can be represented as the sum of two squares since every prime of the form $4k+1$ is also of the form x^2+y^2 and the number of these primes up to N exceeds $c'N/\log N$ for sufficiently large N when $c'<1/2$.

The above theorem is based on the following concept of Besicovitch.[2]

DEFINITION. *A number A (in the base β) is said to be (ϵ, k) normal if any combination of k digits appears consecutively among the digits of A with a relative frequency between $\beta^{-k}-\epsilon$ and $\beta^{-k}+\epsilon$.*

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[1] J. London Math. Soc. vol. 8 (1933) pp. 254–260.

[2] Math. Zeit. vol. 39 (1935) pp. 146–147.

$$|P(x^{(\lambda)})| < 2^{-k}\prod_{\mu=1}^{k}\left[\frac{\mu l-1}{k}\right] \qquad (\lambda = 1, 2, \cdots, l),$$

then $P(x)$ *is irreducible in the field of rational numbers.*

PROOF. If $P(x)$ is reducible, then

$$P(x) = f(x)\cdot g(x).$$

If the degree of $f(x)$ is $k^* \geqq n/2$, then the degree of $g(x)$ is $n-k^*$. It follows from Theorem 3a that $k^* \leqq k-1$, and from Theorem 4a that $n-k^* \leqq h-1$, hence $n \leqq k+h-2$. This gives a contradiction.

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COROLLARY. *Let $h(x)$ be an arbitrary integral polynomial of degree $n-1$ and $x_1, x_2, \cdots, x_l$ different integers. The integral polynomial $P(x)$ of degree n,*

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_l)h(x) + t,$$

is irreducible in the field of rational numbers if $l \geqq 5$, $n > l > n/2$ and $1 \leqq |t| < H$ where $H = [(l+1)/2]$ for $l \geqq 7$ and $H = 3/2$ for $l = 6$ and 5.

For the proof of Theorem 4 for $l > 12$ it is sufficient to prove only (26) instead of (24). We proved here (24) in order to obtain the following theorem.

THEOREM 4a. *Let $P(x)$ be an integral polynomial of degree n; let h and l be integers satisfying the following conditions:*

$$n > l > n/2, \qquad l > h \geqq 2, \quad \text{and} \quad l > 12, \text{ or } l = 11, 9.$$

If for l different integers $x_1, x_2, \cdots, x_l$

$$0 < |P(x_\lambda)| < \phi(h) = 2^{-h} \prod_{\mu=1}^{h} \left[\frac{\mu l - 1}{h}\right],$$

then $P(x)$ cannot contain a factor of degree h^* with $h \leqq h^* \leqq n/2$.

PROOF. If $P(x)$ contains a factor of degree h^*, then by (28) and (24)

$$\max_{\lambda=1,2,\cdots,l} |P(x_\lambda)| \geqq \max_{\lambda=1,2,\cdots,l} |g(x_\lambda)| \geqq \phi(h^*) \geqq \phi(h).$$

This contradicts our assumption.

A similar theorem can be obtained for $l = 12$, 10, 8, and 7.

By combining Theorems 3a and 4a the following theorem is obtained.

THEOREM 5. *Let $P(x)$ be an integral polynomial of degree n, and k, l, and h integers satisfying the following conditions:*

$$k \geqq [(n + 1)/2], \qquad n > l > n/2, \; l > 12 \text{ or } l = 11 \text{ or } 9,$$
$$l > h, \qquad n \geqq k + h - 1.$$

If for n different integers $x_1, x_2, \cdots, x_n$

$$0 < |P(x_\nu)| < 2^{-k} \prod_{\kappa=1}^{k} \left[\frac{\kappa n - 1}{k}\right] \qquad (\nu = 1, 2, \cdots, n),$$

and for l of these x_ν, say $x^{(1)}, x^{(2)}, \cdots, x^{(l)}$, since $6l-2<7(l-1)$. It follows from (22), (23), and (12) that (19) holds for odd $l>7$, too.

Now we have by (16) and (19)

$$\phi(k+1) \geqq \phi(k) \quad \text{for} \quad 2 \leqq k \leqq l-2, \text{ and } l > 12, l = 11, 9. \tag{24}$$

By direct calculation we obtain

$$\begin{aligned} &\phi(5) > \phi(6) > \phi(4) \quad \text{for} \quad l = 12, \\ &\phi(4) > \phi(5) > \phi(3) \quad \text{for} \quad l = 10, \\ &\phi(3) > \phi(4) > \phi(2) \quad \text{for} \quad l = 8 \text{ and } l = 7. \end{aligned} \tag{25}$$

By (24) and (25) we have for $l \geqq 7$

$$\phi(k) \geqq \phi(2) = \left[\frac{2l-1}{2}\right]\left[\frac{l-1}{2}\right] > \left[\frac{l+1}{2}\right] = H \quad (3 \leqq k < l), \tag{26}$$

and by direct calculation

$$\phi(k) \geqq 3/2 = H \quad \text{for} \quad l = 6 \text{ and } 5 \qquad (2 \leqq k < l), \tag{27}$$

hence, by (15), (26), and (27),

$$\max_{\lambda=1,2,\cdots,l} |P(x_\lambda)| \geqq \max_{\lambda=1,2,\cdots,l} |g(x_\lambda)| \geqq \phi(k) \geqq H. \tag{28}$$

This contradicts our assumption, and $P(x)$ must be irreducible.

Theorem 4 cannot be improved further. If we assume instead of

$$0 < |P(x_\lambda)| < [(l+1)/2] \qquad (\lambda = 1, 2, \cdots, l)$$

only

$$0 < |P(x_\lambda)| \leqq [(l+1)/2] \qquad (\lambda = 1, 2, \cdots, l),$$

then $P(x)$ may be reducible.

This is shown by the following examples

$$P(x) = x\left\{h(x) \prod_{\lambda=1}^{l/2} (x^2 - \lambda^2) + 1\right\} \quad \text{for even } l,$$

$$P(x) = x\left\{h(x)\left(x - \frac{l+1}{2}\right) \prod_{\lambda=1}^{(l-1)/2} (x^2 - \lambda^2) + 1\right\} \quad \text{for odd } l,$$

where $h(x)$ is an arbitrary integral polynomial of degree $n-l-1$. We have here $P(x)=x$ for $x=\pm 1, \pm 2, \cdots, \pm l/2$ for even l, and $x=\pm 1, \pm 2, \cdots, \pm(l-1)/2, +(l+1)/2$ for odd l. At l integral points these polynomials take values which are different from zero and, without regard to sign, less than or equal to $[(l+1)/2]$; but they are reducible.

$$\phi(k+1) \geqq \phi(k) \quad \text{for} \quad l > 12 \text{ and } l = 11, 9. \tag{19}$$

For even l we have $k=(l-2)/2$. It follows from (7) and (8) that for $l>12$

$$\begin{aligned}\phi(k+1) = \phi\left\{\left[\frac{l+1}{2}\right]\right\} &\geqq 2^{-l/2}\cdot 1\ 3\cdot 5\cdot 7\cdot \prod_{\kappa=5}^{k+1}\left[\frac{\kappa l-1}{k+1}\right]\\ &= 2^{-l/2}\cdot 1\cdot 3\cdot 5\cdot 7\cdot \prod_{\kappa=4}^{k}\left[\frac{(\kappa+1)l-1}{k+1}\right]\end{aligned} \tag{20}$$

and

$$\begin{aligned}\phi(k) &= \phi\left\{\left[\frac{l-1}{2}\right]\right\}\\ &= 2^{-(l-2)/2}\left[\frac{l-1}{(l-2)/2}\right]\left[\frac{2l-1}{(l-2)/2}\right]\left[\frac{3l-1}{(l-2)/2}\right]\prod_{\kappa=4}^{k}\left[\frac{\kappa l-1}{k}\right]\\ &= 2^{-(l-2)/2}\left[\frac{2l-2}{l-2}\right]\left[\frac{4l-2}{l-2}\right]\left[\frac{6l-2}{l-2}\right]\prod_{\kappa=4}^{k}\left[\frac{\kappa l-1}{k}\right].\end{aligned}$$

Now $[(6l-2)/(l-2)]=6$ since $6l-2<7(l-2)$ for $l>12$. Similarly $[(2l-2)/(l-2)]=2$ and $[(4l-2)/(l-2)]=4$, hence

$$\phi(k) \geqq 2^{-(l-2)/2}\cdot 2\cdot 4\cdot 6\cdot \prod_{\kappa=4}^{k}\left[\frac{\kappa l-1}{k}\right]. \tag{21}$$

Since $3\ 5\ 7>2\cdot 2\cdot 4\cdot 6$, it follows from (20), (21), and (12) that (19) holds for even $l>12$.

Now, let l be odd. Here we have by (7) and (9) for $l>7$

$$\begin{aligned}\phi(k+1) = \phi\left(\frac{l+1}{2}\right) &= 2^{-(l+1)/2}\cdot 1\cdot 3\ 5\cdot 7\cdot \prod_{\kappa=5}^{k+1}\left[\frac{\kappa l-1}{k+1}\right]\\ &= 2^{-(l+1)/2}\cdot 1\cdot 3\cdot 5\cdot 7\cdot \prod_{\kappa=4}^{k}\left[\frac{(\kappa+1)l-1}{k+1}\right]\end{aligned} \tag{22}$$

and

$$\begin{aligned}\phi(k) &= \phi\left(\frac{l-1}{2}\right)\\ &= 2^{-(l-1)/2}\left[\frac{2l-2}{l-1}\right]\left[\frac{4l-2}{l-1}\right]\left[\frac{6l-2}{l-1}\right]\prod_{\kappa=4}^{k}\left[\frac{\kappa l-1}{k}\right]\\ &= 2^{-(l-1)/2}\cdot 2\cdot 4\ 6\cdot \prod_{\kappa=4}^{k}\left[\frac{\kappa l-1}{k}\right]\end{aligned} \tag{23}$$

where $H = [(l+1)/2]$ for $l \geqq 7$, $H = 3/2$ for $l = 6$ and 5, *then* $P(x)$ *is irreducible in the field of rational numbers.*

PROOF. If $P(x)$ is reducible, then it must contain a factor $g(x)$ of degree k less than or equal to $n/2$ with integral coefficients. Here $k > 1$. For a linear polynomial takes each value only once, hence the l integers $g(x_1), g(x_2), \cdots, g(x_l)$ must be different. However, only the $2[(l-1)/2]$ values $\pm 1, \pm 2, \cdots, \pm[(l-1)/2]$ are possible because

$$0 < |g(x_\lambda)| \leqq |P(x_\lambda)| < H \leqq \left[\frac{l+1}{2}\right] \qquad (\lambda = 1, 2, \cdots, l).$$

This gives a contradiction since $l > 2[(l-1)/2]$.

It follows from Theorem 2 that

$$\max_{\lambda=1,2,\cdots,l} |g(x_\lambda)| \geqq 2^{-k} \prod_{\kappa=1}^{k} \left[\frac{\kappa l - 1}{k}\right]. \tag{15}$$

We denote the right-hand side of (15), for a given l, similarly as in (10), by $\phi(k)$ and maintain that

(16) $\phi(k+1) \geqq \phi(k)$ for $2 \leqq k \leqq (l-3)/2$ and for $l/2 \leqq k \leqq l-2$.

It follows from the proof of Theorem 3 that (16) holds for $l/2 \leqq k \leqq l-2$ if we write l instead of n since $[(l+1)/2] = l/2$ for even l and $k \neq l/2$ for odd l.

Now we consider the case $k \leqq (l-3)/2$. Here we have

$$\left[\frac{l-1}{k+1}\right] \geqq \left[\frac{l-1}{(l-1)/2}\right] = 2. \tag{17}$$

Moreover, it follows from (12) that

$$\left[\frac{(\kappa+1)l - 1}{k+1}\right] \geqq \left[\frac{\kappa l - 1}{k}\right] \qquad (\kappa = 1, 2, \cdots, k). \tag{18}$$

We proved (12) only for $\kappa \geqq 2$; but the proof remains correct for $\kappa = 1$. It follows now from (18) and (17) that

$$\begin{aligned} 2^{-k-1} \prod_{\kappa=0}^{k} \left[\frac{(\kappa+1)l - 1}{k+1}\right] &= 2^{-k-1} \prod_{\kappa=1}^{k+1} \left[\frac{\kappa l - 1}{k+1}\right] \\ &\geqq 2^{-k} \prod_{\kappa=1}^{k} \left[\frac{\kappa l - 1}{k}\right], \end{aligned}$$

hence $\phi(k+1) \geqq \phi(k)$ for $k \leqq (l-3)/2$, and (16) is proved.

We now consider the remaining value $k = [(l-1)/2]$. We maintain that also here

Now we have by the formula of Wallis

$$\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots(2m-2)} \sim \left(\frac{4m}{\pi}\right)^{1/2}.$$

For even n we obtain

$$\frac{G}{G_2} = \left(\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots(n-2)}\right)^{1/2} \sim \left(\frac{2n}{\pi}\right)^{1/4},$$

and for odd n

$$\frac{G}{G_2} = \frac{1}{2^{1/2}}\left(\frac{(n-1)\cdot 1\cdot 3\cdot 5\cdots(n-2)}{2\cdot 4\cdot 6\cdots(n-3)}\right)^{1/2} \sim \left(\frac{n}{2}\right)^{1/2}\left(\frac{2n}{\pi}\right)^{1/4} = \left(\frac{n^3}{2\pi}\right)^{1/4}.$$

Moreover, we have $G > G_2$ for $n > 3$.

COROLLARY. *The integral polynomials*

$$A(x - x_1)(x - x_2)\cdots(x - x_n) + t \tag{14}$$

are irreducible in the field of rational numbers if $x_\kappa \neq x_\nu$ *and* $1 \leq |t| < G$.

We have $G > 1$ for $n > 4$. Therefore the polynomials (14) are irreducible for $t = \pm 1$ and $n > 4$. This is, as already mentioned in the introduction, for $A = 1$ a theorem of Schur, and for arbitrary A a theorem of Dorwart and Ore.

We can formulate Theorem 3 also in the following form:

THEOREM 3a. *Let* $P(x)$ *be an integral polynomial of degree* n. *If for* n *different integers* $x_1, x_2, \cdots, x_n$

$$0 < |P(x_\nu)| < 2^{-k}\prod_{\kappa=1}^{k}\left[\frac{\kappa n - 1}{k}\right] = \phi(k) \qquad (\nu = 1, 2, \cdots, n)$$

where $k \geq [(n+1)/2]$, *then* $P(x)$ *cannot contain a factor* $f^*(x)$ *of degree* k^* *with* $k \leq k^* < n$.

PROOF. If $P(x)$ contains such a factor, then by (6), (7), and (11)

$$\max_{\nu=1,2,\cdots,n} |P(x_\nu)| \geq \max_{\nu=1,2,\cdots,n} |f^*(x_\nu)| \geq \phi(k^*) \geq \phi(k).$$

This contradicts our assumption.

THEOREM 4. *Let* $P(x)$ *be a polynomial of degree* n *with integral coefficients,* l *an integer with* $l \geq 5$, *and* $n > l > n/2$. *If for* l *different integers* $x_1, x_2, \cdots, x_l$

$$0 < |P(x_\lambda)| < H \qquad (\lambda = 1, 2, \cdots, l)$$

$$\kappa n - 1 = qk + r \qquad (0 \leqq r < k),$$

hence $\kappa n > qk$ and $n > q$ since $\kappa \leqq k$. It follows that

$$\left[\frac{(\kappa + 1)n - 1}{k + 1}\right] = \left[\frac{qk + r + n}{k + 1}\right] = \left[\frac{q(k + 1) + r + n - q}{k + 1}\right]$$
$$\geqq q = \left[\frac{\kappa n - 1}{k}\right].$$

This proves (12). Moreover we have for $\kappa = 1$

$$\left[\frac{2n - 1}{k + 1}\right] \geqq \left[\frac{2n - 1}{n - 1}\right] = 2 = 2\left[\frac{n - 1}{k}\right] \tag{13}$$

because it is sufficient for the proof of (11) to assume that $k+1 \leqq n-1$ and $k \geqq [(n+1)/2]$. Multiplying (13) and (12) for $\kappa = 2, 3, \cdots, k$ we obtain

$$\prod_{\kappa=1}^{k}\left[\frac{(\kappa + 1)n - 1}{k + 1}\right] \geqq 2\prod_{\kappa=1}^{k}\left[\frac{\kappa n - 1}{k}\right],$$

hence, since $[(n-1)/(k+1)] = 1$,

$$2^{-k-1}\prod_{\kappa=1}^{k+1}\left[\frac{\kappa n - 1}{k + 1}\right] = 2^{-k-1}\left[\frac{n - 1}{k + 1}\right]\prod_{\kappa=1}^{k}\left[\frac{(\kappa + 1)n - 1}{k + 1}\right]$$
$$\geqq 2^{-k}\prod_{\kappa=1}^{k}\left[\frac{\kappa n - 1}{k}\right].$$

This proves (11). It follows now from (6), (7), (11), and (10) that

$$M \geqq G.$$

Since $P(x)/f(x)$ is a polynomial with integral coefficients and $P(x_\nu) \neq 0$, we obtain

$$\max_{\nu=1,2,\cdots,n} |P(x_\nu)| \geqq \max_{\nu=1,2,\cdots,n} |f(x_\nu)| = M \geqq G.$$

This contradicts our assumption, and the theorem is proved.

The bound of Tatuzawa is

$$G_2 = (2^{-n}(n - 1)!)^{1/2}$$

and our bound by (8) and (9)

$$G = \frac{(n - 1)!}{2^{n-1}\{[(n - 2)/2]!\}} = \begin{cases} 2^{-n/2}\cdot 1\cdot 3\cdot 5 \cdots (n - 1) & \text{for even } n, \\ 2^{-(n+1)/2}(n - 1)\cdot 1\cdot 3\cdot 5 \cdots (n - 2) & \\ & \text{for odd } n. \end{cases}$$

$$\left[\frac{\kappa n-1}{k}\right]=\left[\frac{\kappa n-1}{n/2}\right]=\left[\frac{2\kappa n-2}{n}\right]=2\kappa-1\qquad(\kappa=1,2,\cdots,k),$$

hence by (7)

$$\begin{aligned}\phi\{[(n+1)/2]\}&\geqq 2^{-k}\prod_{\kappa=1}^{k}(2\kappa-1)=2^{-n/2}\cdot 1\cdot 3\cdot 5\cdots(n-1)\\&=\frac{(n-1)!}{2^{n/2+(n-2)/2}[(n-2)/2]!}=\frac{(n-1)!}{2^{n-1}\{[(n-2)/2]!\}}=G.\end{aligned}\tag{8}$$

If n is odd, then

$$\begin{aligned}\left[\frac{\kappa n-1}{k}\right]&=\left[\frac{\kappa n-1}{(n+1)/2}\right]=\left[\frac{2\kappa n-2}{n+1}\right]\\&=\left[\frac{(2\kappa-1)(n+1)+n-2\kappa-1}{n+1}\right],\end{aligned}$$

hence

$$\left[\frac{\kappa n-1}{k}\right]=\begin{cases}2\kappa-1 & \text{for}\quad \kappa=1,2,\cdots,(n-1)/2,\\ 2\kappa-2 & \text{for}\quad \kappa=(n+1)/2,\end{cases}$$

and by (7)

$$\begin{aligned}\phi\{[(n+1)/2]\}&\geqq 2^{-(n+1)/2}(n-1)\prod_{\kappa=1}^{(n-1)/2}(2\kappa-1)\\&=\frac{(n-1)!}{2^{(n+1)/2+(n-3)/2}\{[(n-3)/2]!\}}=\frac{(n-1)!}{2^{n-1}\{[(n-2)/2]!\}}=G.\end{aligned}\tag{9}$$

It follows from (8) and (9) that

$$\phi\left(\left[\frac{n+1}{2}\right]\right)\geqq G.\tag{10}$$

Now we maintain that

$$\phi(k+1)\geqq\phi(k)\quad\text{for}\quad\left[\frac{n+1}{2}\right]\leqq k\leqq n-2.\tag{11}$$

For this purpose we want to prove that

$$\left[\frac{(\kappa+1)n-1}{k+1}\right]\geqq\left[\frac{\kappa n-1}{k}\right]\qquad(\kappa=2,3,\cdots,k).\tag{12}$$

We divide $\kappa n-1$ by k:

$$\left[\frac{\beta n-1}{k}\right] \geqq \left[\frac{(\beta-\alpha)n-1}{k}\right]+\left[\frac{\alpha n}{k}\right],$$

hence by (4)

$$\begin{aligned} x'_\beta - x'_\alpha &\geqq \left[\frac{(\beta-\alpha)n-1}{k}\right]+\left[\frac{\alpha n}{k}\right]-\left[\frac{\alpha n-1}{k}\right] \\ &\geqq \left[\frac{(\beta-\alpha)n-1}{k}\right]. \end{aligned} \tag{5}$$

It is obvious that (5) holds also for $\alpha=0$. If the length of the smallest interval which contains $\kappa+1$ of the numbers $x'_0, x'_1, \cdots, x'_k$ is denoted by d'_κ, then by (5)

$$d'_\kappa \geqq \left[\frac{\kappa n-1}{k}\right] \qquad (\kappa = 1, 2, \cdots, k),$$

hence, by Theorem 1,

$$\max_{\nu=1,2,\cdots,n} |f(x_\nu)| \geqq \max_{\kappa=0,1,\cdots,k} |f(x'_\kappa)| \geqq 2^{-k}|a_0|\prod_{\kappa=1}^{k}\left[\frac{\kappa n-1}{k}\right].$$

2. **Criteria for irreducibility.** Theorem 2 will be used now to obtain criteria for irreducibility of polynomials.

THEOREM 3. *Let $P(x)$ be a polynomial of degree n with integral rational coefficients. If for n integral values $x_1, x_2, \cdots, x_n$ the absolute value of $P(x_\nu)$ for $\nu=1, 2, \cdots, n$ is less than*

$$G = \frac{(n-1)!}{2^{n-1}\{[(n-2)/2]!\}},$$

but different from 0, *then $P(x)$ is irreducible in the field of rational numbers.*

PROOF. If $P(x)$ is reducible, then it contains a factor $f(x)$ of degree k with integral coefficients where $n>k\geqq[(n+1)/2]$. It follows now from Theorem 2 that

$$M = \max_{\nu=1,2,\cdots,n} |f(x_\nu)| \geqq 2^{-k}\prod_{\kappa=1}^{k}\left[\frac{\kappa n-1}{k}\right]. \tag{6}$$

We set for fixed n

$$\phi(k) = \phi(k, n) = 2^{-k}\prod_{\kappa=1}^{k}\left[\frac{\kappa n-1}{k}\right]. \tag{7}$$

Let us first assume that $k=[(n+1)/2]$. For even values of n we have

$$\begin{aligned}\max(|b_0|, |c_0|) &\geqq 2^{-1}(|b_0| + |c_0|)\\ &\geqq 2^{-1}|b_0 - c_0| = 2^{-1}|a_0|\, d_k.\end{aligned} \tag{2}$$

On the other hand it follows from (1) that

$$\begin{aligned}\max_{\kappa=1,2,\cdots,k+1} |f(x_\kappa)| &\geqq \max_{\kappa=1,2,\cdots,k} |g(x_\kappa)|,\\ \max_{\kappa=1,2,\cdots,k+1} |f(x_\kappa)| &\geqq \max_{\kappa=2,3,\cdots,k+1} |h(x_\kappa)|.\end{aligned} \tag{3}$$

Since $g(x)$ and $h(x)$ are of lower degree than k, our theorem may be applied to them. The lengths of the smallest intervals which contain κ of the points $x_1, x_2, \cdots, x_k$ or κ of the points $x_2, x_3, \cdots, x_{k+1}$ are both not smaller than $d_{\kappa-1}$. Hence

$$\max_{\kappa=1,2,\cdots,k} |g(x_\kappa)| \geqq 2^{-k+1}|b_0|\, d_1 d_2 \cdots d_{k-1},$$

$$\max_{\kappa=2,3,\cdots,k+1} |h(x_\kappa)| \geqq 2^{-k+1}|c_0|\, d_1 d_2 \cdots d_{k-1},$$

and by (3) and (2)

$$\begin{aligned}\max_{\kappa=1,2,\cdots,k+1} |f(x_\kappa)| &\geqq 2^{-k+1} d_1 d_2 \cdots d_{k-1}\{\max(|b_0|, |c_0|)\}\\ &= 2^{-k}|a_0|\, d_1 d_2 \cdots d_k.\end{aligned}$$

THEOREM 2. *Let* $f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ *be a polynomial of degree* k *and* $x_1 < x_2 < \cdots < x_n$ *a set of more than* k *integers. Then*

$$\max_{\nu=1,2,\cdots,n} |f(x_\nu)| \geqq 2^{-k}|a_0| \prod_{\kappa=1}^{k} \left[\frac{\kappa n - 1}{k}\right].$$

PROOF. We consider those $k+1$ of the integers x_ν whose subscripts are 1 and the k numbers

$$1 + \left[\frac{\rho n - 1}{k}\right] \qquad (\rho = 1, 2, \cdots, k),$$

and denote these x_ν in increasing order by $x_0', x_1', \cdots, x_k'$. The difference of two consecutive elements x_ν is at least 1, hence for $k \geqq \beta > \alpha > 0$

$$x_\beta' - x_\alpha' \geqq \left[\frac{\beta n - 1}{k}\right] - \left[\frac{\alpha n - 1}{k}\right]. \tag{4}$$

Since for every r and s

$$\left[\frac{r+s}{k}\right] \geqq \left[\frac{r}{k}\right] + \left[\frac{s}{k}\right],$$

we have

If for n different integers $x_1, x_2, \cdots, x_n$

$$0 < |P(x_\nu)| < 2^{-k}\prod_{\kappa=1}^{k}\left[\frac{\kappa n - 1}{k}\right] \qquad (\nu = 1, 2, \cdots, n),$$

and for l of these x_ν, say $x^{(1)}, x^{(2)}, \cdots, x^{(l)}$

$$|P(x^{(\lambda)})| < 2^{-k}\prod_{\mu=1}^{k}\left[\frac{\mu l - 1}{k}\right] \qquad (\lambda = 1, 2, \cdots, l),$$

then $P(x)$ is irreducible in the field of rational numbers.

1. Bounds for the absolute value of polynomials at given points. We first prove the following theorem of Tatuzawa.

THEOREM 1. *Let* $f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ *be a polynomial of degree* k, *let* $x_1 < x_2 < \cdots < x_{k+1}$ *be arbitrary numbers, and* d_κ *the length of the smallest interval which contains* $\kappa+1$ *of these numbers* ($\kappa = 1, 2, \cdots, k$). *Then*

$$\max_{\kappa=1,2,\cdots,k+1} |f(x_\kappa)| \geq 2^{-k}|a_0|\, d_1 d_2 \cdots d_k.$$

PROOF. For $k = 1$ we have

$$|f(x_1) - f(x_2)| = |a_0(x_1 - x_2)| = |a_0|\, d_1,$$

hence

$$\begin{aligned} \max \{ |f(x_1)|, |f(x_2)| \} &\geq 2^{-1}\{ |f(x_1)| + |f(x_2)| \} \\ &\geq 2^{-1}|f(x_1) - f(x_2)| = 2^{-1}|a_0|\, d_1. \end{aligned}$$

This proves the theorem for $k = 1$. Let us now assume that it is already proved for polynomials of degree less than k.

We divide $f(x)$ by the polynomials $(x - x_1)(x - x_2) \cdots (x - x_k)$ and $(x - x_2)(x - x_3) \cdots (x - x_{k+1})$, respectively. Then

$$\begin{aligned} f(x) &= a_0(x - x_1)(x - x_2) \cdots (x - x_k) + g(x) \\ &= a_0(x - x_2)(x - x_3) \cdots (x - x_{k+1}) + h(x) \end{aligned} \tag{1}$$

where $g(x)$ and $h(x)$ are polynomials of degree less than k with the highest coefficients

$$b_0 = a_1 + a_0(x_1 + x_2 + \cdots + x_k)$$

and

$$c_0 = a_1 + a_0(x_2 + x_3 + \cdots + x_{k+1}),$$

respectively, hence $b_0 - c_0 = a_0(x_1 - x_{k+1})$, It follows from this theorem in particular that the integral polynomials

$$P(x) = A(x - x_1)(x - x_2) \cdots (x - x_n) + t$$

are irreducible if $x_\mu \neq x_\nu$ and $1 \leqq |t| < G$. This result contains for $n > 4$, $A = 1$, and $t = \pm 1$ a theorem of I. Schur,[4] and for arbitrary A and $t = \pm 1$ a theorem of H. L. Dorwart and O. Ore.[5] For $n \leqq 4$ there are exceptions.[6] The application of Pólya's bound gives this result only for $n > 6$, that of Tatuzawa for $n > 5$, while our bound gives the exact degrees for which these theorems hold.

Moreover, we improve the second theorem of Tatuzawa. Instead of H_1 we obtain the larger bound

$$H = \left[\frac{l+1}{2}\right] \quad \text{for} \quad l \geqq 7, \qquad H = \frac{3}{2} \quad \text{for} \quad l = 6 \text{ and } 5.$$

It is of interest that this bound cannot be improved further. If the absolute value of $P(x_\lambda)$ for $\lambda = 1, 2, \cdots, 1$ is not less than H, but only less than or equal to H, then our result does not remain correct. It will be shown that for every $n > 2$ such polynomials exist which are reducible in the field of rational numbers.

It follows from our theorem that the integral polynomials of degree n

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_l)h(x) + t$$

are irreducible if $l > 4$, $n > l > n/2$, $x_\lambda \neq x_\mu$, and $1 \leqq |t| < H$. This gives for $t = \pm 1$ a theorem of Dorwart and Ore.[7]

Finally a criterion of a new type is obtained. $P(x)$ is irreducible if the absolute value of $P(x_\nu)$ is different from zero and less than a certain bound $S_1 > G$ for n different integers x_ν, but less than another smaller constant S_2 for l of these x_ν. More exactly, the following theorem is proved.

Let $P(x)$ be an integral polynomial of degree n; let k, l, and h be integers satisfying the following conditions: $k \geqq [(n+1)/2]$, $n > l > n/2$, $l > 12$ or $l = 11$ or 9,

$$l > k, \qquad n \geqq k + h - 1.$$

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[4] Aufgabe 226, Archiv der Mathematik und Physik (3) vol. 13 (1908) p. 367. Lösung by W. Flügel, ibid. vol. 15 (1909) pp. 271–272. Cf. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925, pp. 136, 346–347.

[5] *Criteria for the irreducibility of polynomials*, Ann. of Math. vol. 34 (1933) pp. 81–94, 195.

[6] See, for instance, loc. cit. footnote 5, pp. 86, 195.

[7] Loc. cit. footnote 5.

ON THE IRREDUCIBILITY OF CERTAIN POLYNOMIALS

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Introduction. G. Pólya[1] has proved the following theorem:

If for n integral values of x, the integral polynomial $P(x)$ of degree n has values which are different from zero and, without regard to sign, less than

$$G_1 = \frac{(n - [n/2])!}{2^{n-[n/2]}},$$

then $P(x)$ is irreducible in the field of rational numbers. (Here, as in the following, a polynomial with rational integral coefficients is called an "integral polynomial.")

This result was improved for positively definite polynomials by Hildegard Ille[2] and for arbitrary polynomials by T. Tatuzawa.[3] The latter obtained the larger bound

$$G_2 = (2^{-n}(n-1)!)^{1/2}$$

instead of G_1. Moreover he proved the following theorem:

If for l integral values of x where $n > l > n/2$, the integral polynomial $P(x)$ of degree n takes values which are different from zero and, without regard to sign, less than

$$H_1 = (l-1)!^{1/l}/2,$$

then $P(x)$ is irreducible in the field of rational numbers.

In the following, the results of Tatuzawa will be improved further by a slight modification of his method. Instead of G_2 we obtain the larger bound

$$G = \frac{(n-1)!}{2^{n-1}[(n-2)/2]!}.$$

We have

$$\frac{G}{G_2} \sim \left(\frac{2n}{\pi}\right)^{1/4} \quad \text{for even } n \text{ and} \quad \frac{G}{G_2} \sim \left(\frac{n^3}{2\pi}\right)^{1/4} \quad \text{for odd } n.$$

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[1] *Verschiedene Bemerkungen zur Zahlentheorie*, Jber. Deutschen Math. Verein. vol. 28 (1919) pp. 31–40.

[2] *Einige Bemerkungen zu einem von G. Pólya herrührenden Irreduzibilitätskriterium*, Jber. Deutschen Math. Verein. vol. 35 (1926) pp. 204–208.

[3] *Über die Irreduzibilität gewisser ganzzahliger Polynome*, Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 253–254.

$$M = -2^{-1}\pi k(h_1 h_2)^{-1/2}[K' - \mathrm{sn}^{-1}(\rho, k')]. \tag{27}$$

From (10), (15), (23), (25), (27),

$$A = (2/\pi)V_0(1 + r)K/[K' - (1 + r)\,\mathrm{sn}^{-1}(\rho, k')], \tag{28}$$

$$\begin{aligned} B = V_0[(k + h_1)(k + h_2)]^{-1/2}\{1 - 2^{-1}\pi(1 + r)[k(h_1/h_2)^{1/2}K \\ + KE(\mathrm{sn}^{-1}(\rho, k'), k') - (K - E) \\ \cdot \mathrm{sn}^{-1}(\rho, k')]\}/[K' - (1 + r)\,\mathrm{sn}^{-1}(\rho, k')]. \end{aligned} \tag{29}$$

The constants in (28), (29) are given in (1), (2). K and E are the complete elliptic integrals of the first and second kinds, respectively, with respect to the modulus k, and K' is the complete elliptic integral of the first kind with respect to the modulus k'.

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4. Stratton, *Electromagnetic theory*, McGraw-Hill, 1941.
5. E. T. Whittaker and G. N. Watson, *Modern analysis*, Cambridge University Press, 1935.

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$$\begin{aligned} E(a, k) = {} & E + i(1/x)[(x^2 - k^2)(1 - x^2)]^{1/2} \\ & - iE(\mathrm{dn}^{-1}(k/x, k'), k') + i\,\mathrm{dn}^{-1}(k/x, k') \end{aligned}$$

when $k<x<1$.

From (19), (21),

$$\begin{aligned} F(x, k) = {} & x[(k^2 - x^2)(1 - x^2)]^{-1/2}[KE(\mathrm{sn}^{-1}(x/k, k), k) \\ & - E\,\mathrm{sn}^{-1}(x/k, k)], \qquad 0 < x < k, \\ F(x, k) = {} & -K + x[(x^2 - k^2)(1 - x^2)]^{-1/2}[KE(\mathrm{dn}^{-1}(k/x, k'), k') \\ & + (E - K)\,\mathrm{dn}^{-1}(k/x, k')], \qquad k < x < 1. \end{aligned} \tag{22}$$

From (16), (22),

$$\begin{aligned} P_1 = {} & [(h + h_1)/(h + h_2)]^{1/2}K - KE(\mathrm{dn}^{-1}(\tau, k'), k') \\ & + (K - E)\,\mathrm{dn}^{-1}(\tau, k'), \\ Q_1 = {} & k(h_1/h_2)^{1/2}K' + K'E(\mathrm{sn}^{-1}(\rho, k'), k') - E'\,\mathrm{sn}^{-1}(\rho, k'), \\ R_1 = {} & k(h_1/h_2)^{1/2}K + KE(\mathrm{dn}^{-1}(\sigma, k'), k') - (K - E)\,\mathrm{dn}^{-1}(\sigma, k'). \end{aligned} \tag{23}$$

By (1), (2), $\rho=\sigma'/k'$. Therefore, by [5, §22.11],

$$\mathrm{dn}^{-1}(\sigma, k') = \mathrm{sn}^{-1}(\rho, k'). \tag{24}$$

By (11), (15), (23), (24),

$$\begin{aligned} N &= k(h_1h_2)^{-1/2}[K'R_1 - KQ_1] \\ &= k(h_1h_2)^{-1/2}[KE' - KK' + K'E]\,\mathrm{sn}^{-1}(\rho, k'). \end{aligned}$$

By using Legendre's relation [5, §22.735],

$$N = 2^{-1}\pi k(h_1h_2)^{-1/2}\,\mathrm{sn}^{-1}(\rho, k'). \tag{25}$$

By (1), (2), $\sigma\tau=k$. Therefore by [5, §22.31] and (24),

$$\mathrm{dn}^{-1}(\tau, k') = K' - \mathrm{sn}^{-1}(\rho, k').$$

By [5, §22.732],

$$E(\mathrm{dn}^{-1}(\tau, k'), k') = kh(h_1h_2)^{-1/2} - E(\mathrm{sn}^{-1}(\rho, k'), k') + E'.$$

Substituting this into the expression for P_1 in (23), and using Legendre's relation, we obtain

$$P_1 = R_1 - \pi/2. \tag{26}$$

From (11), (15), (26),

$$M = N - 2^{-1}\pi Q_0.$$

From (15), (25),

$$
\begin{aligned}
P_1 &= k(h_1h_2)^{-1/2}[h_1K - hF(\sigma, k)],\\
Q_1 &= k(h_1h_2)^{-1/2}[h_1K' + h_1F(\sigma', k')],\\
R_1 &= k(h_1h_2)^{-1/2}[(h + h_1)K + hF(\tau, k)],
\end{aligned} \tag{16}
$$

where

$$
F(x, k) = \int_0^K \frac{x^2 \operatorname{sn}^2(\phi, k)}{1 - x^2 \operatorname{sn}^2(\phi, k)}\, d\phi. \tag{17}
$$

Define the constant a by

$$
\operatorname{sn}(a, k) = x/k. \tag{18}
$$

a is real when $0<x<k$, but is complex when $k<x<1$. By [5, §22. $F(x, k)$ may be expressed in terms of the elliptic function of the t kind as

$$
F(x, k) = \operatorname{sc}(a, k) \operatorname{nd}(a, k)\Pi(K, a, k).
$$

By [5, §22.74], this may be expressed in terms of theta and zeta f tions as

$$
F(x, k) = \operatorname{sc}(a, k) \operatorname{nd}(a, k)\left\{\frac{1}{2}\ln\frac{\Theta(K-a)}{\Theta(K+a)} + KZ(a)\right\}.
$$

Now the Θ-function is even and periodic of period $2K$, so

$$
F(x, k) = \operatorname{sc}(a, k) \operatorname{nd}(a, k)KZ(a).
$$

By [5, §22.731], F may be expressed in terms of the incom elliptic integral of the second kind as

$$
F(x, k) = \operatorname{sc}(a, k) \operatorname{nd}(a, k)[KE(a, k) - aE]. \tag{19}
$$

By [5, §§22.732, 22.734],

$$
\begin{aligned}
E(c + ib, k) = E(c, k) - iE(b, k') + i\operatorname{sc}(b, k')\\
\cdot[\operatorname{dn}(b, k') - k^2 \operatorname{sn}(c, k) \operatorname{sn}(c + ib, k)] +
\end{aligned} \tag{20}
$$

From (18) and [5, §§22.31, 22.34, 22.41],

$$
\begin{aligned}
a &= \operatorname{sn}^{-1}(x/k, k), && 0 < x\\
a &= K + i\operatorname{dn}^{-1}(k/x, k'), && k < x
\end{aligned} \tag{21}
$$

From (20), (21), and [5, §§22.11, 22.31, 22.34, 22.41],

$$
E(a, k) = E(\operatorname{sn}^{-1}(x/k, k), k)
$$

when $0<x<k$, and

$$E_y(x+i0)=(Ax+B)[x(x-h_1)(x-h-h_1)(h+h_1+h_2-x)]^{-1/2},$$
$$h+h_1<x<h+h_1+h_2.$$

Therefore, from (4), (5),

$$(6)\qquad (P_1+rR_1)A+(P_0+rR_0)B=0,\qquad Q_1A+Q_0B=V_0$$

where

$$(7)\qquad 2P_n=\int_0^{h_1}x^n[x(h_1-x)(h+h_1-x)(h+h_1+h_2-x)]^{-1/2}dx,$$

$$(8)\qquad 2Q_n=\int_{h_1}^{h+h_1}x^n[x(x-h_1)(h+h_1-x)(h+h_1+h_2-x)]^{-1/2}dx,$$

$$(9)\qquad 2R_n=\int_{h+h_1}^{h+h_1+h_2}x^n[x(x-h_1)(x-h-h_1)(h+h_1+h_2-x)]^{-1/2}dx.$$

From (6),

$$(10)\quad A=-V_0(P_0+rR_0)/(M+rN),\ B=V_0(P_1+rR_1)/(M+rN),$$

where

$$(11)\qquad M=P_1Q_0-Q_1P_0,\qquad N=R_1Q_0-Q_1R_0.$$

A and B may be calculated from (10), (11) if P_0, Q_0, R_0, P_1, Q_1, R_1 are known. To determine them, in (7), (8), (9), respectively, make the substitutions

$$x=h+h_1-h/[1-\sigma^2\,\mathrm{sn}^2(\phi,k)],$$
$$x=h_1/[1-\sigma'^2\,\mathrm{sn}^2(\phi,k')],$$
$$x=h_1+h/[1-\tau^2\,\mathrm{sn}^2(\phi,k)].$$

Then

$$(12)\quad P_n=k(h_1h_2)^{-1/2}\int_0^K\{h_1-h\sigma^2\,\mathrm{sn}^2(\phi,k)/[1-\sigma^2\,\mathrm{sn}^2(\phi,k)]\}^n d\phi,$$

$$(13)\quad Q_n=k(h_1h_2)^{-1/2}\cdot\int_0^{K'}\{h_1+h_1\sigma'^2\,\mathrm{sn}^2(\phi,k')/[1-\sigma'^2\,\mathrm{sn}^2(\phi,k')]\}^n d\phi,$$

$$(14)\quad R_n=k(h_1h_2)^{-1/2}\cdot\int_0^{K}\{h+h_1+h\tau^2\,\mathrm{sn}^2(\phi,k)/[1-\tau^2\,\mathrm{sn}^2(\phi,k)]\}^n d\phi,$$

where $K=K(k)$, $K'=K(k')$.

From (12)–(14),

$$(15)\qquad P_0=k(h_1h_2)^{-1/2}K,\quad Q_0=k(h_1h_2)^{-1/2}K',\quad R_0=k(h_1h_2)^{-1/2}K.$$

be at potential $-V_0$, corresponding to Davy's given conditions. Davy's solution corresponds to the case $r=-1$.

2. **The electrostatic field.** Let the two plates have widths h_1, h_2, respectively, and let their adjacent edges be separated by a distance h. Define

$$k = [h_1h_2/(h+h_1)(h+h_2)]^{1/2}, \qquad k' = (1-k^2)^{1/2}, \tag{1}$$

$$\begin{aligned} \sigma^2 &= h_1/(h+h_1), & \sigma'^2 &= h/(h+h_1) = 1-\sigma^2, \\ \tau^2 &= h_2/(h+h_2), & \rho^2 &= (h+h_2)/(h+h_1+h_2). \end{aligned} \tag{2}$$

Let the plate of width h_1 occupy the interval $0<x<h_1$, and the plate of width h_2 occupy the interval $h+h_1<x<h+h_1+h_2$ on the x-axis of a rectangular x, y-coordinate system. Let $z=x+iy$, and let $E_x(z)$, $E_y(z)$ denote the x and y components, respectively, of the electric field. Let the charge per unit length of the plate of width h_1 be r times the charge per unit length on the plate of width h_2.

The method to be used here is that of [3, §4]. Let $x_1=0$, $x_2=h_1$, $x_3=h+h_1$, $x_4=h+h_1+h_2$. Take the "undisturbed" field to be zero, so that $f(x)=0$. Then by [3, §§4.1, 4.7, 4.9, 4.10],

$$\begin{aligned} &E_x(z) - iE_y(z) \\ &\quad = (Az+B)/[z(z-h_1)(z-h-h_1)(z-h-h_1-h_2)]^{1/2}, \end{aligned} \tag{3}$$

where the constants A and B must be determined.

3. **Determination of the constants.** By [4, pp. 24, 483],

$$\int_0^{h_1} E_y(x+i0)dx - r\int_{h+h_1}^{h+h_1+h_2} E_y(x+i0)dx = 0. \tag{4}$$

If the plate of width h_1 is to be at a potential $+V_0$, and the plate of width h_2 is to be at a potential $-V_0$, then

$$\int_h^{h+h_1} E_x(x+i0)dx = -2V_0. \tag{5}$$

By (3),

$$E_y(x+i0) = -(Ax+B)[x(h_1-x)(h+h_1-x)(h+h_1+h_2-x)]^{-1/2}, \qquad 0<x<h,$$

$$E_x(x+i0) = -(Ax+B)[x(x-h_1)(h+h_1-x)(h+h_1+h_2-x)]^{-1/2}, \qquad h_1<x<h+h_1,$$

Under the weaker hypothesis that px merely approaches a constant the same proof shows that pw also approaches a constant.

A later note considers the existence and character of solutions of quasi-linear networks for periodic impressed force.

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THE ELECTROSTATIC FIELD OF TWO COPLANAR PLATES

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1. **Introduction.** In a recent paper in the Philosophical Magazine [1, p. 168],[1] N. Davy published what he called an "attempt" to obtain the electrostatic field about two thin, infinitely long, parallel, coplanar metallic plates of unequal width and at potentials $\pm V_0$. It was this remark, no doubt, that led M. C. Gray [2] to call this solution "tentative."

Actually, the solution given by Davy was but one of infinitely many that might fit his given conditions. The reason is that this is a two-dimensional potential problem; and in two-dimensional potential theory infinity is not a suitable zero point for the potential function as it is in three-dimensional potential theory. Consequently, to make the potential function definite, it is necessary either to specify its zero point or to specify some other condition which effectively does this. Davy made no such specification but chose the particular potential function which corresponds to the case in which the charges on the conductors are equal and opposite in sign.

It is the purpose of this paper to solve the problem fulfilling Davy's conditions, but in which the charge per unit length on one plate bears to the charge per unit length on the other plate a given ratio r.

For any $r \neq 1$, the charges on the plates may be increased until the potential difference between the two plates is $2V_0$. If the zero point for the potential function is then taken as the point between the two plates at which the potential is the arithmetic mean of the potentials on the plates, one plate will be at potential $+V_0$ and the other will

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[1] Numbers in brackets refer to the references cited at the end of the paper.

note $\{x, Rh\} = \{h, Rx\} \leqq \|h\| \, \|Rx\|$. Thus $z\|Lx\|^2+\|Rx\|^2+z^{-1}\|Sx\|^2 \leqq A\|h\| \, \|Rx\|$. This makes it obvious that $\|Rx\|$ is bounded, hence $\|Lx\|$ is bounded and $\|Sx\| = Oz^{1/2}$. Thus $\|(L+R+S)x\| = Oz^{1/2}$. But $(L+R+S)$ is a nonsingular constant matrix, so actually $\|x\| = Oz^{1/2}$. Since h is an arbitrary vector the matrix elements of G, say g_{ij}, must satisfy $|g_{ij}| = Oz^{1/2}$. But g_{ij} is a rational function of z and if it were unbounded it would increase at least as rapidly as z. Thus g_{ij} is bounded at infinity. To show g_{ij} is bounded in the neighborhood of the origin, let $z' = 1/z$ and repeat the above argument. Since g_{ij} has no singularities for $\Re z \geqq 0$ and $|z| > 0$, it follows that g_{ij} is uniformly bounded for $\Re z \geqq 0$. This proves the statement for G.

By what has just been proved, zLG is bounded in the neighborhood of the origin. Write

$$\begin{aligned} zLG &= -RG - z^{-1}SG + (zL + R + z^{-1}S)G \\ &= -RG - z^{-1}SG + R. \end{aligned}$$

The three matrices on the right are uniformly bounded at infinity and this proves the statement for zLG. The remaining part follows by symmetry.

LEMMA 6. *Equation* (4) *has a continuous solution* $pw_1 \subset L_2(0, \infty)$.

PROOF. Let $\phi(z) = \int_0^\infty e^{-zt} f(t)dt$, $\Re z > 0$. The existence of $\phi(z)$ is guaranteed since we have shown $f \subset L_2(0, \infty)$. For brevity write $\phi(z) = \mathfrak{L}f(t)$. Let $\zeta(z) = G\phi(z)$, and since G is a uniformly bounded rational transformation for $\Re z \geqq 0$ it follows from the well known theory of the Laplace transformation that $\zeta(z) = \mathfrak{L}u(t)$, where $u(t) \subset L_2(0, \infty)$ and is continuous. Moreover $z^{-1}\zeta(z) = \mathfrak{L}\int_0^t u dt$ and $z^{-1}S\zeta(z) = \mathfrak{L}S\int_0^t u dt$. Likewise by Lemma 5, $zL\zeta(z) = \mathfrak{L}l(t)$, where $l(t) \subset L_2(0, \infty)$ and is continuous. Thus $\mathfrak{L}\int_0^t l dt = \mathfrak{L}Lu$. By the uniqueness property of the Laplace transform, $\int_0^t l dt = Lu$, so $l = Ldu/dt$. Note that

$$\mathfrak{L}\left(Ldu/dt + Ru + S\int_0^t u dt\right) = (zL + R + z^{-1}S)\zeta = \mathfrak{L}Rf.$$

Thus $Ldu/dt + Ru + S\int_0^t u dt = Rf$. Identifying w_1 with $\int_0^t u dt$ completes the proof.

Clearly $x = w - w_1$ satisfies the linear homogeneous equation and px is continuous so $px \to 0$ as $t \to +\infty$ by hypothesis. Moreover, it is easy to show by the Laplace transformation that $px \subset L_2(0, \infty)$. Thus $w \subset L_2(0, \infty)$ and the proof is completed that pq and pq^* approach each other in mean. We have been unable to show that the solutions also approach in the pointwise sense.

LEMMA 2. *If, for all x, $\{x, Hx\} \leqq A\|x\|^2$ then $\|Hx\|^2 \leqq A\{x, Hx\}$.*

LEMMA 3. *The equation $a = Hx$ is solvable for x if and only if the relation $Hu = 0$ implies $\{a, u\} = 0$.*

LEMMA 4. *There is a matrix P such that, for any x, $Hx = HPx$ and $\|Px\| \leqq A\|Hx\|$, where A is independent of x.*

PROOF. These lemmas are obvious if H is a diagonal matrix. However, the relations are invariant under a rotation of axes (orthogonal transformation) and it is well known that there is a rotation which reduces H to diagonal form.

For each value of t the matrix V' is non-negative definite and $\{x, V'x\} \leqq A\|x\|^2$, where A is independent of t. Thus by Lemma 2, $\int_0^\infty \|V'pw\|^2 \leqq A\int_0^\infty \{pw, V'pw\}dt < \infty$. Also $\{x, Rx\} \leqq k\{x, V'x\}$; so again by Lemma 2, $\int_0^\infty \|Rpw\|^2 dt < \infty$.

From $\{x, V'x\} \leqq k\{x, Rx\}$ it follows by Lemma 2 that $\|V'x\|^2 \leqq Ak\{x, Rx\}$. This implies $V'x = 0$ if $Rx = 0$. Now by Lemma 3 it follows that if $a = V'b$ there is an x such that $a = Rx$. Otherwise there would be a u such that $Ru = 0$ and $\{a, u\} \neq 0$. But then $V'u = 0$; so $\{a, u\} = 0$, a contradiction.

Let $b - x = f_0$; then $Rb - V'b = Rf_0 = RPf_0$. Let $f = Pf_0$; then by Lemma 4

$$\|f\| \leq A\|Rf\| \leq A\|Rb\| + A\|V'b\|.$$

Interpreting b as pw, it follows that $\int_0^\infty \|f\|^2 dt < \infty$, that is, $f(t) \subset L_2(0, \infty)$. Moreover

$$\|f(t_1) - f(t_2)\| \leq A\|Rpw_1 - Rpw_2\| + A\|V_1' pw_1 - V_2' pw_2\|.$$

Since pw and V' are continuous it follows that $f(t)$ is continuous. With f so defined we may write

$$Lp^2w + Rpw + Sw = Rf. \tag{4}$$

LEMMA 5. *Let $G = (zL + R + z^{-1}S)^{-1}R$; then G, zLG and $z^{-1}SG$ are uniformly bounded matrices for $\Re z \geqq 0$.*

PROOF. First suppose z is real and $1 \leqq z < \infty$. Let h be an arbitrary vector and let $x = Gh$. Then

$$(zL + R + z^{-1}S)x = Rh$$

and

$$z\{x, Lx\} + \{x, Rx\} + z^{-1}\{x, Sx\} = \{x, Rh\}.$$

Apply Lemma 2 to each of the terms on the left side. On the right side where θ lies between i_ν^+ and i_ν^*. Thus the matrix V' is defined by the v_ν' the same way that R is defined by the r_ν.

Without loss of generality we may suppose $\delta \leqq r_\nu \leqq \Delta$ also. Then by identity (2)

$$(\delta/\Delta)\{y, Ry\} \leqq \{y, V'y\} \leqq (\Delta/\delta)\{y, Ry\}.$$

Before proceeding with the uniqueness proof it is desirable to give an abstract definition of this nonlinear transformation so that a theorem can be stated which is independent of physical concepts. Let R be a symmetric semi-definite matrix. Then a continuous vector function $V(y)$ is a ***quasi-linear replacement*** of R provided $V(y^+) - V(y^*) = V' \cdot (y^+ - y^*)$, where V' is a symmetric matrix which satisfies $k^{-1}\{y, Ry\} \leqq \{y, V'y\} \leqq k\{y, Ry\}$ for a positive constant k independent of the vectors y^+, y^*, and y.

THEOREM. *Let L, R, and S be symmetric semi-definite matrices and suppose that all solutions of the equation $Ld^2x/dt^2 + Rdx/dt + Sx = 0$ are such that $dx/dt \to 0$ as $t \to +\infty$. Let V be a quasi-linear replacement of R and suppose for $t \geqq 0$ the vectors $q(t)$ and $e(t)$ satisfy the equation $Ld^2q/dt^2 + V(dq/dt) + Sq = e$. Then if q^* is any other solution of this equation, $\int_0^\infty \|dq/dt - dq^*/dt\|^2 dt < \infty$. (The first derivatives of x, q, and q^* are assumed to be continuous.)*

PROOF. Let $w = q - q^*$; then by subtraction

$$Lp^2w + V'pw + Sw = 0. \tag{3}$$

This gives the bilinear form

$$\{pw, Lp^2w\} + \{pw, V'pw\} + \{pw, Sw\} = 0,$$

so

$$-p[\{pw, Lpw\} + \{w, Sw\}] = 2\{pw, V'pw\}.$$

By definition of V' the function on the right is continuous so by a standard theorem of the integral calculus

$$-[\{pw, Lpw\} + \{w, Sw\}] + A = 2\int_0^t \{pw, V'pw\}dt.$$

But L and S are non-negative definite so $\int_0^\infty \{pw, V'pw\}dt < A/2$. The integral is absolutely convergent because the integrand is non-negative. (We shall use the letter A to designate various other constants also.)

The following simple properties of a symmetric semi-definite matrix H are needed.

rent in the network may be uniquely specified in terms of the cyclic currents $y_1, y_2, \cdots, y_n$ flowing in these circuits. Let $e_1, e_2, \cdots, e_n$ be the electromotive forces acting in these circuits. It is convenient to introduce the electric charge variables $q_1, q_2, \cdots, q_n$ such that $y_j = dq_j/dt$ and to let p stand for d/dt. The linear network equations may then be written as

$$(1) \qquad Lp^2q + Rpq + Sq = e.$$

Here L, R, and S are n-way matrices and q and e are vectors with components q_j and e_j.

A condition which must be satisfied in order that there be no undamped solutions is specified by the following well known lemma.

LEMMA 1. *A necessary condition that all solutions $y = pq$ of the homogeneous equation $Lp^2q + Rpq + Sq = 0$ approach zero as $t \to +\infty$ is that the matrix $(z^2L + zR + S)$ be nonsingular for $\Re z \geqq 0$ and $|z| > 0$.*

PROOF. Otherwise there is a constant vector q_0 and a value of z such that $(z^2L + zR + S)q_0 = 0$, so $q = q_0e^{zt}$ is a solution. But $pq = zq_0e^{zt}$ does not approach zero if $\Re z \geqq 0$ and $|z| > 0$.

Well known arguments concerning electric and magnetic energy show that the matrices L, R, and S are symmetric semi-definite. For instance if $\{a, b\}$ indicates the bilinear form $\sum_1^n a_jb_j$ and $\|a\|^2 = \{a, a\}$, then $\{y, Ry\}$ is by definition the rate at which energy is dissipated in the resistors. On the other hand let the resistance of the resistors be $r_1, r_2, \cdots, r_m$ and the corresponding currents through them be $i_1, i_2, \cdots, i_m$. (Each i_ν is, of course, the algebraic sum of those currents y_j which are common to r_ν.) The potential drop across the resistor ν is $r_\nu i_\nu$ and the jth component of Ry is defined as the algebraic sum of those drops which are common to the jth circuit. The following identity must hold for physical reasons

$$(2) \qquad \{y, Ry\} = \sum_1^m r_\nu i_\nu^2.$$

This shows that R is non-negative definite.

In the nonlinear network the linear expression $r_\nu i_\nu$ is replaced by the function $v_\nu(i_\nu)$ where $\delta \leqq v_\nu' \leqq \Delta$ for positive constants δ and Δ. Let $V(y)$ be the vector function which replaces Ry in equation (1). If y^+ and y^* are two arbitrary vectors we can write $V(y^+) - V(y^*) = V' \cdot (y^+ - y^*)$, where V' is a matrix defined as follows. By the mean value theorem

$$v_\nu(\overset{+}{i}_\nu) - v_\nu(\overset{*}{i}_\nu) = v_\nu'(\theta)(\overset{+}{i}_\nu - \overset{*}{i}_\nu),$$

NONLINEAR NETWORKS. I

R. J. DUFFIN

The object of this note is to show that a certain system of nonlinear differential equations has a unique asymptotic solution, that is, all solutions approach each other as the independent variable becomes infinite. The interest of these equations is that they describe the vibrations of electrical networks so we shall first discuss the physical origin of the equations.

A linear network is a collection of linear inductors, linear resistors and linear capacitors arbitrarily interconnected. Suppose that such a network has no undamped free vibration. Then a given impressed force may give rise to more than one response but as time goes on the transient vibrations die out and there is a unique relation between impressed force and response. This, of course, is well known. Our main theorem states that if in such a network the linear resistors are replaced by *quasi-linear* resistors then again, after sufficient time has elapsed, there is a unique relation between impressed force and response.

A quasi-linear resistor is a conductor whose differential resistance lies between positive limits. No other nonlinearity besides this type of nonlinear damping is considered. Quasi-linear resistors have extensive practical application.

For example, consider a linear network with one degree of freedom. An inductor of inductance L, a resistor of resistance R and a capacitor of capacitance S^{-1} are connected in series. The current $y(t)$ flowing in this circuit must satisfy the following differential equation

$$L\frac{dy}{dt} + Ry + S\int y dt = e.$$

Here $e(t)$ is the electromotive force impressed in the circuit and may be an arbitrary function of time.

The corresponding nonlinear equation to be studied is obtained by replacing the linear relation Ry by a function $V(y)$ which for all values of y is such that $\delta \leq V'(y) \leq \Delta$, where δ and Δ are positive constants.

In the general network with n degrees of freedom a set of n independent circuits (meshes) is chosen. Then any distribution of cur-

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$\Gamma^\lambda = c_\lambda + f(\Gamma^{\lambda+1})$. Now since the sequence is not $(p-1)$-terminating, for every λ there is a $c_{\lambda+\mu} \leqq p-2$. Moreover, $\Gamma_{\mu+\nu}{}^\lambda = c_\lambda + f(c_{\lambda+1} + \cdots + f(\Gamma_\nu{}^{\lambda+\mu})$, and $\Gamma^\lambda = c_\lambda + f(c_{\lambda+1} + \cdots + f(\Gamma^{\lambda+\mu})$. By (E), $\Gamma^{\lambda+\mu} \leqq c_{\lambda+\mu} + 1 \leqq p-1$, so that we have

$$\text{(E}'\text{)} \qquad c_\lambda \leqq C^\lambda \leqq \Gamma^\lambda < c_\lambda + 1,$$

and, as already shown,

$$\text{(F)} \qquad C^\lambda = c_\lambda + f(C^{\lambda+1}), \qquad \Gamma^\lambda = c_\lambda + f(\Gamma^{\lambda+1}).$$

But (E$'$, F) imply that, under (A), the numbers C^0 and Γ^0 yield the original sequence $\{c_\nu\}$. We call these the lower and upper limit numbers of the sequence.

If $f(t)$ satisfies (D$'$), *the correspondence* (B) *maps all reals $\gamma_0 \geqq 0$ onto all non-$(p-1)$-terminating sequences. Every such sequence is indeed the map of its limit numbers C^0, Γ^0.*

Now if γ_0 yields $\{c_\nu\}$ under (A), then

$$\text{(G)} \qquad C^0_\nu \leqq \gamma_0 = c_0 + f(c_1 + \cdots + f(c_\nu + f(\gamma_{\nu+1}) < \Gamma^0_\nu \qquad \text{all } \nu,$$

and hence $C^0 \leqq \gamma_0 \leqq \Gamma^0$.

Also, if γ_0' and γ_0'' yield $\{c_\nu\}$ under (A), and if $\gamma_0' \leqq \gamma_0 \leqq \gamma_0''$, then γ_0 yields $\{c_\nu\}$. For

$$c_0 \leqq \gamma_0' = c_0 + f(\gamma_1') \leqq \gamma_0 \leqq \gamma_0'' = c_0 + f(\gamma_1'') < c_0 + 1,$$

hence $\gamma_0 = c_0 + f(\gamma_1)$ and $\gamma_1' \leqq \gamma_1 \leqq \gamma_1''$, and so on.

It follows that γ_0 *yields* $\{c_\nu\}$ *under* (A) *if and only if* $C^0 \leqq \gamma_0 \leqq \Gamma^0$. Thus the correspondence (B) is actually a mapping of disjoint closed sets $[C^0, \Gamma^0]$ on all non-$(p-1)$-terminating sequences. The sequences $\{c_\nu\}$ fall into two classes according as $C^0 < \Gamma^0$ or $C^0 = \Gamma^0$. The correspondence (B) thus splits into two parts:

$$\text{(B}'\text{)} \qquad [C^0, \Gamma^0] \to \{c_\nu\}, \qquad C^0 < \Gamma^0,$$

$$\text{(B}''\text{)} \qquad C^0 = \Gamma^0 \to \{c_\nu\}.$$

In the case (B$''$) the $C_\nu{}^0$ and $\Gamma_\nu{}^0$ converge to $C^0 = \Gamma^0 = \gamma_0$ with errors thus (see G):

$$\text{(H)} \qquad 0 \leqq \gamma_0 - C^0_\nu < \Gamma^0_\nu - C^0_\nu; \qquad 0 < \Gamma^0_\nu - \gamma_0 \leqq \Gamma^0_\nu - C^0_\nu.$$

We note here two properties of the sequence $\{p-1, p-1, \cdots\}$ of later use. Although this sequence does not appear under the algorithm, nevertheless the lim $C_\nu{}^0$ exists and is p. For $p-1 < C_\nu{}^0 < C_{\nu+1}{}^0 < p$ and $t_0 = \lim C_\nu{}^0$ satisfies $p-1 < t_0 \leqq p$. But $C_\nu{}^0 = p-1+f(C_{\nu-1}{}^1) = p-1+f(C_{\nu-1}{}^0)$. Hence $t_0 = p-1+f(t_0)$, and by (D$'$), $t_0 = p$.

Also, $p-1+f(p-1+\cdots+f(p-1+f(p-2) \geqq C_{\nu-1}^0$ where the first expression contains ν $(p-1)$'s. Thus the sequence $p-1+f(p-2)$, $p-1+f(p-1+f(p-2)$, $\cdots$ has limit p.

5. **Terminating sequences.** We call a sequence $\{c_\nu\}$ with $c_\nu=0$, $\nu>N$ for some N, ***terminating***. There exist numbers $\gamma_0>0$ yielding $\{0, 0, \cdots\}$ under (A) if and only if $f(t)$ has the property:

(I) There exists a $\gamma_0=f(\gamma_0)$, $0<\gamma_0<1$.

Clearly such a γ_0 yields $\{0, 0, \cdots\}$ under (A). Suppose that $\gamma_0>0$ yields $\{0, 0, \cdots\}$ and that (I) is false. By continuity of $f(t)$ we have

$$\text{(J)} \qquad f(t) < t \quad \text{for all } t \text{ on } 0 < t \leqq 1,$$

and $0<\gamma_0=f(\gamma_1)<\gamma_1=f(\gamma_2)<\gamma_2 \cdots$. Hence $0<\gamma_0<\gamma_1<\gamma_2<\cdots<1$, and $t_0=\lim \gamma_\nu$ exists with $0<t_0\leqq 1$. But from $\gamma_\nu=f(\gamma_{\nu+1})$ follows $t_0=f(t_0)$, a contradiction.

Obviously "not I" is equivalent to (J), and (J) may be restated in slope form

$$\text{(J}'\text{)} \qquad f(t) - f(0)/t < 1 \qquad \text{on } 0 < t \leqq 1.$$

If a γ_0 exists satisfying (I) then not only γ_0 but also every δ_0 on $0\leqq\delta_0<\gamma_0$ will yield $\{0, 0, \cdots\}$ under (A). Hence for a one-one correspondence (B), (J′) is necessary, and we assume from this point on that $f(t)$ satisfies (D′) and (J′).

Under these restrictions, the sequence $\{0, 0, \cdots\}$ has $C^0=\Gamma^0=0$, and since in any sequence $\{c_\nu\}$, $C^0=c_0+\cdots+f(C^\lambda)$, $\Gamma^0=c_0+\cdots+f(\Gamma^\lambda)$, it follows that every terminating sequence has $\Gamma^0=C^0$ and falls under (B″).

We remark here that if $\{d_\nu\}$ is a terminating sequence $\{d_1, d_2, \cdots, d_\nu, 0, 0, \cdots\}$, then the associated limit numbers $D^0=\Delta^0=d_0+f(d_1+\cdots+f(d_\nu+f(D^{\nu+1})=d_0+f(d_1+\cdots+f(d_\nu)$, since $D^{\nu+1}=0$.

6. **The many-one case.** Suppose then that $f(t)$ satisfies (D') and (J′) and consider the algorithm (A) only as it applies to numbers γ_0 on the interval $[0, p)=(0\leqq t<p)$. The correspondences (B′, B″) then map the interval $[0, p)$ onto all non-$(p-1)$-terminating sequences $\{c_\nu\}$ with $0\leqq c_\nu\leqq p-1$.

Let L be the set of all limit numbers C^0, Γ^0 on $[0, p)$ (equal or not) of all such sequences, and G the complement of L in $[0, p)$. The points of L are then the numbers $C^0=\Gamma^0$ occurring under (B″), including the limits of all terminating sequences, together with the end points $C^0<\Gamma^0$ of the closed intervals under (B′). The points of G are those of all the open intervals (C^0, Γ^0) in (B′). Since G is a union of (nonoverlapping, indeed, non-abutting) intervals, G *is open, and* L *is closed.*

We write $[0, p)=L+G$, and $L=L'+L''$, where L' is the set of end points under (B′) and L'' the set of $C^0=\Gamma^0$ under (B″).

Since *the intervals of* G *are countable, so is the set* L'. We now show that L is dense in itself. It then follows that L *is perfect*, L (*hence also* L'') *has the power of the continuum. Since the limits of terminating sequences are countable, the set of limits of non-terminating sequences for which* $C^0=\Gamma^0$ *is of the power of the continuum* (cf. [3]).

Indeed, every point λ of L is a limit point of limit numbers $D^0=\Delta^0$ of *terminating* sequences $\{d_\nu\}$. First let $\lambda=C^0=\Gamma^0$ for $\{c_\nu\}$. Then $\lambda=\lim C_\nu^0=\lim \Gamma_\nu^0$ and $C_\nu^0<\Gamma_\nu^0$. The numbers C_ν^0 are in L, being limit numbers of terminating sequences. Since the sequence $\{c_\nu\}$ is not $(p-1)$-terminating, a subsequence of $\{\Gamma_\nu^0\}$ has $\Gamma_\nu^0=c_0+f(c_1+\cdots+f(c_\nu+1)$ with $c_\nu+1\leqq p-1$, and these Γ_ν^0 are thus in L, being limit numbers of terminating sequences $\{c_0, c_1, \cdots, c_\nu+1, 0, 0, \cdots\}$ in our class. Hence λ is a limit point of points of L.

Second, let $\lambda=C^0<\Gamma^0$ for $\{c_\nu\}$. Then the sequence $\{c_\nu\}$ is not terminating, and a subsequence of $\{C_\nu^0\}$ is properly increasing to C^0 as a limit point.

Finally, let $C^0<\Gamma^0=\lambda$ for $\{c_\nu\}$. Since $\{c_\nu\}$ is not $(p-1)$-terminating, a proper subsequence of $\{\Gamma_\nu^0\}$ with $c_\nu+1\leqq p-1$ is properly decreasing to Γ^0 as a limit point. Hence L is dense in itself.

If $f(t)$ admits a sequence $\{d_\nu\}$ with $D^0<\Delta^0$, that is, *if the correspondence* (B) *is not one-one, then the set* L *is nondense on* $[0, p)$. If (a, b) is a subinterval: $0\leqq a<b<p$, we show that (a, b) contains a subinterval containing no point of L. If (a, b) itself contains no point of L, (a, b) will serve. However if a point λ of L is in (a, b) and if $\lambda=C^0<\Gamma^0$ or $C^0<\Gamma^0=\lambda$ for some $\{c_\nu\}$ then the interval (a, b) intersects (C^0, Γ^0) in an interval containing only points of G. The only case remaining is $\lambda=C^0=\Gamma^0$ in (a, b), $\lambda=\lim C_\nu^0=\lim \Gamma_\nu^0$. But $C_\nu^0<c_0+f(c_1+\cdots+f(c_\nu+f(D^0)<c_0+f(c_1+\cdots+f(c_\nu+f(\Delta^0)<\Gamma_\nu^0$. The inner numbers define an interval of G, interior to (a, b) for sufficiently large ν.

7. **An example.** Consider for $p=3$ the function $f(t)$ defined by $f(0)=0$, $f(4/3)=1/3$, $f(5/3)=2/3$, $f(3)=1$, and elsewhere by the broken line connecting these points. It is clear that $4/3$ and $5/3$ yield $\{1, 1, \cdots\}$ under the algorithm. Moreover, for this sequence, $C^0=4/3$ and $\Gamma^0=5/3$ as is seen graphically from the sequences $1+f(1)$, $1+f(1+f(1)$, $\cdots$ and $1+f(2)$, $1+f(1+f(2)$, $\cdots$. Imagine that we blacken the intervals $(i+f(C^0), i+f(\Gamma^0))$, $i=0, 1, 2$. The first of these defines three intervals $(j+f(0+f(C^0), j+f(0+f(\Gamma^0))$, $j=0, 1, 2$, and the last similarly, all of which we blacken. (Graphically, the

process amounts to projecting the function values above each black interval onto the three 45° lines and thence down to the t-axis.) Repetition of this process yields a set of open intervals of total length $1/3+2/3+3(2/3)(1/4)+\cdots=1/3+2/3(1+3/4+(3/4)^2+\cdots)=3$. It follows that the set of black intervals exhausts the set G, and the complement L is of measure zero, perfect, and nondense on $[0, 3)$. While this is not quite the Cantor "middle-third" set it has precisely the same structure.

8. Sufficient conditions for one-one correspondence. Let $c_0<\gamma_0<\delta_0<c_0+1$ and $\gamma_0, \gamma_1, \cdots, \gamma_n; \delta_0, \delta_1, \cdots, \delta_n$, be the numbers resulting from the first n steps of the algorithm. We say that the slopes $f(\delta_i)-f(\gamma_i)/\delta_i-\gamma_i$, $i=1, \cdots, n$, are *connected.*

In order that the correspondence (B) *be one-one it is sufficient that*:

(K) *There exists an integer n such that the product of every n connected slopes is less than one.*

Suppose that (B) is not one-one, and let X' be the class of all intervals (C^0, Γ^0) under (B′). Then there must be in X' an interval of maximal length. For this interval, write $\Gamma^0-C^0=(f(\Gamma^1)-f(C^1)/\Gamma^1-C^1)\cdots(f(\Gamma^n)-f(C^n)/\Gamma^n-C^n)(\Gamma^n-C^n)$. The interval (C^n, Γ^n) is in X', hence these n connected slopes have product not less than 1, contradicting (K).

Stronger sufficient conditions are:

$$\text{(K}'\text{)} \qquad f(t_2)-f(t_1)/t_2-t_1<1, \qquad 0\leqq t_1<t_2\leqq p.$$

(K″) There exists a β such that $0<\beta<1$, $f(t_2)-f(t_1)/t_2-t_1\leqq\beta$ on $0\leqq t_1<t_2\leqq p$.

In case (K″) obtains, we note that $\Gamma_r^0-C_r^0=(f(\Gamma_{r-1}^1)-f(C_{r-1}^1))/\Gamma_{r-1}^1-C_{r-1}^1)\cdots(f(\Gamma_1^{r-1})-f(C_1^{r-1})/\Gamma_1^{r-1}-C_1^{r-1})\quad(f(c_r+1)-f(c_r)/(c_r+1-c_r)\leqq\beta^r$, so that from (H) the error in the Γ_r^0 and C_r^0 approximations to $\gamma_0=C^0=\Gamma^0$ is not greater than β^r.

Although the slope condition (K′) is sufficient for one-one (B), it is far from necessary. We shall construct functions of arbitrarily great slope for which (B) is one-one.

Consider the set of all ratios (note: *not* slopes) $f(b+f(a+1)-f(b+f(a))/f(b+1)-f(b)$ where a, b are arbitrary integers on $0, 1, \cdots, p-1$. Of these there are only a finite number, each less than one, since the numerator is the difference of function values on a proper subinterval of $(b, b+1)$. Let M be the maximum of these ratios, $M<1$.

Now consider the intervals $(b+f(a), b+f(a+1))$ and suppose that the ratio of inner to outer slope of $f(t)$ on each of these intervals is bounded above from $1/M$, that is:

(L) There exists a $k<1/M$ such that

$$(f(t_2)-f(t_1)/t_2-t_1)/(f(b+f(a+1)-f(b+f(a)/f(a+1)-f(a))\leqq k$$

for all t_1, t_2 on

$$b+f(a)\leqq t_1<t_2\leqq b+f(a+1),$$

or equivalently:

(L′) There exists a $k<1/M$ such that whenever

$$b+f(a)\leqq t_1<t_2\leqq b+f(a+1) \quad \text{and} \quad t_2-t_1\leqq\tau\cdot(b+f(a+1)-(b+f(a)),$$

we must have $f(t_2)-f(t_1)\leqq k\tau(f(b+f(a+1)-f(b+f(a))$.

The condition (L′) *is sufficient for one-one* (B).

For, by definition of $M, f(b+f(a+1)-f(b+f(a)\leqq M(f(b+1)-f(b))$ and $c+f(b+f(a+1)-(c+f(b+f(a))\leqq M((c+f(b+1)-(c+f(b))$. Now use (L′) on the interval $(c+f(b), c+f(b+1))$ and we have

$$\begin{aligned} f(c+f(b+f(a+1)-f(c+f(b+f(a) &\leqq kM(f(c+f(b+1)-f(c+f(b)) \\ &\leqq kM^2(f(c+1)-f(c)). \end{aligned}$$

By iteration of this process, one obtains

$$\begin{aligned} &f(c_1+\cdots+f(c_r+1)-f(c_1+\cdots+f(c_r) \\ &\qquad\qquad \leqq k^{r-2}M^{r-1}(f(c_1+1)-f(c_1))\leqq(kM)^{r-2}M, \end{aligned}$$

which approaches zero since $k<1/M$.

While this discussion is cumbersome, it nevertheless shows that a function $f(t)$ defined arbitrarily (consistent with monotonicity) at $t=0, 1, 2, \cdots, p-1, p$, and then at all $t=b+f(a)$, a, b on $0, \cdots, p-1$, and elsewhere by the broken line connecting these points, must satisfy (L) with $k=1$, since the ratio of inner to outer slope on the straight segments is unity.

Thus the broken line function connecting $f(0)=0$, $f(1)=e>0$, (e arbitrarily small constant), $f(1+e)=1-e$, $f(2)=1$ (for $p=2$) yields a one-one (B). The slope on $(1, 1+e)$ however is $(1-2e)/e$, which may be arbitrarily large.

9. **Algebraic examples of the one-one case.** *Example* 1. Let $f(t)=t^n/p^n$ for an integer $p\geqq 2$ and an integer n on $1\leqq n<p$. One verifies the properties of §2, and condition (K″) with $\beta=n/p$. For $n=1$, our theory reduces to the classical decimals with base p. In the general case let q be an integer not greater than $p-2$, and let C^q be the limit for sequence $\{q, q, \cdots\}$. Then $C^q=q+f(C^q)$, and the number $\alpha=C^q/p$ satisfies $p\alpha=q+\alpha^n$ or $\alpha^n-p\alpha+q=0$, where $0\leqq q/p\leqq\alpha<(q+1)/p\leqq p-1/p$. Thus the equation $x^n-px+q=0$, $1\leqq n<p$, $0\leqq q$

$\leqq p-2$, has exactly one real root on $[0, 1)$, namely $\alpha=(1/p)\cdot(q+f(q+f(q+\cdots$.

In particular, for $n=2$, $p=3$, $q=1$, $x^2-3x+1=0$, $\alpha=(3-5^{1/2})/2$ is approximately

$$(1/3)C_3^0=(1/3)\left(1+\frac{1}{9}\left(1+\frac{1}{9}\left(1+\frac{1}{9}\right)\right)\right),$$

with error not greater than $\beta^3=(2/3)^3$.

Example 2. For $f(t)=(1+t)^{1/n}-1$, $p=2^n-1$, $n>1$, one has slope on $(0, p)$ not greater than $1/n$. We consider $\gamma=1+f(q+f(q+\cdots)$ where $0<q\leqq 2^n-2$. We have $\gamma=1+f(q+\gamma-1)=1+(q+\gamma)^{1/n}-1$ or $\gamma^n-\gamma-q=0$. Thus, the equation $x^n-x-q=0$, $n>1$, $0<q\leqq 2^n-2$, has only one real root γ on $(1, 2]$, namely the number γ above.

For instance, $n=2$, $p=3$, $q=1$, $x^2-x-1=0$, $\gamma=1+f(1+f(1+\cdots$. The successive C_ν^0 are $1+f(1)=2^{1/2}=(1+1^{1/2})^{1/2}$ (from here on radicals are "nested"), $1+f(1+f(1)=(1+(1+1^{1/2})^{1/2})^{1/2}$, and so on. Hence $(1+5^{1/2})/2=(1+(1+(1+\cdots)^{1/2}$.

Recalling the remark at the end of §4, and using $n=2$, $p=3$, $q=2$, $x^2-x-2=0$, $\gamma=2=1+f(2+f(2+\cdots)$, the successive approximations being $1+f(2)=3^{1/2}$, $1+f(2+f(2)=(2+3^{1/2})^{1/2}$, $1+f2+f2+f2=(2+(2+3^{1/2})^{1/2})^{1/2}$, and so on. But using the sequence $p-1+f(p-1+\cdots+f(p-2)$, we have $3=2+f2+f2+\cdots$ with approximations $2+f(1)=1+2^{1/2}$, $2+f2+f1=1+(2+2^{1/2})^{1/2}$, whence $2=(2+(2+(2+\cdots)^{1/2}$ which is the classical limit occurring in the inscribed polygon theory [2].

Finally, for $n=3$, $p=2^n-1=7$, $q=6$, $x^3-x-6=0$, $\gamma=2=1+f(6+f(6+\cdots$, the approximations being $1+f6=7^{1/3}$, $1+f6+f6=[6+7^{1/3}]^{1/3}$, or again using the $(p-2)$-terminating approximations, $2=\{6+[6+(6+\cdots)^{1/3}]^{1/3}\}^{1/3}$.

10. "Spectra" of the topological maps of the unit interval. Let $T=\{F_1(t)\}$ be the class of all continuous increasing functions on $0\leqq t\leqq 1$ with $F_1(0)=0$, $F_1(1)=1$. These are the topological mappings of the unit interval onto itself [5]. If p is any integer not less than 2 and $f(t)$ is of the type in §2:

(M) $f(t)$ continuous increasing on $0\leqq t\leqq p$; $f(0)=0$, $f(p)=1$, then $F(t)=f(pt)$ is in the class T. Thus all our functions may be regarded as magnifications of the functions of T by a factor p in the t-direction. Conversely, if $F_1(t)$ is in T and $p\geqq 2$, then $F_p(t)=F_1(t/p)$ is a function of type (M). Hence for every $F_1(t)$ in T we regard the sequence of functions $\{F_1(t), F_2(t), F_3(t), \cdots\}$ where $F_p(t)=F_1(t/p)$ for $p\geqq 2$. The associated sequence of perfect sets L_p of limit numbers of $F_p(t)$ is a curious sort of "spectrum" for F_1.

For a fixed $F_1(t)$ in T, the correspondence $(t, F_1(t)) \leftrightarrow (nt, F_1(t))$, $0 \leqq t \leqq 1$, is a one-one correspondence of the points on the curves $y = F_1(t)$ and $y = F_n(t)$. This induces a one-one correspondence between the points of the curves F_n and F_{n+1}, namely, $(nt, F_1(t)) \leftrightarrow ((n+1)t, F_1(t))$, $0 \leqq t \leqq 1$. The latter may be used to show that the slopes s_n, s_{n+1} of the chords at corresponding points of F_n and F_{n+1} satisfy $s_n > s_{n+1}$. Hence if some F_n satisfies (K′) so do all succeeding F_n, and thus $L_p = [0, p)$, $p \geqq n$. If F_1 is of bounded slope, there will exist an F_n of slope everywhere less than one. Moreover, one can show that if F_n satisfies (D′) and (J′), so does F_{n+1}. This leads to the question whether $L_p = [0, p)$ implies $L_{p+1} = [0, p+1)$. This is in fact *not* the case.

Example. The broken line function F_1 defined by $F_1(0) = 0$, $F_1(4/9) = 1/3$, $F_1(5/9) = 2/3$, $F_1(1) = 1$ has $L_2 = [0, 2)$, since the product of every two connected slopes of F_2 is less than one (condition (K) with $n = 2$; note that the test (L) fails). But $F_3(t)$ is the function of §7 with L_3 of measure zero. However $L_p = [0, p)$, $p \geqq 4$, since the maximum slope of F_4 is one and all successors therefore have slope less than one.

11. **Unsolved problems.** (1) State simple necessary and sufficient conditions on $f(t)$ such that (B) be one-one. (2) Do there exist functions $f(t)$ which give $C^0 < \Gamma^0$ for non-terminally-periodic sequences $\{c_r\}$? (3) Do functions exist with sets L of every measure between 0 and p? (4) The limits of periodic sequences of period k are algebraic numbers of degree n^k at most for the function of Example 1, §9. Characterize algebraically all such limits.

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UNIVERSITY OF WISCONSIN

A NOTE ON LIE GROUPS

HANS SAMELSON

1. **Introduction.** The following theorem, which plays a role in the classification of Lie groups, was first proved by H. Weyl [1, 2]:[1]

THEOREM A. *If G is a real compact connected semi-simple Lie group, then any connected group G' locally isomorphic with G is also compact.*

It is well known and easily seen by considering the simply connected covering group that Theorem A can also be formulated as follows:

THEOREM B. *The fundamental group of a real connected compact semi-simple Lie group is finite.*

In this note we present two proofs of Theorems A and B; one proof uses differential forms, the other, which is somewhat more elementary, is based on differential geometry.[2]

Let then G be a real connected compact Lie group and assume that the fundamental group of G is infinite. We have to prove that G is not semi-simple. We note that for compact groups "semi-simple" means that the center of G is finite [2, p. 282].

2. **Proof by differential forms.** Since for group manifolds the fundamental group and the one-dimensional homology group coincide, our assumption means that the one-dimensional Betti number is not 0. Let Z denote a 1-cycle, which is not homologous to 0 (with rational or real coefficients). By de Rhams theorem there exists an exact differential form ω of degree one such that $\int_Z \omega \neq 0$. It is well known from Cartan's investigations that we can replace ω by a form $\bar{\omega}$ which is invariant under the right and left translations of G. We denote by $a \cdot \theta$ resp. $\theta \cdot b$ the transform of the differential form θ under left resp. right translation so that $a \cdot \theta(x, dx) = \theta(a \cdot x, a \cdot dx)$, where $a \cdot x$ means the group product of the elements a and x of G and $a \cdot dx$ means the image of the vector dx under the left translation by a, and similarly for $\theta \cdot b$. With Haar measure on G we form the expression $\bar{\omega} = \iint_G a \cdot \omega \cdot b \, da db$; this is an invariant form on G of degree 1. We consider now

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The first proof has also been known to C. Chevalley and G. de Rham for some time, and is given here mainly for completeness' sake.

$$\int_Z \bar{\omega} = \int\int_G \int_Z a \cdot \omega \cdot b \, dadb.$$

By the formula for transformation of integrals we have $\int_Z a \cdot \omega \cdot b = \int_{a \cdot Z} \omega \cdot b$, where $a \cdot Z$ is the image of Z under left translation by a. But $a \cdot Z$ is homotopic to Z since a can be connected with the unit element e by a continuous curve. Therefore $\int_Z a \cdot \omega \cdot b = \int_{a \cdot Z} \omega \cdot b = \int_Z \omega \cdot b$, and by the same reasoning on b we find $\int_Z a \cdot \omega \cdot b = \int_Z \omega$, and therefore finally

$$\int_Z \bar{\omega} = \int\int_G \int_Z \omega \, dadb = \int_Z \omega \cdot \int\int_G dadb = \int_Z \omega \neq 0.$$

The form $\bar{\omega}$ is in particular invariant under the inner automorphisms $a^{-1} \cdot x \cdot a$ of G. Considering $\bar{\omega}$ at the unit element e we have then a nonzero linear function on the tangent space at e which is invariant under the linear transformations of the adjoint group. Since G is compact we can introduce in the tangent space at e an inner product which is invariant under the adjoint group. In a space with an inner product a linear function can be identified with a vector and so $\bar{\omega}$ gives us a vector at e invariant under the adjoint group. (If we write $\bar{\omega}(e, dx) = \sum \alpha_i dx_i$ and assume that the adjoint group is represented by orthogonal matrices, this is simply the vector with components α_i.) But then the one-parameter subgroup in direction of this vector is invariant under the adjoint group also, and lies therefore in the center of G, which shows that G is not semi-simple.

3. **Proof by differential geometry.** The second proof rests on the consideration of geodesics. We assume again that the fundamental group of G is infinite. We introduce in G an invariant differential geometry; this is possible since G is compact; "invariant" means that the right and left translations are isometries. It is well known that the geodesics going through e are the one-parameter subgroups.

Let $\bar{G}$ be the simply connected covering group of G; we introduce the "covering" differential geometry on $\bar{G}$ by requiring that the local isomorphism between G and $\bar{G}$ be an isometry. This differential geometry will also be invariant. Because of the assumption on the fundamental group, $\bar{G}$ is not compact.

As covering space of a compact space, $\bar{G}$ is a "complete" Riemannian space; any two points in it can be connected by a shortest geodesic, that is, by one which realizes the absolute minimum of curve length between the two points (see [3, 4, 5]). In $\bar{G}$ there exist therefore arbitrarily long geodesic segments which are the shortest connections of their end points. By moving the midpoint of each such segment to $\bar{e}$ (the unit element of $\bar{G}$) by means of a left translation, and by considering the limit of a properly chosen sequence we can find a geodesic $\bar{\gamma}$ through $\bar{e}$, which is a "straight line," that is, which realizes the shortest distance in $\bar{G}$ between any two of its points. We shall prove that $\bar{\gamma}$ belongs to the center of $\bar{G}$.

Consider the image $\gamma = c(\bar{\gamma})$ in G of $\bar{\gamma}$ under the covering mapping $c: \bar{G} \rightarrow G$. The group γ may or may not be closed; the closure of γ is a connected compact Abelian Lie group, therefore a torus group T of a certain dimension. We introduce arclength s $(-\infty < s < +\infty)$ on $\bar{\gamma}$, and write $\bar{\gamma}(s)$ for the point on $\bar{\gamma}$ with parameter value s; we can assume $\bar{\gamma}(0) = \bar{e}$.

We determine now a sequence s_n of values of s, such that $s_n \rightarrow +\infty$, and $c(\bar{\gamma}(s_n)) \rightarrow e$. This is possible since T is compact. From a certain n on we can find points $\bar{e}_n$ in $\bar{G}$ such that (1) $c(\bar{e}_n) = e$ and (2) $d(\bar{\gamma}(s_n), \bar{e}_n) = d(c(\bar{\gamma}(s_n)), e)$ (where we denote by d the distance in G and in $\bar{G}$); this is possible because the covering mapping c is a local isometry. It could happen that $\bar{\gamma}(s_n) = \bar{e}_n$. The points $\bar{e}_n$ are in the center of $\bar{G}$, as (1) shows.

Now let a be any element of $\bar{G}$, and consider the transform $\bar{\delta} = a^{-1} \cdot \bar{\gamma} \cdot a$ of $\bar{\gamma}$; transformation by a being an isometry the parameter s on $\bar{\gamma}$ can also be used as arclength on $\bar{\delta}$. Suppose now that $\bar{\delta}$ is different from $\bar{\gamma}$; then in particular the tangent vectors to $\bar{\gamma}$ and $\bar{\delta}$ at $\bar{e}$ must determine an angle different from zero. Let b^+ denote a point with positive s-value on $\bar{\delta}$, and b^- a point with negative s-value on $\bar{\gamma}$. It is well known that the triangle inequality holds for b^+, b^-, and $\bar{e}$, that is, $d(b^+, b^-) < d(b^+, \bar{e}) + d(b^-, \bar{e})$, provided b^+ and b^- are sufficiently close to $\bar{e}$ (see [6]). We choose b^+ and b^- accordingly; let $d(b^+, \bar{e}) + d(b^-, \bar{e}) - d(b^+, b^-) = \eta$; we have then $\eta > 0$.

We determine n such that $d(\bar{\gamma}(s_n), \bar{e}_n) < \eta/3$; the inequality $d(\bar{\delta}(s_n), \bar{e}_n) < \eta/3$ follows then from the fact that the isometrical transformation by the element a transforms $\bar{\gamma}(s_n)$ into $\bar{\delta}(s_n)$, but has $\bar{e}_n$ as fixed point, since $\bar{e}_n$ belongs to the center of $\bar{G}$. We consider now the following broken path $\bar{\zeta}$: from b^- to b^+ on the shortest geodesic joining those two points, from b^+ to $\bar{\delta}(s_n)$ on $\bar{\delta}$, from $\bar{\delta}(s_n)$ to $\bar{e}_n$ on the shortest geodesic, and from $\bar{e}_n$ to $\bar{\gamma}(s_n)$ on the shortest geodesic. It is clear that the length of $\bar{\zeta}$ is less than the distance between b^- and $\bar{\gamma}(s_n)$ as measured on $\bar{\gamma}$—the difference being at least $\eta/3$. But by construction $\bar{\gamma}$ realizes the shortest distance between any two of its points.

Therefore $\bar{\delta}$ cannot be different from $\bar{\gamma}$. But a being an arbitrary element of $\bar{G}$ this means that $\bar{\gamma}$ is in the center of $\bar{G}$; it follows that the torus T, the closure of γ, is in the center of G, and G is shown not to be semi-simple, which finishes the proof.

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EQUIVALENCE IN A CLASS OF DIVISION ALGEBRAS OF ORDER 16

R. D. SCHAFER

Let $\mathfrak{C}$ be a Cayley-Dickson division algebra over an arbitrary field $\mathfrak{F}$ with principal equation

$$x^2 - t(x)x + n(x) = 0 \tag{1}$$

and involution

$$S:\quad x \leftrightarrow xS = t(x) - x. \tag{2}$$

We are concerned with division algebras $\mathfrak{A}$ of order 16 over $\mathfrak{F}$ defined in the following way: let $\mathfrak{C}_*$ be a division algebra (of order 8) over $\mathfrak{F}$ with the same elements as $\mathfrak{C}$ but with multiplication denoted by xoy; further let $\mathfrak{A} = \mathfrak{C} + v\mathfrak{C}$, multiplication[1] in $\mathfrak{A}$ being defined by

$$cx = (a + vb)(x + vy) = (ax + yobS) + v(aS\cdot y + xb) \tag{3}$$

for a, b, x y, in $\mathfrak{C}$.

In the original form of this paper, the author considered the problem of equivalence in the class of algebras $\mathfrak{A} = \mathfrak{C} + v\mathfrak{C}$ with multiplication defined by

$$cx = (a + vb)(x + vy) = (ax + g\cdot ybS) + v(aS\cdot y + xb) \tag{4}$$

for a, b, x, y in $\mathfrak{C}$ where g is a fixed element of $\mathfrak{C}$, $g \notin \mathfrak{F}$. The author had shown in [5] that $\mathfrak{A}$ is a division algebra in case g is chosen with $n(g)$ not a square in $\mathfrak{F}$; in particular, such a choice of g can be made when $\mathfrak{F}$ is the field R of rational numbers. R. H. Bruck, the referee of the paper in its original form, suggested a study of the wider class of algebras defined by (3). Theorems 1 and 2 are generalizations of the result in [5] and are due[2] to R. H. Bruck. By their use the class of algebras studied here has been considerably enlarged.[3]

In §2 we shall determine conditions for the equivalence of two alge-

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[1] This modification of the Cayley-Dickson process was originally presented by R. H. Bruck in [2], Theorem 16C, to obtain non-alternative division algebras of orders 4 and 8. Numbers in brackers refer to the references cited at the end of the paper.

[2] Theorem 2 was communicated to the author by Bruck, complete except for the proof of the equivalence of $\mathfrak{C}_*$ and $\mathfrak{C}$, a fact which Bruck conjectured.

[3] See the comment following the corollary to Theorem 4.

bras $\mathfrak{A}$, $\mathfrak{A}_*$ of this class. These conditions lead directly to a determination of the automorphisms of $\mathfrak{A}$.

1. **A class of division algebras of order 16.** An algebra $\mathfrak{Q}$ is called a quaternion algebra if $\mathfrak{Q}=(1, u_2, u_3, u_4)$, $u_4=u_3u_2$, $u_2^2=u_2+\alpha$, $u_3^2=\beta$, $u_2u_3=u_3(1-u_2)$, where α and $\beta\neq 0$ are in $\mathfrak{F}$, $-4\alpha\neq 1$. An algebra $\mathfrak{C}$ is called a Cayley-Dickson algebra if $\mathfrak{C}=\mathfrak{Q}+u_5\mathfrak{Q}$, with elements $x=p+u_5q$, where p, q are quaternions and multiplication is defined by $(p_1+u_5q_1)(p_2+u_5q_2)=(p_1p_2+\gamma q_2\cdot q_1S)+u_5(p_1S\cdot q_2+p_2q_1)$ where $u_5^2=\gamma\neq 0$ in $\mathfrak{F}$ and S is the involution (2) of $\mathfrak{Q}$.

Among the well known properties of $\mathfrak{C}$ which we shall use are that $\mathfrak{C}$ is an alternative algebra (see [4]), that $\mathfrak{C}$ is central simple, that (1) holds for x in $\mathfrak{C}$, where

$$x+xS=t(x),\qquad x(xS)=(xS)x=n(x) \tag{5}$$

for the involution (2) of $\mathfrak{C}$. The norm form $n(x)$ permits composition—that is, $n(xy)=n(x)n(y)$ for x, y in $\mathfrak{C}$.

THEOREM 1. *Let $\mathfrak{C}$ be a Cayley-Dickson division algebra over $\mathfrak{F}$ with involution S. Let $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$ have multiplication defined by* (3), *where $\mathfrak{C}_o$ is chosen so that*

$$n(xoy)=\lambda n(x)n(y),\qquad \lambda\neq 0 \text{ in } \mathfrak{F}, \tag{6}$$

for all x, y in $\mathfrak{C}$. If λ is not a square in $\mathfrak{F}$, then $\mathfrak{A}$ is a division algebra over $\mathfrak{F}$.

A necessary and sufficient condition that $\mathfrak{A}$ be a division algebra is that

$$ax+yobS=0,\qquad aS\cdot y+xb=0\qquad (a, b, x, y \text{ in } \mathfrak{C}) \tag{7}$$

imply either $c=0$ or $z=0$. If we assume on the contrary that (7) hold for nonzero c, z it follows that a, b, x, y are all nonzero. Use of (5) then shows (7) essentially equivalent to $yobS=a[(aS\cdot y)bS]/n(b)$. Taking norms, we get

$$n(yobS)=[n(a)/n(b)]^2n(y)n(bS). \tag{8}$$

If λ in (6) is not a square in $\mathfrak{F}$, then (8) cannot be satisfied for any a, b, y in $\mathfrak{C}$, and $\mathfrak{A}$ is a division algebra over $\mathfrak{F}$.

In order to complete the proof of Theorem 2 we require two simple lemmas concerning Cayley-Dickson algebras.

LEMMA 1. *Let $\mathfrak{C}$ and $\mathfrak{C}_*$ be Cayley-Dickson algebras with respective* *unity elements* 1 *and* e *and principal equations* $x^2 - t(x)x + n(x)1 = 0$ *and*

$$x * x - t_*(x)x + n_*(x)e = 0. \tag{9}$$

If $\mathfrak{C}$ *and* $\mathfrak{C}_*$ *are equivalent, the equivalence being given by* $x \leftrightarrow xP$, x *in* $\mathfrak{C}_*$, xP *in* $\mathfrak{C}$, *then*

$$S_*P = PS \tag{10}$$

where S_* *is the involution*

$$S_*: \quad x \leftrightarrow xS_* = t_*(x)e - x \tag{11}$$

of $\mathfrak{C}_*$ *and* S *is the involution* (2) *of* $\mathfrak{C}$.

For (9) implies that $(x * x)P - t_*(x)xP + n_*(x)eP = (xP)^2 - t_*(x)xP + n_*(x)1 = 0$ since $eP = 1$. But $(xP)^2 - t(xP)xP + n(xP)1 = 0$. Hence

$$t_*(x) = t(xP), \qquad n_*(x) = n(xP). \tag{12}$$

But then $(xS_*)P = [t_*(x)e - x]P = t_*(x)1 - xP = t(xP) - xP = (xP)S$ for all x in $\mathfrak{C}_*$, or $S_*P = PS$.

LEMMA 2. *Let* $\mathfrak{C}$ *and* $\mathfrak{C}_*$ *be Cayley-Dickson division algebras over* $\mathfrak{F}$ *with norm functions* $n(x)$ *and* $n_*(x)$. *If*

$$n(x * y) = n(x)n(y), \qquad x, y \text{ in } \mathfrak{C}, \tag{13}$$

then

$$n_*(x) = n(x). \tag{14}$$

We resort to the matrix representation of quadratic forms, regarding x as a one-rowed matrix, so that the matrix product $xAx' = n(x)$, where A is an 8×8 matrix with elements in $\mathfrak{F}$ ($'$ denoting transpose). Let R_y, R_y^* be the right multiplications defined by $xR_y = xy$, $xR_y^* = x * y$. Then $n(x * y) = n(xy)$ or $xR_y^*AR_y^{*\prime}x' = xR_yAR_y'x'$, from which it follows that $R_y^*AR_y^{*\prime} - R_yAR_y'$ is a skew-symmetric matrix, and $R_y^*(A + A')R_y^{*\prime} = R_y(A + A')R_y'$. Taking determinants, and noting that $|A + A'| \neq 0$ since $\mathfrak{C}$ is a division algebra, we obtain $|R_y^*|^2 = |R_y|^2$ or $[n_*(y)]^8 = [n(y)]^8$. Thus $n_*(y) = \epsilon n(y)$, $\epsilon^8 = 1$. Take $y = 1$. Then $\epsilon = 1$ and (14) follows.

THEOREM 2. *Let* $\mathfrak{C}$ *be a Cayley-Dickson division algebra over* $\mathfrak{F}$ *with unity element* 1, *and* $\mathfrak{C}_o$ *be a division algebra with the same elements as* $\mathfrak{C}$ *and with multiplication* $x \circ y$. *Then* (6) *holds if and only if, first,* $\mathfrak{C}_o$ *is isotopic to* $\mathfrak{C}$ (*that is, there exist nonsingular linear transformations* U, V, W *on* $\mathfrak{C}$ *such that*

$$x \circ y = (xU \cdot yV)W \tag{15}$$

for x, y *in* $\mathfrak{C}$) *and, second,*[4]

$$n(xT) = n(x)n(1T), \qquad T = U, V, W. \tag{16}$$

If $\mathfrak{A} = \mathfrak{C} + v\mathfrak{C}$, *multiplication defined by* (3), (15), (16), *and if* $n(1U)n(1V)n(1W)$ *is not a square in* $\mathfrak{F}$, *then* $\mathfrak{A}$ *is a division algebra over* $\mathfrak{F}$.

Define two nonsingular linear transformations H, J on $\mathfrak{C}$ in the following way: let $xH = xo1$, $xJ = 1ox$, for x in $\mathfrak{C}$. Then, if (6) holds, $n(xH^{-1}o1) = n(x) = \lambda n(xH^{-1})$ and

$$n(xH^{-1}) = n(x)/\lambda. \tag{17}$$

Also $n(1oxHJ^{-1}) = n(xH) = n(xo1) = \lambda n(x) = \lambda n(xHJ^{-1})$ or

$$n(xHJ^{-1}) = n(x). \tag{18}$$

Let $\mathfrak{C}_*$ be the isotope of $\mathfrak{C}_o$ defined by

$$x * y = [xo(yHJ^{-1})]H^{-1}. \tag{19}$$

Then 1 is a unity element for $\mathfrak{C}_*$. Moreover, $n(x * y) = n[xo(yHJ^{-1})]/\lambda = n(x)n(yHJ^{-1}) = n(x)n(y)$, or (13) holds. That is, $n(x)$ is a quadratic form permitting composition with respect to the multiplication $x * y$ of $\mathfrak{C}_*$; hence[5] $\mathfrak{C}_*$ is a Cayley-Dickson algebra over $\mathfrak{F}$ and $n(x)$ is equivalent in $\mathfrak{F}$ to the norm form $n_*(x)$ of $\mathfrak{C}_*$. (Actually $n(x)$ is identical with $n_*(x)$ by (14), but that fact is immaterial at this point of the proof.) Since any two Cayley-Dickson algebras with equivalent norm forms are equivalent,[6] it follows that $\mathfrak{C}_* \cong \mathfrak{C}$ with

$$x * y = (xU \cdot yU)U^{-1}, \tag{20}$$

for U a nonsingular linear transformation on $\mathfrak{C}$. Moreover, $\mathfrak{C}_o$ is isotopic to $\mathfrak{C}$. Let

$$V = JH^{-1}U, \qquad W = U^{-1}H. \tag{21}$$

Then (15) follows from (19), (20), (21). Also (16) holds, for $n(xU) = n_*(x) = n(x)$ by (12) and (14), while $1U = 1 = n(1U)$; also $n(xV)$

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[4] Any nonsingular linear transformation T satisfying (16) is closely related to a norm-preserving transformation U. For set $1T = t$ and $U = TR_t^{-1}$. Then $T = UR_t$ and $xT = xU \cdot t$, $n(xT) = n(xU)n(t)$. Hence $n(xU) = n(x)$. Conversely if $T = UR_t$ where U is norm-preserving, then T has property (16).

[5] See the principal theorem of [1, p. 161], for the relationship between quadratic forms permitting composition and the norm forms of certain alternative algebras.

[6] [3, p. 777]. The proof in Jacobson's paper is for $\mathfrak{F}$ of characteristic not two. The author has verified the theorem for arbitrary $\mathfrak{F}$, but will not take the space to include a proof here.

$=n(xJH^{-1}U)=n(xJH^{-1})=n(x)$ by (18) while $1V=1=n(1V)$, and $n(xW)=n(xU^{-1}H)=\lambda n(xU^{-1})=\lambda n(x)$ by (17) while $n(1W)=\lambda n(1)=\lambda$.

Conversely, if multiplication in $\mathfrak{C}$ is defined by (15) and (16), then (6) holds. For $n(xoy)=n[(xU\cdot yV)W]=n(xU\cdot yV)n(1W)=n(xU)n(yV)n(1W)=\lambda n(x)n(y)$ where $\lambda=n(1U)n(1V)n(1W)$. By Theorem 1, a sufficient condition that $\mathfrak{A}$ be a division algebra is that λ be not a square in $\mathfrak{F}$.

2. Equivalence in this class of algebras.

The algebras of §1 are quite general, and are at the same time a concrete realization of the hypotheses in the theorems of this section. In order to show that these theorems, which are actually proved for a more general class of division algebras, apply to the algebras described in §1, we note that if $\mathfrak{C}$ is a Cayley-Dickson division algebra over $\mathfrak{F}$, and $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$ has multiplication defined by (3) and (6), λ not a square in $\mathfrak{F}$, then, for any nonzero element y of $\mathfrak{C}$, the product $yoyS$ is not in $\mathfrak{F}$. For if $yoyS\in F$, then $(yoyS)^2=n(yoyS)=\lambda n(y)n(yS)=\lambda[n(y)]^2$, and λ is a square in $\mathfrak{F}$, a contradiction.

THEOREM 3. *Let* $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$, *with multiplication defined by* (3) *where* $yoyS$ *is not in* $\mathfrak{F}$ *for nonzero* y *in* $\mathfrak{C}$. *Then an element* z *of* $\mathfrak{A}$ *satisfies a quadratic equation with coefficients in* $\mathfrak{F}$ *if and only if* z *is in* $\mathfrak{C}$.

For $z^2=(x+vy)^2=(x^2+yoyS)+v(xS\cdot y+xy)=t(x)x-n(x)+yoyS+t(x)vy=t(x)z-n(x)+yoyS$. If $z^2-T(z)z+N(z)=0$ for some $T(z)$ and $N(z)$ in $\mathfrak{F}$, then $t(x)z-n(x)+yoyS=T(z)z-N(z)$, or $\{t(x)-T(z)\}z=n(x)-yoyS-N(z)$ in $\mathfrak{C}$. Either z is in $\mathfrak{C}$, or $t(x)-T(z)=0=n(x)-yoyS-N(z)$ and $yoyS=n(x)-N(z)$ in $\mathfrak{F}$, contrary to hypothesis. Therefore the only elements of $\mathfrak{A}$ satisfying quadratic equations are the elements $z=x$ of $\mathfrak{C}$.

Let $\mathfrak{A}_*=\mathfrak{C}_*+v'*\mathfrak{C}_*$ where $\mathfrak{C}_*$ is a Cayley-Dickson algebra with principal equation (9), $\mathfrak{C}_1$ is a division algebra with the same elements as $\mathfrak{C}_*$ but with multiplication $[x, y]$. Let multiplication in $\mathfrak{A}_*$ be defined by

$$(22)\quad (a+v'*b)*(x+v'*y)=\{a*x+[y,bS_*]\}+v'*(aS_* * y+x*b)$$

for a, b, x, y in $\mathfrak{C}_*$, where S_* is the involution (11).

THEOREM 4. *A division algebra* $\mathfrak{A}_*=\mathfrak{C}_*+v'*\mathfrak{C}_*$ *with multiplication defined by* (22), $[y, yS_*]$ *not in* $\mathfrak{F}$ *for nonzero* y *in* $\mathfrak{C}_*$, *is equivalent to a division algebra* $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$ *with multiplication defined by* (3), $yoyS$ *not in* $\mathfrak{F}$ *for nonzero* y *in* $\mathfrak{C}$, *if and only if, first,* $\mathfrak{C}_*\cong\mathfrak{C}$, *the equivalence being given by* $x\leftrightarrow xP$, x *in* $\mathfrak{C}_*$, xP *in* $\mathfrak{C}$, *and, second,* $\mathfrak{C}_1\cong\mathfrak{C}_*$ *by the* *specific equivalence* $x \leftrightarrow \delta^2 xP$, x *in* $\mathfrak{C}_1$, $\delta^2 xP$ *in* $\mathfrak{C}_*$, *for some* $\delta \neq 0$ *in* $\mathfrak{F}$. *(This second condition may be stated as*

$$[x, y]P = \delta^2(xPoyP) \tag{23}$$

for some $\delta \neq 0$ *in* $\mathfrak{F}$.*) The equivalence between* $\mathfrak{A}_*$ *and* $\mathfrak{A}$ *is* $x+v' * y \leftrightarrow xP+\delta v(yP)$.

Suppose $\mathfrak{A}_* \cong \mathfrak{A}$. Since $\mathfrak{A}_*$ contains a Cayley-Dickson subalgebra $\mathfrak{C}_*$ containing all of the elements of $\mathfrak{A}_*$ which satisfy quadratic equations with coefficients in $\mathfrak{F}$, it follows that $\mathfrak{C}_* \cong \mathfrak{C}$. Let the equivalence H between $\mathfrak{A}_*$ and $\mathfrak{A}$, $z \leftrightarrow zH$, z in $\mathfrak{A}_*$, zH in $\mathfrak{A}$, have matrix

$$H = \begin{pmatrix} P & Q \\ T & D \end{pmatrix}. \tag{24}$$

Then $Q=0$. For if $a \in \mathfrak{C}_*$, both a and aH satisfy quadratic equations with coefficients in $\mathfrak{F}$, $aH \in \mathfrak{C}$ and $Q=0$ in (24). The nonsingularity of H implies that P and D are nonsingular. Moreover, the equivalence H between $\mathfrak{A}_*$ and $\mathfrak{A}$ induces the equivalence P between $\mathfrak{C}_*$ and $\mathfrak{C}$.

Let L_x, L_x^*, R_y, R_y^* be the left and right multiplications of $\mathfrak{C}$ and $\mathfrak{C}_*$ defined by $xy=yL_x=xR_y$, $x * y=yL_x^*=xR_y^*$. Then multiplication is defined in $\mathfrak{A}$ by $cz=cR_z$,

$$R_z = \begin{pmatrix} R_x & SR_y \\ SL_y^{(o)} & L_x \end{pmatrix}, \qquad aL_y^{(o)} = yoa,$$

and in $\mathfrak{A}_*$ by $c * z=cR_z^*$,

$$R_z^* = \begin{pmatrix} R_x^* & S_*R_y^* \\ S_*L_y^{(1)} & L_x^* \end{pmatrix}, \qquad aL_y^{(1)} = [y, a],$$

where $R_x^*=PR_{xP}P^{-1}$, $L_x^*=PL_{xP}P^{-1}$, and so on.

The equivalence of $\mathfrak{A}_*$ and $\mathfrak{A}$ is given by $R_z^*H=HR_{zH}$ for all z in $\mathfrak{A}_*$, or

$$\begin{pmatrix} PR_{xP}P^{-1} & S_*PR_{yP}P^{-1} \\ S_*L_y^{(1)} & PL_{xP}P^{-1} \end{pmatrix} \begin{pmatrix} P & 0 \\ T & D \end{pmatrix} = \begin{pmatrix} P & 0 \\ T & D \end{pmatrix} \begin{pmatrix} R_{xP+yT} & SR_{yD} \\ SL_{yD}^{(o)} & L_{xP+yT} \end{pmatrix} \tag{25}$$

for all x, y in $\mathfrak{C}_*$.

It follows from (25) that $PR_{xP}+S_*PR_{yP}P^{-1}T=PR_{xP+yT}$ or $S_*PR_{yP}P^{-1}T=PR_{yT}$. Then (10) implies that $PSR_{yP}P^{-1}T=PR_{yT}$ or $SR_{yP}P^{-1}T=R_{yT}$. Let $y=e$ and denote eT by t. Since $eP=1$, it follows that $SP^{-1}T=R_{eT}=R_t$, and $T=PSR_t$. Therefore $SR_{yP}P^{-1}PSR_t=R_{yPSR_t}$ or $L_{yPS}R_t=R_{yPSR_t}$. By a lemma of Moufang [4, Lemma 1],

$L_{yPS}R_{yPS}R_t = R_{yPSR_t}L_{yPS}$. Hence $L_{yPS}R_{yPS}R_t = L_{yPS}R_tL_{yPS}$, and for all $y \neq 0$, $R_{yPS}R_t = R_tL_{yPS}$.

If $t \neq 0$, let $y = tP^{-1}$. Then $R_{tS}R_t = R_tL_{tS} = n(t)I = R_tR_{tS}$ and $L_{tS} = R_{tS}$, $t = \xi$ in $\mathfrak{F}$. That is, if $t \neq 0$, then $\xi R_{yPS} = \xi L_{yPS}$, $R_{yPS} = L_{yPS}$, $yPS \in \mathfrak{F}$ for every $y \neq 0$ in $\mathfrak{C}_*$, and P is singular, a contradiction. Hence $eT = t = 0$ and $T = PSR_t = 0$ in (24).

With this result and (10), we may write (25) as

$$(26) \qquad \begin{pmatrix} PR_{xP}P^{-1} & PSR_{yP}P^{-1} \\ PSP^{-1}L_y^{(1)} & PL_{xP}P^{-1} \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} R_{xP} & SR_{yD} \\ SL_{yD}^{(e)} & L_{xP} \end{pmatrix},$$

from which it follows that $PSR_{yP}P^{-1}D = PSR_{yD}$, or $R_{yP}P^{-1}D = R_{yD}$. Let $y = e$ and denote eD by d. Since $eP = 1$, it follows that $P^{-1}D = R_{eD} = R_d$, or $D = PR_d$. Thus $R_{yP}P^{-1}PR_d = R_{yP}R_d$ or $R_{yP}R_d = R_{yPR_d}$. Let $y = xP^{-1}$. Then $R_xR_d = R_{xd}$ for every x in $\mathfrak{C}$. Therefore d is a scalar δ in $\mathfrak{F}$, or $D = \delta P$, $\delta \neq 0$. Thus

$$H = \begin{pmatrix} P & 0 \\ 0 & \delta P \end{pmatrix}$$

and (26) reduces to three identities and the final relationship $PSP^{-1}L_y^{(1)}P = \delta^2 PSL_{yP}^{(e)}$ or $L_y^{(1)}P = \delta^2 PL_{yP}^{(e)}$ from which (23) follows.

COROLLARY. *Let $\mathfrak{A} = \mathfrak{C} + v\mathfrak{C}$ be a division algebra with multiplication defined by* (4), $g \notin \mathfrak{F}$, *and $\mathfrak{A}_* = \mathfrak{C}_* + v' * \mathfrak{C}_*$ with multiplication defined by*

$$(a + v' * b) * (x + v' * y) = \{a * x + g_1 * (y * bS_*)\} + v' * (aS_* * y + x * b)$$

for a, b, x, y in $\mathfrak{C}_*$, where S_* is the involution (11) *of $\mathfrak{C}_*$ and g_1 is a fixed element of $\mathfrak{C}_*$, $g_1 \notin \mathfrak{F}$. Then $\mathfrak{A}_* \cong \mathfrak{A}$ if and only if, first, $\mathfrak{C}_*$ is equivalent to $\mathfrak{C}$, the equivalence being given by $x \leftrightarrow xP$, x in $\mathfrak{C}_*$, xP in $\mathfrak{C}$, and, second, $g_1P = \delta^2 g$ for some $\delta \neq 0$ in $\mathfrak{F}$. The equivalence between $\mathfrak{A}_*$ and $\mathfrak{A}$ is $x + v' * y \leftrightarrow xP + \delta v(yP)$.*

For $[x, y] = g_1 * (x * y)$ and $x \circ y = g(xy)$ in Theorem 4. Then (23) becomes $\{g_1 * (x * y)\}P = g_1P(xP \cdot yP) = \delta^2 g(xP \cdot yP)$, or simply $g_1P = \delta^2 g$.

By Theorem 4 we see that the class of algebras described in §1 is actually considerably larger than that defined by (4), which was the class originally studied. For any such algebra is the particular case $U = I$, $V = I$, $W = L_g$ of (3), (15), (16). Even such a simple variation $\mathfrak{A}_*$ of $\mathfrak{A}$ as that in which multiplication is defined by

$$(27) (a + v' * b) * (x + v' * y) = \{a * x + (y * bS_*) * g_1\} + v' * (aS_* * y + x * b),$$

$g_1 \notin \mathfrak{F}$, cannot be equivalent to $\mathfrak{A}$, since (27) is the case $U = I$, $V = I$,

$W=R_{g_1}^*$ of (3), (15), (16), and $\mathfrak{C}_1$ cannot be equivalent to $\mathfrak{C}_*$ as required by Theorem 4. For $\mathfrak{C}_*$ has a left unity quantity g^{-1}, while $\mathfrak{C}_1$ has a right unity quantity (the inverse of g_1 with respect to multiplication $x*y$ in $\mathfrak{C}_*$). If $\mathfrak{C}_*\cong\mathfrak{C}_1$, then $\mathfrak{C}_*$ has a right unity quantity h, and $h=g^{-1}$, $\mathfrak{C}_*$ has unity quantity g^{-1}. Then $x=x\circ g^{-1}=gxg^{-1}$, or $xg=gx$ for all x in $\mathfrak{C}$, $g\in\mathfrak{F}$, a contradiction. Actually $\mathfrak{C}_1$ is anti-isomorphic to $\mathfrak{C}_*$ in the case $\mathfrak{C}_*\cong\mathfrak{C}$ (that is, $R_y^*=PR_{yP}P^{-1}$) and $g_1P=gS$, for then $R_y^{(1)}=R_y^*R_{g_1}^*=PR_{yP}R_{g_1P}P^{-1}=PS^2R_{yP}R_{gS}P^{-1}=PSL_{yPS}L_gSP^{-1}=(PS)L_{yPS}^{(g)}(PS)^{-1}$.

The automorphisms of the division algebras of order 16 over $\mathfrak{F}$ which are studied in this paper are given directly by the conditions of Theorem 4, $\mathfrak{A}_*=\mathfrak{A}$.

THEOREM 5. *A nonsingular linear transformation H on a division algebra $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$ with multiplication defined by* (3), *$y\circ yS$ not in $\mathfrak{F}$ for nonzero y in $\mathfrak{C}$, is an automorphism of $\mathfrak{A}$ if and only if H induces an automorphism P on $\mathfrak{C}$ and δ^2P is an automorphism of $\mathfrak{C}_*$ for some $\delta\neq 0$ in $\mathfrak{F}$. Such an automorphism H of $\mathfrak{A}$ has the form $x+vy\leftrightarrow xP+\delta v(yP)$.*

COROLLARY. *A nonsingular linear transformation H on a division algebra $\mathfrak{A}=\mathfrak{C}+v\mathfrak{C}$ with multiplication defined by* (4), *$g\notin\mathfrak{F}$, is an automorphism of $\mathfrak{A}$ if and only if H induces an automorphism P on $\mathfrak{C}$ such that $gP=\delta^2g$ for some $\delta\neq 0$ in $\mathfrak{F}$. Such an automorphism H has the form $x+vy\leftrightarrow xP+\delta v(yP)$. If $t(g)\neq 0$, then $gP=g$ and either $\delta=1$ or $\delta=-1$. If $t(g)=0$, then $\delta^4=1$.*

For $g_1=g$ in the corollary to Theorem 4. Moreover, $t(gP)=t(g)$, $n(gP)=n(g)$. If $t(g)\neq 0$, then $t(gP)=t(\delta^2g)=\delta^2t(g)=t(g)$ implies that $\delta^2=1$, $\delta=\pm 1$. If $t(g)=0$, then $n(gP)=n(\delta^2g)=\delta^4n(g)=n(g)$ implies that $\delta^4=1$.

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A SUMMATION MATRIX WITH A GOVERNOR

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1. Introduction and summary. The present paper describes a new method of summation with a triangular matrix. The innovation consists in the fact that the elements in the nth row of the matrix are constructed from the first n terms of the series to be summed. The dependence of the matrix on the series is such as to allow fairly rapid convergence of the summation process in the case of "well-behaved" series, yet to make the method applicable to some series whose partial sums oscillate violently. The method is included in the Abel method.

2. Summability (*G*). Corresponding to any formal series $\sum_{n=0}^{\infty} a_n$ we form the sequence $\{S_n\}$ determined by the relation

$$S_n = \frac{\sum_{r=0}^{n} s_r \sum_{k=0}^{n-r} | a_k |}{\sum_{r=0}^{n} \sum_{k=0}^{r} | a_k |}, \tag{1}$$

where $s_r=\sum_{k=0}^{r} a_k$ ($r=0, 1, 2, \cdots$) (the right member of equation (1) has meaning except when $a_k=0$ ($k=0, 1, 2, \cdots, n$), in which case we write $S_n=0$).

DEFINITION. *If*

$$\lim_{n\to\infty} \sum_{k=0}^{n} | a_k | \Big/ \sum_{r=0}^{n} \sum_{k=0}^{r} | a_k | = 0 \tag{A}$$

and

$$\lim_{n\to\infty} S_n = S,$$

the series $\sum a_n$ *is summable* (*G*) *to* S; if we wish to be more explicit, we say that the series is summable $(G, \sum_{k=0}^{n-r} | a_k |)$ to S (read: summable with the governor $\sum_{k=0}^{n-r} | a_k |$).

It is evident that equation (1) can be written in the form

$$S_n = \sum_{r=0}^{n} s_r p_{n-r}/P_n,$$

where $p_k=\sum_{h=0}^{k} | a_h |$ and $P_n=\sum_{k=0}^{n} p_k$. In other words, equation (1) represents a Nörlund transformation [3][1] with positive increasing co-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

efficients, provided that condition (A) is satisfied.

The notation of the last paragraph will be used throughout this paper. Where more than one series is under discussion, symbols such as $S_n\{a\}$, $P_n\{a\}$ will be used to indicate quantities associated with the series $\sum a_n$.

THEOREM 1. *Summability (G) is consistent with every Nörlund method with positive coefficients.*

COROLLARY. *Summability (G) is consistent with the Cesàro methods of all orders.*

THEOREM 2. *Summability (G) is included in the Abel method.*

THEOREM 3. *If the series $\sum a_n$, $\sum b_n$, $\sum(a_n+b_n)$ are summable (G) to A, B, and C, respectively, then $A+B=C$.*

Theorem 1 follows from the fact that all Nörlund methods with positive coefficients are consistent with respect to each other [2], and its corollary from the fact that all Cesàro methods of integral order are Nörlund methods [1]. Theorems 2 and 3 are consequences of Silverman and Tamarkin's theorem [4] to the effect that every Nörlund method with positive coefficients is contained in the Abel method.

LEMMA. *A necessary condition for condition* A *to be satisfied is that* $\liminf_{n\to\infty}|a_n|/p_n=0$; *a sufficient condition is that* $\lim_{n\to\infty}a_n/p_n=0$.

To prove our lemma, we write,

$$p_n/P_n = p_t/P_n + (p_n - p_t)/P_n = p_t/P_n + \frac{p_n - p_t}{P_n - P_t}(1 - P_t/P_n),$$

and we recall that if b_i, c_i are non-negative numbers such that $H\leqq b_i/c_i\leqq K$ $(i=1, 2, \cdots, s)$, then $H\leqq\sum_{i=1}^{s}b_i/\sum_{i=1}^{s}c_i\leqq K$. If $\liminf_{n\to\infty}|a_n|/p_n>h>0$, the value of the fraction $(p -p_t)/(P_n-P_t)$ exceeds h if t is sufficiently large and n is greater than t. Since p_r is a non-decreasing function of r, the values of p_t/P_n and P_t/P_n tend to zero as n becomes large, and condition (A) is not satisfied.

If on the other hand $\lim_{n\to\infty}a_n/p_n=0$, a similar argument shows that condition (A) is satisfied, and the proof of the lemma is complete.

THEOREM 4. *The method of summability (G) includes the method of convergence.*

This theorem follows from the preceding lemma and the fact that every Nörlund method includes the method of convergence (a trivial exception to the theorem exists: if $a_n=0$ $(n=0, 1, 2, \cdots)$, $\lim_{n\to\infty}S_n$

$=0$, but condition (A) fails to be satisfied, and a slight modification of our definition is necessary for the series $\sum a_n$ to be summable (G)).

It should here be remarked that the condition $\lim_{n\to\infty} a_n/p_n=0$ is not necessary for condition (A) to be satisfied, as can be seen from the following example: Let

$$a_0 = 0,$$
$$|a_n| = 1 \qquad (n \neq 0, 1, 2, 4, \cdots),$$
$$|a_n| = p_{n-1} \qquad (n = 1, 2, 4, \cdots).$$

We demonstrate, by induction, that

$$p_n \leqq n \log_2 n \qquad (n = 1, 2, 3, \cdots),$$

where the subscript 2 indicates the base of the logarithm. Suppose that

$$p_r \leqq r \log_2 r$$

where $r=2^h$ (h a certain positive integer). Then

$$p_{r+t} \leqq r \log_2 r + t < (r+t) \log_2 (r+t) \qquad (t<r),$$
$$p_{2r-1} \leqq r \log_2 r + r - 1,$$
$$p_{2r} < 2r \log_2 r + 2r = 2r \log_2 (2r);$$

that is, the inequality holds when $r=2^h+1,\ 2^h+2, \cdots, 2^{h+1}$. Since the inequality holds when $r=1$, and since

$$P_n \geqq \sum_{r=2}^{n} (r-2) = (n-2)(n-1)/2,$$

it follows that

$$p_n/P_n \leqq \frac{2n \log_2 n}{(n-2)(n-1)},$$

that is, condition (A) is satisfied while $\limsup_{n\to\infty} |a_n|/p_n = 1/2$.

3. **Special properties.** In this section we consider some of the properties of the sequence S_n that are of interest even when the sequence fails to converge.

THEOREM 5. *If* $b_n = ha_n$ $(n=0, 1, 2, \cdots)$ *where* h *is a constant, then* $S_n\{b\} = hS_n\{a\}$.

This theorem follows immediately from the fact that $p_n\{b\} = |h| p_n\{a\}$, $P_n\{b\} = |h| P_n\{a\}$, and $s_n\{b\} = hs_n\{a\}$.

The next theorem concerns the effect of altering the values of a finite number of terms of the series $\sum a_n$. Since summability (G) is a regular method, it suffices to consider the case of nonconvergent series.

If $b_n = a_n$ $(n \neq i)$ and $b_i = a_i + k$,

$$\begin{aligned}\sum_{r=0}^{n} s_r\{b\}p_{n-r}\{b\} &= \sum_{r=0}^{n} s_r\{a\}p_{n-r}\{b\} + k\sum_{r=i}^{n} p_{n-r}\{b\} \\ &= \sum_{r=0}^{n} s_r\{a\}p_{n-r}\{a\} \\ &\quad + (|a_i + k| - |a_i|)\sum_{r=0}^{n-i} s_r\{a\} + kP_{n-i}\{b\};\end{aligned}$$

it follows that

$$\begin{aligned}S_n\{b\} &= S_n\{a\}P_n\{a\}/P_n\{b\} + (|a_i + k| - |a_i|)\sum_{r=0}^{n-i} s_r\{a\}/P_n\{b\} \\ &\quad + k\left(1 - \sum_{n-i+1}^{n} p_r\{b\}/P_n\{b\}\right).\end{aligned}$$

If condition (A) is satisfied by the series $\sum a_n$,

$$\begin{aligned}P_n\{b\} &= P_n\{a\} + (n - i + 1)(|a_i + k| - |a_i|) \\ &= (1 + \epsilon_n)P_n\{a\},\end{aligned}$$

where $\lim_{n\to\infty}\epsilon_n = 0$ (nonconvergence of the series $\sum a_n$ implies that $\lim_{n\to\infty} p_n\{a\} = \infty$ and therefore that $\lim_{n\to\infty} P_n\{a\}/n = \infty$). From this it follows that the series $\sum b_n$ also satisfies condition (A), and that

$$S_n\{b\} = (1 + \epsilon_n')S_n\{a\} + (1 + \epsilon_n'')k + (|a_i + k| - |a_i|)\theta_n$$

where $\lim_{n\to\infty}\epsilon_n' = 0 = \lim_{n\to\infty}\epsilon_n''$ and $|\theta_n| \leq \max_{r \leq n-i}|s_r\{a\}|/p_{r+i}\{b\}$. Imposing the further requirement that $\lim_{n\to\infty}\theta_n = 0$, we now obtain the following result:

THEOREM 6. *If the series* $\sum a_n$ *satisfies condition* (A); *if*

(B) $$\lim_{n\to\infty} \sum_{r=0}^{n} s_r\{a\}/P_n\{a\} = 0;$$

and if

$$b_n = a_n \qquad (n \neq n_1, n_2, \cdots, n_k),$$
$$b_{n_i} = a_{n_i} + k_i \qquad (i = 1, 2, \cdots, k);$$

then

$$S_n\{b\} = (1 + \epsilon_n)S_n\{a\} + \sum_{i=1}^{k} k_i + \eta_n,$$

where $\lim_{n\to\infty}\epsilon_n = 0 = \lim_{n\to\infty}\eta_n$.

Corollary 1. *If the set of partial sums* $s_n\{a\}$ *is bounded*, $\lim (S_n\{b\} - S_n\{a\}) = \sum_{i=1}^{k} h_i$.

Corollary 2. *If the series* $\sum a_n$ *satisfies conditions* (A) *and* (B) *and is* (*is not*) *summable* (*G*), *every series differing from it in a finite number of terms only is* (*is not*) *summable* (*G*).

Theorem 7. *If*

$$\begin{aligned} b_n &= a_n && (n = 0, 1, \cdots, i-1),\\ b_i &= 0, &&\\ b_n &= a_{n-1} && (n = i+1, i+2, \cdots) \end{aligned}$$

and the series $\sum a_n$ *satisfies condition* (A), *then*

$$S_n\{b\} - S_{n-1}\{a\} = \epsilon_n S_{n-1}\{a\} + \epsilon_n'$$

where $\lim_{n\to\infty} \epsilon_n = 0 = \lim_{n\to\infty} \epsilon_n'$.

This theorem asserts that the insertion of a finite number of terms whose value is zero into a series satisfying condition (A) cannot destroy or bring into existence summability (*G*) for the series or change the value of the sum (*G*) of the series. A stronger theorem is of course desirable; we may ask, for example, whether there exists an infinite sequence of integers $\{n_i\}$ such that the insertion of a term of value zero after each of the terms a_{n_i} in any series $\sum a_n$ satisfying condition A cannot destroy or bring into existence summability (*G*) for the series or change the value of the sum (*G*) of the series.

To prove Theorem 7, we observe that (for $n > 2i$)

$$P_n\{b\} = P_{n-1}\{a\} + p_{n-1}\{a\} + p_{i-1}\{a\}$$

and

$$\begin{aligned} &\sum_{r=0}^{n} s_r\{b\} p_{n-r}\{b\}\\ &\qquad = \sum_{r=0}^{i-1} s_r\{b\} p_{n-r}\{b\} + s_i\{b\} p_{n-i}\{b\} + \sum_{r=i+1}^{n} s_r\{b\} p_{n-r}\{b\}\\ &\qquad = \sum_{r=0}^{i-1} s_r\{b\} p_{n-r}\{b\} + s_i\{b\} p_{n-i}\{b\} + \sum_{r=i}^{n-1} s_r\{a\} p_{n-r-1}\{b\}\\ &\qquad = \sum_{r=0}^{n-1} s_r\{a\} p_{n-1-r}\{a\} + \eta_n P_n\{b\} \qquad \left(\lim_{n\to\infty} \eta_n = 0\right). \end{aligned}$$

Division of $\sum_{r=0}^{n} s_r\{b\} p_{n-r}\{b\}$ by $P_n\{b\}$ gives the desired result.

The following theorem regarding the summability (*G*) of certain alternating series provides a first indication of the strength of the method (G):

THEOREM 8. *If $f(z)$ is a polynomial in z with real coefficients, then the series $\sum_{n=0}^{\infty}(-1)^n f(n)$ is summable* (G).

We assume, first, that $f(z) \geqq 0$ when $z=0, 1, 2, \cdots$. If $f(z)$ is of degree q, the quantities p_n and P_n are then polynomials in n of degree $q+1$ and $q+2$, respectively. We shall show that for each n the sum $\sum_{r=0}^{n} s_r p_{n-r}$ can be expressed as one of two polynomials of degree $q+2$ or less in n, and that the leading terms of the two polynomials have identical coefficients.

Writing

$$\sum_{r=0}^{n} s_r p_{n-r} = \sum_{r=0}^{n} \sum_{k=0}^{r} a_k p_{n-r} = \sum_{k=0}^{n} a_k \sum_{r=k}^{n} p_{n-r} = \sum_{k=0}^{n} a_k P_{n-k},$$

and $P_n = f_2(n)$, we have the relation

$$\sum_{r=0}^{n} s_r p_{n-r} = \sum_{r=0}^{n} (-1)^r f(r) f_2(n-r).$$

Now, if $g(z)$ is holomorphic in the region $-1/2 \leqq x \leqq n+1/2$ $(z=x+iy)$ and if, for some pair of constants (C, α) with $\alpha<1$, the inequality $|g(z)| < Ce^{\alpha|y|}$ holds in this region, then [2, p. 53]

$$\sum_{r=0}^{n} (-1)^r g(r) = \frac{1}{2i} \int_{n+1/2-i\infty}^{n+1/2+i\infty} \frac{g(z)}{\sin \pi z}\, dz - \frac{1}{2i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{g(z)}{\sin \pi z}\, dz.$$

In our case, this identity takes the form

$$\begin{aligned} \sum_{r=0}^{n} s_r p_{n-r} &= \frac{1}{2i} \int_{n+1/2-i\infty}^{n+1/2+i\infty} \frac{f(z) f_2(n-z)}{\sin \pi z}\, dz - \frac{1}{2i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{f(z) f_2(n-z)}{\sin \pi z}\, dz \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{(-1)^n f(n+1/2+it) f_2(-1/2-it) + f(-1/2+it) f_2(n+1/2-it)}{\cosh \pi t}\, dt, \end{aligned}$$

and the theorem is proved for the case where $f(z) \geqq 0$ when $z=0, 1, 2, \cdots$.

If the polynomial $f(z)$ is negative for some of the values $z=0, 1, 2, \cdots, k-1$, and positive for $z=k, k+1, \cdots$, we consider first the series

$$\sum_{n=0}^{\infty} (-1)^{n+k} f(n+k) = \sum_{n=k}^{\infty} (-1)^n f(n).$$

This series is summable (G), and by Theorem 7 it is still summable (G)

after k terms of value zero are prefixed to it. It follows from Theorem 6 that the series $\sum(-1)^n f(n)$ is summable, and the proof of Theorem 8 is complete.

We note, incidentally, that in the special case under consideration in Theorem 8,

$$\lim_{n\to\infty} S_n = \frac{1}{2}\int_{-\infty}^{\infty} \frac{f(-1/2+it)}{\cosh \pi t}\, dt.$$

If $f(z)$ is non-negative $(z=0, 1, 2, \cdots)$, this is obvious from the fact that in the quantity $\sum_{r=0}^{n} s_r p_{n-r}$ the term of highest degree in n is the same as in the quantity

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{f(-1/2+it)f_1(n)}{\cosh \pi t}\, dt.$$

In the more general case, we choose an arbitrary polynomial $g(z)$ so that $f(z)+g(z)\geqq 0$, $g(z)\geqq 0$ $(n=0, 1, 2, \cdots)$. Since the two series $\sum(-1)^n[f(n)+g(n)]$ and $\sum(-1)^n g(z)$ are summable to

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{f(-1/2+it)+g(-1/2+it)}{\cosh \pi t}\, dt$$

and

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{g(-1/2+it)}{\cosh \pi t}\, dt,$$

respectively, the result follows at once from Theorem 3.

The following proposition is now a consequence of Theorem 2:

THEOREM 9. *If $f(z)$ is a polynomial with real coefficients,*

$$\lim_{x\to 1+0} \sum_{n=0}^{\infty} x^n f(n) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{f(-1/2+it)}{\cosh \pi t}\, dt.$$

4. **An unsolved problem.** The method of the present paper can be extended to matrices with governors other than $\sum_{k=0}^{n-r}|a_k|$. Any relation of the form

$$p_n = p(a_0, a_1, \cdots, a_n) \tag{2}$$

or, more generally,

$$p_n = p(n; a_0, a_1, \cdots) \tag{3}$$

determines a Nörlund method, corresponding to each series $\sum a_n$ satisfying the appropriate analogue of condition (A). It would obviously be of value to know a simple relation of the form (2) or (3) generating a "universal Nörlund method (G^*)," that is, a method which sums the series $\sum a_n$ whenever the series is summable by any Nörlund method.

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THE LOCATION OF THE ZEROS OF POLYNOMIALS WITH COMPLEX COEFFICIENTS

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1. Introduction. In a recent paper *On the zeros of polynomials with complex coefficients* [2],[1] we gave an algorithm for determining the number of zeros in any half-plane. We here extend these methods to the actual computation of the zeros by successive approximation. The computation involves only the repetitive processes of synthetic division and cross-multiplication, and *computation with complex numbers has been eliminated.* This method is therefore especially suitable where the computing is to be done by a clerical staff with the use of ordinary calculating machines.

While relatively few methods for finding the zeros of complex polynomials have been given, there are many methods for real polynomials. Runge [12], in 1898, made a comprehensive survey of known processes, and recently this summary was brought up to date by Fry [3]. Among the various methods, Lagrange [10] was one of the first to use continued fractions in the solution. On the other hand, at Bell Telephone Laboratories, the solution of the problem has been reduced to a mechanical one by means of the isograph [1], a machine which solves real polynomials up to the tenth degree.

The method described in this paper is similar to that of G. R. Stibitz [3], who divides the complex plane into horizontal and vertical strips, and to that of F. L. Hitchcock [5], who also uses successive approximation and the greatest common divisor process, but entirely by means of rectangular coordinates. Hitchcock's process consists of Horner's method to approximate the real part of the zero and the euclidean algorithm for the greatest common divisor to find the imaginary part. However, the computational routine is relatively complicated compared with the simple processes in the method given below. This simplification is partly due to the formulas given in §5 for the continued fraction expansion of a rational function.

2. The method. (i) *The separation of the zeros into annular rings.* After bounds for the moduli of the zeros of a polynomial have been obtained (see §6), the next step in the computation is the separation of the zeros into different annular rings. In this way, closer bounds for the moduli of the individual zeros are found.

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[1] Numbers in brackers refer to the bibliography at the end of the paper.

Let

$$P(z) = A_0z^n + A_1z^{n-1} + A_2z^{n-2} + \cdots + A_n \tag{2.1}$$

be a polynomial with complex coefficients. By the substitution

$$z = r\frac{w-1}{w+1}, \qquad r > 0, \tag{2.2}$$

we obtain

$$\begin{aligned} P_r(w) &= (w+1)^nP\left(r\frac{w-1}{w+1}\right) \\ &= (p_0 + iq_0)w^n + (p_1 + iq_1)w^{n-1} + (p_2 + iq_2)w^{n-2} + \cdots \\ &\quad + (p_n + iq_n). \end{aligned} \tag{2.3}$$

We form

$$\begin{aligned} Q_r(w) &= \frac{P_r(w) \pm \bar{P}_r(-w)}{2} \\ &= iq_0w^n + p_1w^{n-1} + iq_2w^{n-2} + p_3w^{n-3} + \cdots . \end{aligned} \tag{2.4}$$

Here the positive or negative sign is used according as the degree of $P_r(w)$ is odd or even, respectively, and $\bar{P}_r(w)$ is obtained from $P_r(w)$ by the replacement of all its coefficients by their complex conjugates.

To obtain an algorithm for determining the number of zeros of $P(z)$ interior and exterior to $|z| = r$, we reformulate Theorem 4.1 of [2]: *If* $P_r(w)$ *and* $Q_r(w)$ *are of the form* (2.3) *and* (2.4), *respectively, and there exists a J-fraction expansion*

$$\frac{Q_r(w)}{P_r(w) - Q_r(w)} = \frac{1}{c_1w + k_1 +}\,\frac{1}{c_2w + k_2 +}\cdots + \frac{1}{c_nw + k_n}, \qquad c_p \neq 0, \tag{2.5}$$

where the c_p *are real and the* k_p *are pure imaginary or zero, then* $P_r(w)$ *has k zeros in* $R(w) > 0$ *and* $(n-k)$ *zeros in* $R(w) < 0$ *if k of the coefficients* c_p *are negative and the remaining* $(n-k)$ *are positive. The expansion* (2.5) *exists only if* $q_0 = 0$. We may prove in the same way a similar theorem for the case where $q_0 \neq 0$. The J-fraction expansion is of the form

$$\begin{aligned} &\frac{Q_r(w)}{P_r(w) - Q_r(w)} \\ &= \frac{1}{k_0 +}\,\frac{1}{c_1w + k_1 +}\,\frac{1}{c_2w + k_2 +}\cdots + \frac{1}{c_nw + k_n}, \quad k_0 \neq 0. \end{aligned} \tag{2.6}$$

Now, by the transformation (2.2), the interior of $|z| = r$ is mapped onto $R(w) > 0$, the exterior of $|z| = r$ onto $R(w) < 0$, and $|z| = r$ onto $R(w) = 0$. We then have the following application of the above theorem:

THEOREM 2.1. *If either the expansion* (2.5) *or* (2.6) *exists, the polynomial* $P(z)$ (2.1) *has* k *zeros within* $|z| = r$ *and* $(n-k)$ *outside if* k *of the coefficients* c_p *are negative and the remaining* $(n-k)$ *are positive.*

By immediate extension of this theorem, we obtain the following corollary:

THEOREM 2.2. *The number of zeros of* $P(z)$ *in the annular ring* $r_2 > r > r_1$ *is equal to* $k(r_2) - k(r_1)$, *where* $k(r_2)$ *and* $k(r_1)$ *denote the number of negative coefficients* c_p *for* $r = r_2$ *and* $r = r_1$, *respectively.*

In order to eliminate computation with complex numbers, we form, as in [2],

$$P'(w) = i^n P_r(-iw) = U(w) + iV(w), \tag{2.7}$$

where

$$\begin{aligned} U(w) &= p_0 w^n - q_1 w^{n-1} - p_2 w^{n-2} + q_3 w^{n-3} + p_4 w^{n-4} - \cdots, \\ V(w) &= q_0 w^n + p_1 w^{n-1} - q_2 w^{n-2} - p_3 w^{n-3} + q_4 w^{n-4} + \cdots. \end{aligned} \tag{2.8}$$

Then, if $q_0 = 0$, we have in general

$$\begin{aligned} \frac{V(w)}{U(w)} &= \frac{1}{d_1 w + m_1} \, \underset{+}{} \, \frac{1}{d_2 w + m_2} \, \underset{+}{} \cdots \underset{+}{} \, \frac{1}{d_n w + m_n} \\ &= \frac{1}{c_1 w + i k_1} \, \underset{-}{} \, \frac{1}{c_2 w + i k_2} \, \underset{-}{} \cdots \underset{-}{} \, \frac{1}{c_n w + i k_n}, \end{aligned} \tag{2.9}$$

where

$$c_p = (-1)^{p+1} d_p, \quad i k_p = (-1)^{p+1} m_p, \qquad p = 1, 2, \cdots, n. \tag{2.10}$$

If $q_0 \neq 0$, the J-fraction expansion in general is of the form

$$\begin{aligned} \frac{V(w)}{U(w)} &= \frac{1}{m_0} \, \underset{+}{} \, \frac{1}{d_1 w + m_1} \, \underset{+}{} \, \frac{1}{d_2 w + m_2} \, \underset{+}{} \cdots \underset{+}{} \, \frac{1}{d_n w + m_n} \\ &= \frac{1}{i k_0} \, \underset{-}{} \, \frac{1}{c_1 w + i k_1} \, \underset{-}{} \, \frac{1}{c_2 w + i k_2} \, \underset{-}{} \cdots \underset{-}{} \, \frac{1}{c_n w + i k_n}, \end{aligned} \tag{2.11}$$

where

$$c_p = (-1)^p d_p, \quad i k_0 = m_0, \quad i k_p = (-1)^p m_p, \quad p = 1, 2, \cdots, n. \tag{2.12}$$

In §5 explicit formulas are given for finding the expansions (2.9) and (2.11).

(ii) *Computation of the zeros of $P(z)$ of known modulus.* From the mapping (2.2) we see that

$$z' = r\frac{w' - 1}{w' + 1} \tag{2.13}$$

is a zero of $P(z)$ of modulus r if w' is a pure imaginary zero of $P(r(w'-1)/(w'+1))$, where 0 and ∞ are included. But the pure imaginary zeros of $(w+1)^n P(r(w-1)/(w+1))$ are precisely the pure imaginary zeros of $D_r(w)$ which is the greatest common divisor of $P_r(w)$ and $Q_r(w)$. We may obtain $D_r(w)$ by the euclidean algorithm and therefore the zeros of $P(z)$ by (2.13). As in (i), we may reduce our problem to that of computation with real numbers by finding $D_r'(w)$, the greatest common divisor of $U(w)$ and $V(w)$, and then set $D_r(w) = D_r'(iw)$.

In particular, if $-r$ is a zero of $P(z)$, then $w=0$ is a common zero of $P_r(w)$ and $Q_r(w)$; if $+r$ is a zero of $P(z)$, then $w=\infty$ is a common zero of $P_r(w)$ and $Q_r(w)$.

(iii) *Computation of the zeros of $P(z)$ if the moduli are not known.* By successively varying the values of r such that the last remainder approaches zero in the euclidean algorithm and such that simultaneously there is a change in the number of positive and negative signs of the c_p in the continued fraction expansions (2.5) or (2.6), we may approximate as closely as we wish the value of $D_r(w)$ from the last nonzero remainder. From the zeros w' of this approximation to $D_r(w)$ we may obtain the zeros of $P(z)$, namely $z' = r(w'-1)/(w'+1)$, as closely as desired. A simple method to obtain a correction in the value of r and also in the value of z' is by interpolation between the values of the remainders corresponding to a change in the number of positive and negative signs of the c_p. The computation may be conveniently arranged in tabular form as shown in §5.

3. Cases where the J-fraction expansion does not exist. If, in the euclidean algorithm for the greatest common divisor of $Q_r(w)$ and $P_r(w) - Q_r(w)$ (see formulas (5.1) or (5.3)), a zero remainder is obtained before n steps have been carried out, then we find the zeros of the first nonzero remainder $D_r(w)$.

If, however, one of the c_p, $p \neq n$, is zero and yet the remainder corresponding to this c_p is not zero, we again cannot expand $Q_r(w)/(P_r(w) - Q_r(w))$ into a J-fraction. This case occurs when certain determinants vanish (cf. [2]). We may then proceed as follows: If, for r

near r_1, the J-fraction expansion shows a change in the number of positive and negative signs of the c_p and for $r=r_1$ one of the c_p, $p\neq n$, vanishes, then $r=r_1$ is the modulus of the desired zero. However, the corresponding value of w cannot always be found from the next to the last remainder. We then make a transformation $z=z'+k$, where k is a constant, and find in the same way the modulus $r=r_2$ of the same zero of $P(z')$. From the triangle thus formed, we compute $z=x+iy$, the desired zero of $P(z)$.

4. Computation of the zeros of polynomials with real coefficients.

If the coefficients of $P(z)$ are real, the continued fraction expansion (2.9) may be computed from table (5.3). However, it is advisable to first determine whether the zeros of $P(z)$ are all real by the following theorem (cf., for example, Grommer [4, pp. 123–125]).

THEOREM 4.1. *If*

$$P(z) = A_0z^n + A_1z^{n-1} + A_2z^{n-2} + \cdots + A_n, \qquad A_0 > 0,$$

has real coefficients, all its zeros are real and different if and only if

$$\frac{P'(z)}{P(z)} = \frac{1}{r_1z + s_1} + \frac{1}{r_2z + s_2} + \cdots + \frac{1}{r_nz + s_n}, \tag{4.1}$$

where the r_p and s_p are real and the r_p are positive or negative according as p is odd or even, respectively.

The proof easily follows by an application of Sturm sequences.

That $P(z)$ has no multiple zeros is not an essential restriction, for, in the case of multiple zeros, $P(z)$ and $P'(z)$ have common factors which may first be removed by the euclidean algorithm.

The following theorem of Van Vleck [16] gives a method to ascertain whether a real polynomial contains the maximum number of imaginary zeros.

THEOREM 4.2. *If the A_p are real and the terms of the sequence*

$$A_0, \quad \begin{vmatrix} A_0 & A_1 \\ A_1 & A_2 \end{vmatrix}, \quad \begin{vmatrix} A_0 & A_1 & A_2 \\ A_1 & A_2 & A_3 \\ A_2 & A_3 & A_4 \end{vmatrix}, \cdots,$$

$$\Delta_n = \begin{vmatrix} A_0 & A_1 & A_2 & \cdots A_n \\ A_1 & A_2 & A_3 & \cdots A_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ A_n & A_{n+1} & A_{n+2} & \cdots A_{2n} \end{vmatrix} \tag{4.2}$$

are positive, then all the zeros of

$$P_{2n}(z) = A_0 + A_1 z + A_2 z^2 + \cdots + A_{2n} z^{2n}$$

are imaginary, and all but one of the zeros of

$$P_{2n+1}(z) = A_0 + A_1 z + A_2 z^2 + \cdots + A_{2n+1} z^{2n+1}.$$

5. Formulas for the expansion of a rational function into a J-fraction.

In [2] we presented an algorithm for the expansion of a rational function

$$\frac{f_1}{f_0} = \frac{\alpha_{11} w^{n-1} + \alpha_{12} w^{n-2} + \cdots + \alpha_{1n}}{\alpha_{00} w^n + \alpha_{01} w^{n-1} + \cdots + \alpha_{0n}}$$

into a J-fraction of the form (2.5) (see table (2.4), p. 146), and we showed that the expansion (2.5) exists if and only if $\alpha_{00} \neq 0$ and certain determinants of *odd* order are different from zero (formulas (2.8)).

In a similar way, it can be shown that

$$\frac{F_1}{F_0} = \frac{\alpha_{00} w^n + \alpha_{01} w^{n-1} + \cdots + \alpha_{0n}}{\beta_{00} w^n + \beta_{01} w^{n-1} + \cdots + \beta_{0n}}$$

may be expanded into a J-fraction of the form (2.6) by the following formulas:

$$\tag{5.1}\begin{array}{llll}
\alpha_{00} = q_0, & \alpha_{01} = p_1, & \alpha_{02} = -q_2, & \cdots, \\
\beta_{00} = p_0, & \beta_{01} = -q_1, & \beta_{02} = -p_2, & \cdots, \\
\alpha_{11} = \dfrac{\alpha_{00}\beta_{01} - \beta_{00}\alpha_{01}}{\alpha_{00}}, & \alpha_{12} = \dfrac{\alpha_{00}\beta_{02} - \beta_{00}\alpha_{02}}{\alpha_{00}}, & \alpha_{13} = \dfrac{\alpha_{00}\beta_{03} - \beta_{00}\alpha_{03}}{\alpha_{00}}, & \cdots, \\
\beta_{11} = \dfrac{\alpha_{11}\alpha_{01} - \alpha_{00}\alpha_{12}}{\alpha_{11}}, & \beta_{12} = \dfrac{\alpha_{11}\alpha_{02} - \alpha_{00}\alpha_{13}}{\alpha_{11}}, & \beta_{13} = \dfrac{\alpha_{11}\alpha_{03} - \alpha_{00}\alpha_{14}}{\alpha_{11}}, & \cdots, \\
\alpha_{22} = \dfrac{\alpha_{11}\beta_{12} - \beta_{11}\alpha_{12}}{\alpha_{11}}, & \alpha_{23} = \dfrac{\alpha_{11}\beta_{13} - \beta_{11}\alpha_{13}}{\alpha_{11}}, & \alpha_{24} = \dfrac{\alpha_{11}\beta_{14} - \beta_{11}\alpha_{14}}{\alpha_{11}}, & \cdots, \\
\beta_{22} = \dfrac{\alpha_{22}\alpha_{12} - \alpha_{11}\alpha_{23}}{\alpha_{22}}, & \beta_{23} = \dfrac{\alpha_{22}\alpha_{13} - \alpha_{11}\alpha_{24}}{\alpha_{22}}, & \beta_{24} = \dfrac{\alpha_{22}\alpha_{14} - \alpha_{11}\alpha_{25}}{\alpha_{22}}, & \cdots, \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots,
\end{array}$$

$$d_p = \frac{\alpha_{p-1,p-1}}{\alpha_{p,p}}, \quad m_0 = \frac{\beta_{00}}{\alpha_{00}}, \quad m_p = \frac{\beta_{p,p}}{\alpha_{p,p}}, \qquad p = 1, 2, \cdots, n.$$

The expansion (2.6) exists if and only if $D_0 = \alpha_{00} \neq 0$ and the determinants D_p, $p = 1, 2, \cdots, n$, formed from the first n principal minors of *even* order in the array

$$(5.2)\qquad \begin{array}{cccccccc}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} & \cdots \\
\beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} & \beta_{04} & \beta_{05} & \beta_{06} & \cdots \\
0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \cdots \\
0 & \beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} & \beta_{04} & \beta_{05} & \cdots \\
0 & 0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \cdots \\
0 & 0 & \beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} & \beta_{04} & \cdots \\
0 & 0 & 0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

are different from zero, where $\alpha_{0p}=\beta_{0p}=0$ if $p>n$.

As a special case of table (2.4) of [2], the rational function

$$\frac{\phi_1}{\phi_0}=\frac{\alpha_{11}w^{n-1}+\alpha_{12}w^{n-3}+\alpha_{13}w^{n-5}+\cdots}{\alpha_{00}w^{n}+\alpha_{01}w^{n-2}+\alpha_{02}w^{n-4}+\cdots}$$

may be expanded into a J-fraction of the form (2.5) by the formulas

$$(5.3)\qquad \begin{array}{llll}
\alpha_{00}, & \alpha_{01}, & \alpha_{02}, & \cdots, \\
\alpha_{11}, & \alpha_{12}, & \alpha_{13}, & \cdots, \\
\alpha_{22}=\dfrac{\alpha_{11}\alpha_{01}-\alpha_{00}\alpha_{12}}{\alpha_{11}}, & \alpha_{23}=\dfrac{\alpha_{11}\alpha_{02}-\alpha_{00}\alpha_{13}}{\alpha_{11}}, & \alpha_{24}=\dfrac{\alpha_{11}\alpha_{03}-\alpha_{00}\alpha_{14}}{\alpha_{11}}, & \cdots, \\
\alpha_{33}=\dfrac{\alpha_{22}\alpha_{12}-\alpha_{11}\alpha_{23}}{\alpha_{22}}, & \alpha_{34}=\dfrac{\alpha_{22}\alpha_{13}-\alpha_{11}\alpha_{24}}{\alpha_{22}}, & \alpha_{35}=\dfrac{\alpha_{22}\alpha_{14}-\alpha_{11}\alpha_{25}}{\alpha_{22}}, & \cdots, \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots,
\end{array}$$

$$d_p=\frac{\alpha_{p-1,p-1}}{\alpha_{p,p}},\ m_p=0,\qquad p=1,2,3,\cdots,$$

provided $\alpha_{00}\neq 0$ and certain determinants of *odd* and *even* order are different from zero (the principal minors in the array (2.8) of [2]).

6. **Upper and lower bounds for the moduli of the zeros of $P(z)$.** In [2] a method was given for finding bounds for the moduli of the zeros of a polynomial. We apply the principles of this method to obtain better bounds by means of Cassinian ovals.

Let

$$(6.1)\qquad P(z)=z^n+(a_1+ib_1)z^{n-1}+(a_2+ib_2)z^{n-2}+\cdots+(a_n+ib_n),$$

where the coefficient of z^n is taken to be unity since the polynomial may always be reduced to this form. The expansion (2.5) will in general exist and, after simple transformations, it may be written in the form

$$\frac{Q(z)}{P(z)} = \frac{(c_1 z + k_1 + 1)^{-1}}{1} \; \frac{h_1}{+\,1} \; \frac{h_2}{+\,1} + \cdots + \frac{h_{n-1}}{1}, \tag{6.2}$$

where

$$\begin{gathered} h_1 = \frac{1}{(c_1 z + k_1 + 1)(c_2 z + k_2)}, \quad h_2 = \frac{1}{(c_2 z + k_2)(c_3 z + k_3)}, \\ \cdots, \; h_{n-1} = \frac{1}{(c_{n-1} z + k_{n-1})(c_n z + k_n)}. \end{gathered} \tag{6.3}$$

If $g_1, g_2, \cdots, g_{n-1}$ are numbers such that $0<g_p<1$, $p=1, 2, \cdots, n-1$, then $P(z)\neq 0$ if z satisfies the inequalities $|h_1| \leqq g_1$, $|h_2| \leqq (1-g_1)g_2$, $|h_3| \leqq (1-g_2)g_3, \cdots, |h_{n-1}| \leqq (1-g_{n-2})g_{n-1}$. Hence each zero of $P(z)$ must lie within at least one of the Cassinian ovals

$$\begin{gathered} \left| (c_1 z + k_1 + 1)(c_2 z + k_2) \right| = \frac{1}{g_1}, \\ \left| (c_2 z + k_2)(c_3 z + k_3) \right| = \frac{1}{(1-g_1)g_2}, \cdots, \\ \left| (c_{n-1} z + k_{n-1})(c_n z + k_n) \right| = \frac{1}{(1-g_{n-2})g_{n-1}}. \end{gathered} \tag{6.4}$$

Then an upper bound for the moduli of the zeros is obtained from the point of maximum modulus of all these curves. The point with largest modulus on each oval is on the line joining its two fixed points.

The methods of this paper may be applied to advantage to a polynomial whose zeros are relatively close together, as, for example, in the case of the polynomial (cf. Hitchcock [5])

$$P(z) = z^3 + (.4 - i)z^2 + (6.67 - .2i)z + (.0667 - 6.67i). \tag{6.5}$$

An upper bound of 2.866 for the moduli of the zeros is found by the use of Cassinian ovals with $g_1=56/59$, $g_2=160/168$, and a lower bound of .47 with $g_1=85/175$, $g_2=175/204$. By the methods of §2, the moduli are actually found to be 2.590596, 2.580912, .997642, and the corresponding zeros are $-.254757+2.578039i$, $-.168394-2.575413i$, $.023149+.997373i$.

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THE BOUNDEDNESS OF ORTHONORMAL POLYNOMIALS ON CERTAIN CURVES OF THE THIRD DEGREE

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1. **Introduction.** Properties of boundedness of systems of orthonormal polynomials, significant because of their relation to the convergence of the corresponding developments of "arbitrary" functions in series, constitute a subject for detailed investigation in themselves. In the case of the orthonormal systems associated with algebraic curves no method appears to be readily available for dealing with the problem as formulated in general terms, while on the other hand special methods of some degree of variety throw light on the facts relating to curves of particular types [6, 7, 4, 5].[1] An earlier paper by the present writer [4] is concerned with loci of the second degree; a more recent one [5] includes reference to the curves of the third degree represented by the equations $y = Ax^3 + Bx^2 + Cx + D$ and $y^2 = x^3$. Although indefinite multiplication of particular instances would be unprofitable, some additional illustrations may serve to suggest a more adequate picture of the general situation. The following paragraphs present extensions of the reasoning to other curves with equations of the form $y^2 = F(x)$, where $F(x)$ is a polynomial of the third degree.

2. **The curve $y^2 = x^2(x+1)$.** This curve has a double point with two real branches intersecting at the origin. It has a parametric representation in which the coordinates are not merely rational functions but more specifically polynomials in the auxiliary variable,

$$x = t^2 - 1, \qquad y = t(t^2 - 1).$$

Since $x^3 = y^2 - x^2$, any monomial in terms of the coordinates containing x^3 as a factor can be replaced on the curve by an expression of lower degree. A fundamental sequence of monomials for application of the Schmidt process in constructing the orthogonal system on the curve is

$$1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^2y,\ xy^2,\ y^3,\ x^2y^2,\ xy^3,\ y^4,\ \cdots . \tag{1}$$

In terms of the parameter these are respectively

$$1,\ t^2 - 1,\ t(t^2 - 1),\ (t^2 - 1)^2,\ t(t^2 - 1)^2,\ t^2(t^2 - 1)^2,\ \cdots . \tag{2}$$

An arbitrary linear combination of a finite number of the above

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[1] Numbers in brackets refer to the references cited at the end of the paper.

expressions in t is *an arbitrary polynomial in t taking on equal values for $t = \pm 1$.* Each of the expressions enumerated does assume equal values for $t = \pm 1$, the value being 0 except in the case of the constant which heads the list. Any polynomial can be expressed as a sum of one containing only even powers and one containing only odd powers; the successive even powers of t are expressible in terms of 1 and the second, fourth, $\cdots$ members of the sequence (2); if an odd polynomial takes on equal values for $t = \pm 1$ the common value must be 0, and the polynomial must be divisible by $t^2 - 1$; and any odd polynomial divisible by $t^2 - 1$ is expressible in terms of the third, fifth, $\cdots$ members of the sequence (2), since the products of the successive odd powers of t by $t^2 - 1$ are thus expressible. The orthonormal polynomials in x and y on an arc of the curve marked off by the inequalities $a \leqq t \leqq b$ are, with suitably adjusted weight function, the orthonormal polynomials $q_n(t)$ on the same interval of values for t, subject to the auxiliary condition that $q_n(1) = q_n(-1)$ (see [1]). The form of the auxiliary condition does not carry any implication as to the relative positions of the intervals (a, b) and $(-1, 1)$.

Let $q_0(t)$, $q_2(t)$, $q_3(t)$, $\cdots$ specifically be the orthonormal polynomials for integration with respect to t and weight function unity. Generalization of the weight function by application of Korous's theorem [8, p. 157; 2, pp. 205–208; 3; 4, pp. 352–353] will not be discussed here. The orthonormal system contains no polynomial of the first degree, since no such polynomial satisfies the auxiliary condition [1, pp. 72–73].

The proof of boundedness of the q's is similar in principle to others which have been given elsewhere [1, 5], but possesses enough individuality in detail to justify brief attention. Let $p_0(t)$, $p_1(t)$, $p_2(t)$, $\cdots$ be the normalized Legendre polynomials for the interval $a \leqq t \leqq b$, orthogonal for weight unity, with all non-negative degrees represented. Let

$$q_n(t) = \sum_{k=0}^{n} c_{nk} p_k(t),$$

with

$$c_{nk} = \int_a^b q_n(t) p_k(t) dt,$$

a representation of this form being possible for any polynomial of the nth degree. Since $q_n(t)$ is orthogonal to every polynomial of lower degree which satisfies the auxiliary condition, and since

$$p_k(t) + 2^{-1}t[p_k(-1) - p_k(1)]$$

is such a polynomial, it follows that

$$c_{nk} = 2^{-1}[p_k(1) - p_k(-1)] \int_a^b tq_n(t)dt$$

for $k<n$.

Inasmuch as both the p's and the q's are normalized

$$1 = \int_a^b [q_n(t)]^2 dt = \sum_{k=0}^{n} c_{nk}^2,$$

in consequence of which 1 is an upper bound for each of the quantities $|c_{nk}|$, while at the same time

$$\sum_{k=0}^{n-1} c_{nk}^2 = 4^{-1}\left[\int_a^b tq_n(t)dt\right]^2 \sum_{k=0}^{n-1} [p_k(1) - p_k(-1)]^2 \leq 1,$$

$$\left|\int_a^b tq_n(t)dt\right| \leq 2\left\{\sum_{k=0}^{n-1} [p_k(1) - p_k(-1)]^2\right\}^{-1/2}. \tag{3}$$

In comparison with the quantity in braces, inequalities will be needed presently for $p_k(1)$ and $p_k(-1)$ separately. The recurrence formula for the p's can be written in the form

$$tp_k(t) = \alpha_{k,k+1}p_{k+1}(t) + \alpha_{kk}p_k(t) + \alpha_{k,k-1}p_{k-1}(t),$$

with coefficients α_{kj} each of which has G as an upper bound for its absolute value, if G is the larger of the numbers $|a|$, $|b|$ [5, p. 178]. For $t = \pm 1$ the formula reads

$$p_k(1) = \alpha_{k,k+1}p_{k+1}(1) + \alpha_{kk}p_k(1) + \alpha_{k,k-1}p_{k-1}(1),$$
$$-p_k(-1) = \alpha_{k,k+1}p_{k+1}(-1) + \alpha_{kk}p_k(-1) + \alpha_{k,k-1}p_{k-1}(-1),$$

whence by subtraction

$$\begin{aligned} p_k(1) + p_k(-1) = \alpha_{k,k+1}&[p_{k+1}(1) - p_{k+1}(-1)] \\ &+ \alpha_{kk}[p_k(1) - p_k(-1)] \\ &+ \alpha_{k,k-1}[p_{k-1}(1) - p_{k-1}(-1)]. \end{aligned}$$

By further addition or subtraction of $p_k(1) - p_k(-1)$,

$$\begin{aligned} 2p_k(\pm 1) = \alpha_{k,k+1}&[p_{k+1}(1) - p_{k+1}(-1)] \\ &+ (\alpha_{kk} \pm 1)[p_k(1) - p_k(-1)] \\ &+ \alpha_{k,k-1}[p_{k-1}(1) - p_{k-1}(-1)]. \end{aligned}$$

There is consequently a number G_1, independent of k, such that

$$|p_k(\pm 1)| \leqq G_1 \sum_{j=k-1}^{k+1} |p_j(1) - p_j(-1)|.$$

Since furthermore, if $j \leqq n-1$,

$$\begin{aligned} |p_j(1) - p_j(-1)| &= \{[p_j(1) - p_j(-1)]^2\}^{1/2} \\ &\leqq \left\{\sum_{j=0}^{n-1} [p_j(1) - p_j(-1)]^2\right\}^{1/2} \end{aligned}$$

it follows that

$$|p_k(\pm 1)| \leqq 3G_1 \left\{\sum_{j=0}^{n-1} [p_j(1) - p_j(-1)]^2\right\}^{1/2}$$

if $k \leqq n-2$.

In the representation

$$q_n(t) = \sum_{k=n-2}^{n} c_{nk}p_k(t) + \sum_{k=0}^{n-3} c_{nk}p_k(t) \tag{4}$$

the sum from 0 to $n-3$ can be written as

$$2^{-1}\int_a^b tq_n(t)dt \left\{\sum_{k=0}^{n-3} p_k(1)p_k(t) - \sum_{k=0}^{n-3} p_k(-1)p_k(t)\right\}. \tag{5}$$

By the Christoffel-Darboux identity

$$\sum_{k=0}^{n-3} p_k(1)p_k(t) = \alpha_{n-3,n-2} \frac{p_{n-2}(1)p_{n-3}(t) - p_{n-3}(1)p_{n-2}(t)}{1-t},$$

the coefficient $\alpha_{n-3,n-2}$ being defined in agreement with the notation of the preceding paragraph. This coefficient then has an upper bound independent of n. The quantities $p_{n-2}(1)$, $p_{n-3}(1)$ are subject to the inequality at the end of the preceding paragraph. The Legendre polynomials $p_{n-3}(t)$, $p_{n-2}(t)$ are bounded, uniformly with respect to n, throughout any closed interval interior to (a, b), and $1/(1-t)$ is bounded except in the neighborhood of $t=1$. Similar observations apply in the case of the identity

$$\sum_{k=0}^{n-3} p_k(-1)p_k(t) = \alpha_{n-3,n-2} \frac{p_{n-2}(-1)p_{n-3}(t) - p_{n-3}(-1)p_{n-2}(t)}{-1-t}.$$

These facts together with (3) show that the whole expression (5) has an upper bound independent of n for its absolute value when t is kept away from the values a, b, ± 1. The first sum in the right-hand member of (4) is similarly bounded, without special reference to the points $t = \pm 1$. *The orthonormal polynomials* $q_n(t)$ *are uniformly bounded throughout any closed interval which is interior to* (a, b) *and excludes the points* $t = \pm 1$, the last restriction being naturally redundant if the values $t = \pm 1$ do not belong to the interval (a, b).

3. **The curve** $y^2 = x^2(x-1)$. Except for a conjugate point at the origin, which is of no concern for the orthonormal system, this curve is real only for $x \geqq 1$. It has the parametric representation

$$x = t^2 + 1, \qquad y = t(t^2 + 1).$$

Fundamental monomials in terms of x and y are the same as in (1). As functions of t these become

$$1,\ t^2 + 1,\ t(t^2 + 1),\ (t^2 + 1)^2,\ t(t^2 + 1)^2,\ t^2(t^2 + 1)^2,\ \cdots .$$

An arbitrary linear combination of a finite number of them is an arbitrary polynomial taking on equal values for $t = \pm i$, or, in alternative characterization, an arbitrary even polynomial plus the product of an arbitrary odd polynomial by t^2+1.

Let $p_0(t)$, $p_1(t)$, $p_2(t)$, $\cdots$ as before be the normalized Legendre polynomials on an interval (a, b), and let $q_0(t)$, $q_1(t)$, $q_2(t)$, $\cdots$ be the orthonormal polynomials on the same interval associated with the auxiliary condition $q_n(i) = q_n(-i)$. The polynomial

$$p_k(t) + 2^{-1}it[p_k(i) - p_k(-i)],$$

which is real for real values of t, satisfies the auxiliary condition, and $q_n(t)$ is orthogonal to it for $n > k$. Consequently, in the representation

$$q_n(t) = \sum_{k=0}^{n} c_{nk}p_k(t), \qquad c_{nk} = \int_a^b q_n(t)p_k(t)dt,$$

the form

$$c_{nk} = 2^{-1}i[p_k(-i) - p_k(i)]\int_a^b tq_n(t)dt$$

can be used for the general coefficient when $k < n$. In analogy with the reasoning of the preceding section it is found that

$$\left|\int_a^b tq_n(t)dt\right| \leqq 2\left\{-\sum_{k=0}^{n-1}[p_k(-i) - p_k(i)]^2\right\}^{-1/2},$$

the quantity in braces being real and positive; with regard to the question whether it might conceivably vanish it is sufficient to note that $p_1(-i)-p_1(i)\neq 0$. The reasoning of the preceding section can then be adapted step by step to the discussion of the boundedness of the present polynomials $q_n(t)$, in spite of the degree of novelty associated with the appearance of imaginaries in the formulas; the only material difference is that since the denominators $\pm i-t$ in the Christoffel-Darboux identity can not vanish on the real interval of values of t considered, there is nothing corresponding to the exceptional status of the values $t=\pm 1$ in the earlier passage. The significance of this difference is of course that the singular point of the curve is now definitely excluded from the range of integration. *The orthonormal polynomials $q_n(t)$ on the interval (a, b) are uniformly bounded except near the ends of the interval.*

4. **The curve $y^2=x^3-x=(x+1)x(x-1)$.** This curve has no singular point, and does not admit the sort of parametric representation used above. The discussion of the orthonormal system will be based on properties of geometric symmetry, and will be restricted to particular domains of integration for which the properties of symmetry in question are realized.

Fundamental monomials in x and y for the application of Schmidt's process are still given by (1). In setting up the orthogonal system for a portion of the curve which is symmetric with respect to the x-axis, with arc length as variable of integration and with unity or any other even function of y as weight function, the monomials containing even powers of y and those containing odd powers of y can be considered separately, since under these conditions any even function of y and any odd function of y are orthogonal to each other.

The even monomials in y from the sequence (1) are

$$1,\ x,\ x^2,\ y^2,\ xy^2,\ x^2y^2,\ y^4,\ xy^4,\ \cdots .$$

In terms of x these are equal on the curve to

$$1,\ x,\ x^2,\ x^3-x,\ x(x^3-x),\ x^2(x^3-x),\ (x^3-x)^2,\ x(x^3-x)^2,\ \cdots ,$$

and linear combinations of the first $n+1$ of them are linear combinations of the first $n+1$ members of the sequence

$$1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6,\ x^7,\ \cdots ,$$

that is, the polynomials of the orthogonal system on the curve which are even functions of y are merely orthogonal polynomials in x on the corresponding interval with the appropriate weight function. Integration with respect to arc length s and with unit weight function over portions of the curve symmetric to each other above and below the x-axis is equivalent to a single integration with respect to x and with weight

$$2\frac{ds}{dx} = 2\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} = \frac{[4y^2+(3x^2-1)^2]^{1/2}}{(x^3-x)^{1/2}}.$$

Since y and $3x^2-1$ do not vanish simultaneously on the curve, the numerator in the last expression is a positive differentiable function of x. If the domain of integration is the entire closed loop of the curve, corresponding to the range $-1 \leqq x \leqq 0$, the weight function is equivalent [4, pp. 352–353] from the point of view of Korous's theorem to $[-x(1+x)]^{-1/2}$, and the orthonormal polynomials are uniformly bounded over the entire range, those which belong to the last-named weight function being reducible by a linear transformation of the independent variable to the cosine polynomials on the interval $(-1, 1)$ (see, for example [2, p. 191]). If the domain is a symmetric portion of the loop designated by the inequalities $-1 \leqq x \leqq a < 0$ or by the inequalities $-1 < a \leqq x \leqq 0$, or a symmetric arc of the infinite branch marked off by the specifications $1 \leqq x \leqq b$, the weight function is equivalent to $(1+x)^{-1/2}$ or $(-x)^{-1/2}$ or $(x-1)^{-1/2}$ in the various cases respectively, which means that the orthonormal polynomials are uniformly bounded on the interval of the x-axis except near the end of the interval at which the weight function remains finite (see [2, pp. 200–201]), or, on the curve, uniformly bounded except near the ends of the arc.

The monomials in (1) containing odd powers of y are

$$y,\ xy,\ x^2y,\ y^3,\ xy^3,\ x^2y^3,\ y^5,\ xy^5,\ \cdots,$$

expressible on the curve in the form

$$\begin{gathered} y,\ yx,\ yx^2,\ y(x^3-x),\ yx(x^3-x),\ yx^2(x^3-x), \\ y(x^3-x)^2,\ yx(x^3-x)^2,\ \cdots. \end{gathered}$$

Linear combinations of these are linear combinations of the successive terms

$$y,\ yx,\ yx^2,\ yx^3,\ yx^4,\ yx^5,\ yx^6,\ yx^7,\ \cdots.$$

Orthonormal combinations for unit weight with respect to x are orthonormal polynomials in x for weight y^2, each multiplied by y. For unit weight with respect to s and for integration over a pair of arcs symmetric with respect to the x-axis, y^2 as weight function is to be replaced by

$$2y^2 \frac{ds}{dx} = (x^3 - x)^{1/2}[4y^2 + (3x^2 - 1)^2]^{1/2}$$

if the formulas are still written with x as variable of integration.

For the oval as domain of orthogonality the weight is equivalent to $[-x(1+x)]^{1/2}$ for the purposes of Korous's theorem. The orthonormal polynomials in terms of x are essentially the sine polynomials (see [2, p. 192]) and on multiplication by $[-x(1+x)]^{1/2}$ or by

$$y = \pm\ [(1 - x)(- x)(1 + x)]^{1/2}$$

give products which are uniformly bounded on the entire domain. For the symmetric arcs limited above and below the x-axis by the specifications $-1 \leqq x \leqq a < 0$, $-1 < a \leqq x \leqq 0$, $1 \leqq x \leqq b$, the weight functions for which orthonormal polynomials in x are to be constructed are equivalent respectively to $(1+x)^{1/2}$, $(-x)^{1/2}$, $(x-1)^{1/2}$, and the products obtained on multiplication by y, regarded as polynomials in x and y, are uniformly bounded near the points where $y=0$ and elsewhere on the range of integration except near the ends of the respective arcs (see [4, pp. 352–353, 356]).

The complete orthonormal system being composed in each case of the polynomials which are even in y and those which are odd in y taken together, *the orthonormal polynomials on the closed loop of the curve, for arc length as variable of integration and unit weight function, are uniformly bounded on the entire loop; the orthonormal polynomials on each of the other symmetric arcs specified are uniformly bounded except near the ends of the arc.*

5. **The curve $y^2 = x^3 + x = x(x^2+1)$.** This curve has a single real branch, of infinite extent, for all points of which $x \geqq 0$. Fundamental monomials in x and y are once more those containing no power of x above the second. Analysis closely patterned after that of the preceding section shows that *the orthonormal polynomials on an arc of the curve symmetric with respect to the x-axis, for arc length as variable of integration and unit weight function, are uniformly bounded except near the ends of the arc.*

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THE REPRESENTATION OF e^{-x^λ} AS A LAPLACE INTEGRAL

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According to a theorem of Bochner [1, p. 498][1] the function e^{-x^λ}, for any fixed value of λ in $0<\lambda<1$, is completely monotonic and admits a unique representation

$$e^{-x^\lambda} = \int_0^\infty e^{-xt} d\alpha_\lambda(t), \qquad 0 \leqq x < \infty,$$

where $\alpha_\lambda(t)$ is bounded and increasing. It follows further from a criterion of Hille and Tamarkin [2, p. 903] that the function also has the form

$$e^{-x^\lambda} = \int_0^\infty e^{-xt}\phi_\lambda(t)\, dt. \tag{1}$$

One can conclude therefore, since $\alpha_\lambda'(t) = \phi_\lambda(t)$, that $\phi_\lambda(t)$ is positive almost everywhere and that

$$\int_0^\infty \phi_\lambda(t) dt < \infty.$$

For this last integral is the total variation of $\alpha_\lambda(t)$, suitably normalized.

Further information concerning $\phi_\lambda(t)$ may be derived from some general results of Post. Let γ be the contour

$$\frac{x}{a} + \frac{|y|}{b} = 1$$

where a and b are fixed and positive; their precise values are a matter of indifference. The principal branch of e^{-s^λ} is holomorphic in the sector to the right of γ, and is moreover of zero type there since $0<\lambda<1$. If e^{-s^λ} is denoted by $f(s)$, the theory of Post [3, p. 730] shows that the limit

$$L[f;t] = \lim_{kh\to t,\, h\to 0+} \frac{(-1)^k}{k!} h^{-k-1} f^{(k)}\left(\frac{1}{h}\right) = \frac{1}{2\pi i}\int_\gamma e^{st}e^{-s^\lambda} ds$$

exists for all $t>0$; γ must be traced so that the origin is at the left. But according to the Post-Widder inversion theorem [4, p. 288] $L[f;t]$ is the inverse Laplace transform of e^{-s^λ}, and so must be equal

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[1] Numbers in brackets refer to the references cited at the end of the paper.

to $\phi_\lambda(t)$ almost everywhere. Explicitly then

$$\phi_\lambda(t) = \frac{1}{2\pi i}\int_\gamma e^{st}e^{-s^\lambda}ds \tag{2}$$

almost everywhere.

A further advantage of the approach we have used is this. Post has shown [3, p. 733] that the existence everywhere of $L[f; t]$ implies its continuity. It follows that by altering $\phi_\lambda(t)$ on a set of measure zero at most it becomes continuous for $t>0$. With this alteration (2) is valid for all $t>0$. The explicit evaluation of the contour integral in (2) is of considerable interest, and this is our next problem.

Certain special cases already appear in the literature in one form or another. For example, if $\lambda=1/2$ it is known [5, pp. 401–402] that

$$\phi_{1/2}(t) = \frac{1}{\pi}\int_0^\infty e^{-tu}\sin u^{1/2}du = \frac{1}{2\pi^{1/2}}\,\frac{1}{t^{3/2}}\,e^{-1/4t}. \tag{3}$$

Very recently Humbert [6] has discussed the problem of representing the derivative of e^{-s^λ} as a Laplace integral. His method, which is purely formal, leads to the following expression for $\phi_\lambda(t)$:

$$\phi_\lambda(t) = -\frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\sin\pi\lambda k\,\frac{\Gamma(\lambda k+1)}{t^{\lambda k+1}}. \tag{4}$$

In particular, if $\lambda=2/3$

$$\phi_{2/3}(t) = -\frac{1}{2(3\pi)^{1/2}}\,\frac{1}{t}\,e^{-2/27t^2}W_{-1/2,-1/6}\left(-\frac{4}{27t^2}\right).$$

The present note provides a rigorous justification for the formula (4)

We shall prove first that (1) is inverted by

$$\phi_\lambda(t) = \frac{1}{\pi}\int_0^\infty e^{-tu}e^{-u^\lambda\cos\pi\lambda}\sin(u^\lambda\sin\pi\lambda)du, \tag{5}$$

the proof being the simple matter of demonstrating the equality of the right-hand members of (2) and (5). To this end consider a contour δ defined as follows. Let A and B be the intersections in the second and third quadrants respectively of the contour $\gamma=\gamma(a, b)$ with a circle ζ_1, center at the origin, radius $R>\max(a, b)$. Let ζ_2 be a circle of radius $\rho<ab(a^2+b^2)^{-1/2}$, center also at the origin. The intersections of ζ_1 and ζ_2 with the negative real axis will be denoted by C and D respectively. δ is then the contour obtained by starting at C, following the negative real axis to the right as far as D, circumnavigating the

origin once around the circle ζ_2, retracing the negative axis from D back to C, then tracing in order the arc CB of ζ_1, the portion of γ between B and the point $(a, 0)$, then between $(a, 0)$ and A, returning finally to C along the arc AC of ζ_1. Then for any $t>0$

$$\frac{1}{2\pi i}\int_\delta e^{zt}e^{-z^\lambda}dz = 0. \tag{6}$$

Now let $R\to\infty$, $\rho\to 0$. The parts of the integrals taken along the curved portions of δ vanish. What is left is the integral along γ, and along the negative real axis traced twice. The first of these is the right-hand member of (2). The second, taking into account the fact that e^{-z^λ} changed branches when the origin was circumnavigated, is precisely the negative of the right-hand member of (5). In view of (6) the two right-hand members in question must therefore be equal. This completes the proof of (5).

To derive (4) observe that by (5)

$$\begin{aligned}\phi_\lambda(t) &= \frac{1}{\pi} I\left\{\int_0^\infty e^{-tu}e^{-(e^{-i\pi}u)^\lambda}du\right\}\\ &= \frac{1}{\pi} I\left\{\sum_{k=0}^\infty \frac{(-1)^k}{k!}e^{-i\pi\lambda k}\frac{\Gamma(\lambda k+1)}{t^{\lambda k+1}}\right\},\end{aligned}$$

where "I" denotes "imaginary part of."

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SEMI-GROUPS OF OPERATORS AND THE WEIERSTRASS THEOREM

NELSON DUNFORD AND I. E. SEGAL

In this note we give a short proof of the generalization[1] of M. H. Stone's representation of groups of unitary operators in Hilbert space and show how it yields the theorem of Weierstrass on uniform approximation by polynomials. This classical result in turn yields fairly easily the more recent algebraic-topological formulation of the Weierstrass theorem as given by M. H. Stone for real algebras and by I. Gelfand and G. Šilov[2] for complex algebras.

For $0 \leqq s < \infty$ let T_s be a linear operation in the real or complex Banach space X satisfying the conditions

$$T_{s+t} = T_s T_t, \quad T_0 = I, \quad |T_s| \leqq 1, \quad \lim_{s\to t} T_s x = T_t x, \quad x \in X. \tag{1}$$

Let $A_h = h^{-1}[T_h - I]$ and $D(A)$ be the domain of definition of the operator $Ax = \lim_{h\to 0} A_h x$.

THEOREM 1. *If T_s, $0 \leqq s < \infty$, is a semi-group of operators in X satisfying* (1) *then $D(A)$ is dense in X and uniformly for s in any finite interval we have*

$$T_s x = \lim_{h\to 0} e^{sA_h} x, \qquad x \in X. \tag{2}$$

Note that $x_s = s^{-1}\int_0^s T_u x\,du \to x$ and that

$$A_h x_s = (sh)^{-1}\left[\int_s^{s+h} T_u x\,du - \int_0^h T_u x\,du\right] \to A_s x \quad \text{as} \quad h \to 0.$$

Thus $D(A)$ is dense in X. Since $|e^{sA_h}| = |e^{(s/h)T_h} e^{-s/h}| \leqq e^{s/h} e^{-s/h} = 1$ it suffices to prove (2) for $x \in D(A)$. If $x \in D(A)$ we have

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[1] See I. Gelfand, C. R. (Doklady) Acad. Sci. URSS. vol. 25 (1939) pp. 713–718; M. Fukamiya, Proc. Imp. Acad. Tokyo vol. 16 (1940) pp. 262–265; E. Hille, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 175–178; E. Hille, ibid. pp. 421–424. The particular form of the theorem given here is that of E. Hille.

[2] M. H. Stone, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481; I. Gelfand and G. Šilov, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 25–39. Our result differs from theirs in applying to spaces which are locally compact, and to subalgebras which are not assumed to contain a unit.

$$
\begin{aligned}
|T_s x - e^{sA_h}x| &= \left|\int_0^s \frac{d}{du}\left(e^{(s-u)A_h}T_u x\right)du\right| \\
&= \left|\int_0^s e^{(s-u)A_h}T_u(Ax - A_h x)du\right| \\
&\leqq s\,|Ax - A_h x| \to 0, \qquad \text{as } h \to 0,
\end{aligned}
$$

which completes the proof.[3]

There is an n-parameter analogue of Theorem 1. Suppose that E_n^+ is the set of points $s=(s_1, \cdots, s_n)$ in Euclidean n-space whose coordinates s_i are non-negative. Let T_s, $s\in E_n^+$, be a linear operator in X satisfying (1). Let $h_i=(0, \cdots, 0, h, 0, \cdots, 0)$ be the vector in E_n^+ with $h>0$ in the ith place and zeros elsewhere. Let $A(h, i)=h^{-1}[T_{h_i}-I]$ and $D(A_i)$ be the domain of definition of the operator $A_i x=\lim_{h\to 0}A(h, i)x$.

THEOREM 2. *If T_s, $s\in E_n^+$, is a semi-group of operators in X satisfying* (1) *then $\bigcap_i D(A_i)$ is dense in X and uniformly for s in a bounded closed set we have*

$$
(3) \qquad T_s x = \lim_{h\to 0} \exp\left(s_1 A(h, 1) + \cdots + s_n A(h, n)\right)x, \qquad x \in X.
$$

Let C be the cube $0\leqq s_i \leqq \delta$ and $x_\delta=\delta^{-n}\int_C T_u x du$. Then $x_\delta \to x$ as as $\delta\to 0$. Just as before it follows that $x_\delta \in \bigcap_i D(A_i)$ so that this set is dense in X. Since $\exp\left(\sum_i s_i A(h, i)\right)=\prod_i \exp\left(s_i A(h, i)\right)$ equation (3) follows from Theorem 1.

THEOREM 3. *Every real or complex continuous function $x(s)$ defined on a bounded closed subset E of Euclidean space is the uniform limit of polynomials in the coordinates $s_1, \cdots, s_n$ of s.*

Let X be the Banach space of real or complex bounded uniformly continuous functions $x(\lambda)$ defined on E_n ($|x|\equiv\sup|x(\lambda)|$). We may and shall assume that $E\subset E_n^+$ and that the function $x(s)=x$ of the theorem is a point of X. For $s\in E_n^+$ define $T_s x=x(\lambda+s)$. By (3) then we have uniformly for $\lambda\in E_n$ and $s\in E$

$$
x(\lambda+s) = \lim_{h\to 0}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(s_1 A(h, 1) + \cdots + s_n A(h, n)\right)^\nu x.
$$

Since $A(h, i)x$ is independent of s and the series $\sum_{\nu=0}^{\infty}$ converges uniformly for $s\in E$ the desired conclusion follows by placing $\lambda=0$.

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[3] In case T_s is continuous in s in the uniform topology of operators, obvious modifications of the above proof show that A is a bounded operator with $|A_h - A| \to 0$ and $T_s = e^{sA}$.

THEOREM 4. *Let* $\mathfrak{A}$ *be a closed (in the topology* $|x| = \sup|x(\gamma)|$*) subalgebra of the Banach algebra* $C(\Gamma)$ *of all continuous (real or complex) functions on the locally compact space which vanish at infinity.*[4] *In the case of the complex algebra, suppose also that if* $x \in \mathfrak{A}$ *then there exists a* y *in* $\mathfrak{A}$ *such that* $\bar{x}(\gamma) = y(\gamma)$.[5] *If for every* $\gamma_1 \neq \gamma_2$ *there is an* $x \in \mathfrak{A}$ *such that* $x(\gamma_1) \neq 0$, $x(\gamma_2) = 0$, *then* $\mathfrak{A} = C(\gamma)$.

To begin with, suppose that the algebra is real and that Γ is compact. In view of the known form of the general linear functional on $C(\Gamma)$, it suffices to show that a regular countably additive numerically-valued set function μ on the Borel sets of Γ is identically zero provided

$$(4) \qquad \int_\Gamma x(\gamma) d\mu(\gamma) = 0, \qquad x \in \mathfrak{A}.$$

Let Γ_1 be Γ with the neighborhood system defined for $\epsilon > 0$ and $x_i \in \mathfrak{A}$ with $x_i(\gamma_0) = 0$ by

$$(5) \qquad N(\gamma_0) = [\gamma \,|\, |x_i(\gamma)| < \epsilon, i = 1, \cdots, k].$$

Such neighborhoods are, by the continuity of x_i, open in Γ and hence Γ_1 is compact. By hypothesis these neighborhoods separate points and since a continuous 1-1 map between compact spaces is a homeomorphism, we see that the neighborhoods $N(\gamma_0)$ form a basis for Γ. Let $F(s) = F(s_1, \cdots, s_n)$ be a bounded uniformly continuous function defined for $s = (s_1, \cdots, s_n) \in E_n$. Let $x_1, \cdots, x_n$ be elements of $\mathfrak{A}$. If $F(x_1, \cdots, x_n)$ is defined by the equation

$$F(x_1, \cdots, x_n)(\gamma) = F(x_1(\gamma), \cdots, x_n(\gamma)), \qquad \gamma \in \Gamma,$$

it is plain that $F(x_1, \cdots, x_n) \in C(\Gamma)$. If $F(s)$ coincides with a polynomial (with $F(0) = 0$) in $s_1, \cdots, s_n$ on the set $(x_1(\Gamma), \cdots, x_n(\Gamma))$, then clearly $F(x_1, \cdots, x_n) \in \mathfrak{A}$ But since $\mathfrak{A}$ is closed, Theorem 3 shows that $F(x_1, \cdots, x_n) \in \mathfrak{A}$ for an arbitrary continuous function $F(s)$ such that $F(0) = 0$. Let N be the neighborhood in E_n defined by the inequalities $|s_i| < \epsilon$, $i = 1, \cdots, n$, and let $\Phi(s)$ be the characteristic function of its complement. Clearly $(x_1(\gamma), \cdots, x_n(\gamma)) \in N$ if and only if γ is in the neighborhood $N(\gamma_0)$ given by (5). Let $F_r(s)$ be a uniformly bounded sequence of continuous functions such that $F_r(s) \to \Phi(s)$, $s \in E_n$. Then $F_r(x_1, \cdots, x_n)(\gamma)$ approaches 1 for $\gamma \notin N(\gamma_0)$ and 0 for $\gamma \in N(\gamma_0)$ and so

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[4] A function x vanishes at infinity if for $\epsilon > 0$ there is a compact set outside of which $|x(\gamma)| < \epsilon$.

[5] $\bar{x}(\gamma)$ is the complex conjugate of $x(\gamma)$.

$$\mu(\Gamma - N(\gamma_0)) = \lim_{\nu} \int_{\Gamma} F_\nu(x_1, \cdots, x_n)(\gamma) d\mu(\gamma) = 0,$$

from which it readily follows that $\mu = 0$.

In case $\mathfrak{A}$ is complex the proof is the same with the exception that $N(\gamma_0)$ is defined by

$$N(\gamma_0) = [\gamma \mid | x_i(\gamma) |^2 < \epsilon, \ i = 1, 2, \cdots, n].$$

The proof for the case of Γ locally compact and not compact can be reduced to the case of Γ compact by compactification with a single point and addition of the constants to $\mathfrak{A}$, with removal of the point and the constants after the result just obtained has been applied.

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SPACES CONGRUENT WITH BOUNDED SUBSETS OF THE LINE

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This note is devoted to a characterization of bounded subsets of the line from the viewpoint of metric geometry. Such spaces have been characterized by Menger and others,[1] usually from the consideration of conditions on imbeddability of finite subsets. The theorem given below differs from these in making use of a minimality property of the linear metric as the condition for congruence.

THEOREM. *Let S be a metric space with distance function $d(xy)$ and diameter t, possibly infinite. A necessary and sufficient condition that S be congruent with a bounded subset of the real line is that if $d'(xy)$ is another metric for S which is topologically equivalent*[2] *to $d(xy)$ and in which S has diameter t, then there are two points a and b of S such that $d'(ab) > d(ab)$.*

PROOF. The condition is necessary. For suppose that S is a bounded subset of the real line with diameter t, and that there exists a metric $d'(xy)$ for S such that (1) for some two points a and b, $d'(ab) < d(ab)$, $d(xy)$ denoting the Euclidean metric for S; (2) for each two points x and y, $d'(xy) \leqq d(xy)$; and (3) there exist sequences $\{x_n\}$, $\{y_n\}$, of points such that $d'(x_ny_n)$ approaches t as n increases. From (2), $\lim d(x_ny_n)$ is also t. There is no loss of generality in assuming that

$$\begin{aligned} d(x_ny_n) &= d(x_na) + d(ab) + d(by_n) \\ &= d(x_na) + d'(ab) + d(by_n) + [d(ab) - d'(ab)] \\ &\geqq d'(x_na) + d'(ab) + d'(by_n) + [d(ab) - d'(ab)] \\ &\geqq d'(x_ny_n) + [d(ab) - d'(ab)]. \end{aligned}$$

By taking the limit of both sides of this last inequality, we get an immediate contradiction of (1).

The condition is sufficient. There exist sequences $\{x_n'\}$, $\{y_n'\}$, of points of S such that $\lim d(x_n'y_n') = t$. Let $\{x_n\}$ be any subsequence of $\{x_n'\}$. For each two points x and y let

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[1] For definitions and bibliography, see L. M. Blumenthal, *Distance geometries*, University of Missouri Studies, 1938. This theorem had its beginning in discussions of some problems in measure with Dr. Dorothy Maharam.

[2] That is, in order that $d(x_nx)$ approach zero, it is necessary and sufficient that $d'(x_nx)$ approach zero.

$$(4)\qquad h(xy) = \sup \left| d(xx_n) - d(yx_n) \right|.$$

By the triangular inequality, for each n both $d(xx_n) - d(yx_n)$ and $d(yx_n) - d(xx_n)$ are less than $d(xy)$, so that (5) $h(xy) \leqq d(xy)$. Clearly $h(xx) = 0$ and $h(xy) = h(yx)$. Since, for each n,

$$\left| d(xx_n) - d(zx_n) \right| \leqq \left| d(xx_n) - d(yx_n) \right| + \left| d(yx_n) - d(zx_n) \right|,$$

we have

$$(6)\qquad h(xz) \leqq h(xy) + h(yz).$$

Now, let $\{y_n\}$ be the subsequence of $\{y_n'\}$ corresponding to $\{x_n\}$. By (5), $h(xy) \leqq t$; on the other hand, m being fixed, and n varying,

$$\sup \left| d(x_m x_n) - d(y_m x_n) \right| \geqq \left| d(x_m x_m) - d(x_m y_m) \right| = d(x_m y_m),$$

so that (7) $\sup h(xy) = t$. Finally, from (5) we have that (8) a necessary condition that a sequence $\{a_n\}$ converge to a point a is that $h(a_n a)$ approach zero.

It is not in general true that $h(xy)$ vanishes only if $x = y$, so that in general h is not a metric. I shall, however, show that under our hypotheses it is.

Let $d'(xy) = (1/2)\left[d(xy) + h(xy)\right]$. Using the properties (4)–(8) of h and the corresponding properties for d, it is easily seen that d' is a metric equivalent to the metric d, that $d'(xy) \leqq d(xy)$, and that $\sup d'(xy) = t$. It follows from the condition that $h(xy) = d(xy)$, and hence that $h(xy)$ is independent of the particular subsequence of $\{x_n'\}$ chosen.

This last statement implies that for each two points x and y

$$(9)\qquad h(xy) = \lim \left| d(xx_n') - d(yx_n') \right| = d(xy).$$

For if this limit does not exist it is possible to find two subsequences $\{x_{1n}\}$, $\{x_{2n}\}$, of $\{x_n'\}$ such that for each n

$$(10)\qquad \left| d(xx_{1n}) - d(yx_{1n}) \right| < \left| d(xx_{2n}) - d(yx_{2n}) \right|$$

and such that the upper limit of the left-hand side of (10) is less than the lower limit of the right-hand side. This implies that $h(xy)$ as defined by one of these sequences is less than $h(xy)$ as defined by the other. By considering the subsequences $\{x_n'; n > j\}$ for $j = 1, 2, 3, \cdots$, it is easily seen that (9) holds.

Now let a, b, c be any three points of S. By changing the lettering and taking a subsequence $\{x_n\}$ of $\{x_n'\}$, we can assume that, for each n,

$$d(ax_n) - d(bx_n) \geqq 0; \qquad d(bx_n) - d(cx_n) \geqq 0. \tag{11}$$

Adding the two members of (11) gives

$$\left| d(ax_n) - d(cx_n) \right| = \left| d(ax_n) - d(bx_n) \right| + \left| d(bx_n) - d(cx_n) \right|, \tag{12}$$

and letting n increase in (12), we have by (9),

$$d(ac) = d(ab) + d(bc). \tag{13}$$

Hence of any three points, one is between the other two, in the metric sense. By a well known result of Menger,[3] if S has more than four points it is congruent with a subset, S', of the line. If S consists of exactly four points and is not a subset of the line, it is easy to reduce the distance between two of the points while leaving the diameter unchanged.

All that need be shown now is that S' is bounded. Suppose that it is not. We may assume that the origin belongs to S'. Then the transformation sending the point of S' with abscissa x into the point with abscissa $x/2$ reduces each distance while leaving the diameter infinite, which contradicts our condition, and completes our proof.

Various special sets can be characterized by adding additional hypotheses to the condition. For example, an interval can be characterized as, say, a connected space in which the diameter is actually assumed by the distance between some two points, and which satisfies the condition of the theorem.

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[3] Karl Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. vol. 100 (1928) pp. 75–163. See in particular p. 128.

SYMBOLIC SOLUTION OF CARD MATCHING PROBLEMS

N. S. MENDELSOHN

The main problem to be discussed here is the following. Find the number of arrangements of n cards marked 1, 2, $\cdots$, n subject to conditions of the type: the card marked "i" shall not be jth, the card marked "k" shall not be rth, and so on. A generalization of this problem is also discussed.

A solution of the card matching problem has been given by Kaplansky in [2].[1] The present solution depends on a somewhat different approach to the problem. Both Kaplansky and I make use of the finite difference operator E, defined by $Ef(n)=f(n+1)$: Kaplansky's solution is based on a symbolic interpretation of the method of inclusion and exclusion; my solution gives a recurrence formula expressing the solution of the problem of matching n cards in terms of the solution of the problems of matching less than n cards. The solution proposed here is capable of giving explicit formulae for several particular cases, for example, the "problème des ménages." Furthermore, it is capable of being extended to problems of considerably greater generality.

Suppose we have $a_1, a_2, \cdots, a_n$ cards, all considered distinct, of which a_r are marked r. It is required to find the number of arrangements of these cards in which none of the cards marked "r" appear in any of p_r specified places. As an immediate corollary, we also obtain the number of arrangements in which these conditions are violated (1) exactly s times and (2) at most s times.

Let p_{rs} be the number of places simultaneously forbidden to cards marked r or s; p_{rst} the number of places simultaneously forbidden to cards marked r, s or t, and so on. The form our solution takes depends on the $p_{rst} \ldots$ with the largest number of subscripts which does not vanish. We give the following examples.

Case I. All $p_i=0$. The number of suitable arrangements is $E^{a_1+\cdots+a_n}$. This is obvious.

Case II. Some $p_i\neq 0$, but all $p_{ij}=0$. The number of suitable arrangements is $F_1(a_1;\ p_1)F_1(a_2;\ p_2)\cdots F_1(a_n;\ p_n)0!$ where $F_1(a;\ p)=\sum(-1)^r[a,r][p,r]E^{a-r}$, the summation being carried out with respect to r which ranges from 0 to min (a, p). The symbol $[a, r]$ is used

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[1] Numbers in brackets refer to the references cited at the end of the paper.

for the binomial coefficient $a(a-1)\cdots(a-r+1)/r!$. This problem was solved by Kaplansky in [1].

Case III. Some $p_{ij}\neq 0$, all $p_{ijk}=0$. The number of suitable arrangement is $F_n(a_1,\cdots,a_n;p_1,\cdots,p_n;p_{12},\cdots,p_{1n},p_{23},\cdots,p_{n-1,n})0!$ (written as $F_n(a_i;p_i;p_{ij})0!$ if no ambiguity occurs). Here $F_1(a;p)$ is the same as defined in case II while $F_n(a_i;p_i;p_{ij})$ is defined inductively by the recurrence

$$F_n(a_i;p_i;p_{ij}) = \sum(-1)^{k_1+\cdots+k_{n-1}}k_1!\cdots k_{n-1}![a_1,k_1][p_{1n},k_1]\cdots[a_{n-1},k_{n-1}][p_{n-1,n},k_{n-1}]BC$$

where the summation is carried out with respect to each of the indices $k_1,k_2,\cdots,k_{n-1}$, the index k_r ranging from 0 to min (a_r,p_{rn}). Here B and C represent the following expressions: $B=F_{n-1}(a_r-k_r;p_r-p_{rn};p_{rs})$ where r and s range from 1 to $n-1$; $C=F_1(a_n;p_n-\{k_1+\cdots+k_{n-1}\})$.

Case IV. Some $p_{ijk}\neq 0$, all $p_{ijkl}=0$. The number of suitable arrangements is $F_n(a_i;p_i;p_{ij};p_{ijk})0!$ $(i,j,k=1,2,\cdots,n;i<j<k)$ where $F_n(a_i;p_i;p_{ij};p_{ijk})=\sum(-1)^{k_1+\cdots+k_{n-2}}k_1!\cdots k_{n-2}![a_1,k_1]\cdot[p_{1,n-1,n},k_1]\cdots[a_{n-2},k_{n-2}][p_{n-2,n-1,n},k_{n-2}]F_n^*$, where the summation is carried out with respect to each of the indices $k_1,k_2,\cdots,k_{n-2}$. Here k_r ranges from 0 to min $(a_r,p_{r,n-1,n})$ and $F_n^*=F_n(a_i^*;p_i^*;p_{ij}^*;p_{ijk}^*)$, where:

$$a_i^* = a_i - k_i \qquad (i=1,2,\cdots,n-2),$$

$$a_{n-1}^* = a_{n-1}, \qquad a_n^* = a_n,$$

$$p_i^* = p_i - p_{i,n-1,n} \qquad (i=1,2,\cdots,n-2),$$

$$p_{n-1}^* = p_{n-1} - (k_1+k_2+\cdots+k_{n-2}),$$

$$p_n^* = p_n - (k_1+k_2+\cdots+k_{n-2}),$$

$$p_{ij}^* = p_{ij} \qquad (i,j=1,2,\cdots,n-2;j>i),$$

$$p_{i,n-1}^* = p_{i,n-1} - p_{i,n-1,n} \qquad (i=1,2,\cdots,n-2),$$

$$p_{in}^* = p_{in} - p_{i,n-1,n} \qquad (i=1,2,\cdots,n-2),$$

$$p_{n-1,n}^* = p_{n-1,n} - (k_1+k_2+\cdots+k_{n-2}),$$

$$p^*_{ijk} = p_{ijk} \qquad \text{unless } j = n-1,\ k = n,$$

$$p^*_{i,n-1,n} = 0.$$

Note that this gives a reduction formula in which $p_{i,n-1,n}$ are reduced to 0. By permuting the a_i, repeated applications of the above formula reduce all the p_{ijk} to 0, after which the formulas of case III apply.

The general case can now be written down by analogy.

If we wish to find the number of arrangements in which the conditions are violated (1) exactly s times and (2) at most s times, we replace $0!$ by $(-1)^s[m-0,\ s]0!$ and by $(-1)^s[m-1-0,\ s]0!$ respectively, where $m=\sum a_i$ (see [2] and [4]).

We give a proof for the case III; the extension of the proof to the other cases is obvious.

In the formulas as they stand certain conventions must be made for the cases where some a_i or p_i reduce to 0. These are:

$$F_n(0, 0, \cdots, 0; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n}) = 1,$$

$$F_n(0, a_2, \cdots, a_n; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n})$$
$$= F_{n-1}(a_2, \cdots, a_n; p_2, \cdots, p_n; p_{23}, \cdots, p_{n-1,n}),$$

$$F_n(a_1, \cdots, a_n; 0, p_2, \cdots, p_n; p_{12}, \cdots, p_{n-1,n})$$
$$= E^{a_1}F_{n-1}(a_2, \cdots, a_n; p_2, \cdots, p_n; p_{23}, \cdots, p_{n-1,n}).$$

With these conventions the proof of the theorem is obtained by induction on the sum $\sum p_{ij}$. If $\sum p_{ij}=0$, the formula quoted for F_n in case III becomes

$$F_n(a_1, \cdots, a_n; p_1, \cdots, p_n; 0, \cdots, 0)$$
$$= F_{n-1}(a_1, \cdots, a_{n-1}; p_1, \cdots, p_{n-1}; 0, \cdots, 0)F_1(a_n; p_n).$$

Hence by induction,

$$F_n(a_1, \cdots, a_n; p_1, \cdots, p_n; 0, \cdots, 0) = \prod F_1(a_i; p_i),$$

the product ranging over all terms $i=1, 2, \cdots, n$.

This is precisely the formula quoted in case II and was given by Kaplansky in [1]. We write the number of suitable arrangements in the form

$$F_n(a_1, \cdots, a_n; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n})0!.$$

Suppose now that $p_{n-1,n}\neq 0$. (This can be brought about by a change of notation if necessary, since we are now considering the case where not all $p_{ij}=0$.) Consider one of the places which is forbidden simultaneously to cards marked $(n-1)$ and to cards marked n. If we lift the prohibition on the cards marked $(n-1)$ to this place, the number of suitable arrangements is

$$F_n(a_1, \cdots, a_{n-1}, a_n; p_1, \cdots, p_{n-1}-1, p_n; p_{12}, \cdots, p_{n-1,n}-1)0!.$$

Of these arrangements,

$$\begin{aligned} a_{n-1}F_n(a_1, \cdots, a_{n-1}-1, a_n; p_1, \cdots, p_{n-1}-1, p_n-1; \\ p_{12}, \cdots, p_{n-1,n}-1)0! \end{aligned}$$

have a card marked $(n-1)$ in the specified place. Thus

$$\begin{aligned} &F_n(a_1, \cdots, a_n; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n}) \\ &\quad = F_n(a_1, \cdots, a_n; p_1, \cdots, p_{n-1}-1, p_n; p_{12}, \cdots, p_{n-1,n}-1) \\ &\qquad - a_{n-1}F_n(a_1, \cdots, a_{n-1}-1, a_n; p_1, \cdots, p_{n-1}-1, p_n-1; \\ &\qquad\qquad p_{12}, \cdots, p_{n-1,n}-1). \end{aligned}$$

The induction is established by substituting the values in the right-hand side of this equation into the recurrence stated for F_n.

For computing purposes, the following remark is useful. If $p_{ij}=0$ $(i=1, 2, \cdots, r; j=r+1, r+2, \cdots, n)$, then

$$\begin{aligned} &F_n(a_1, \cdots, a_r, a_{r+1}, \cdots, a_n; p_1, \cdots, p_r, p_{r+1}, \cdots, p_n; p_{12}, \cdots, p_{n-1,n}) \\ &\quad = F_r(a_1, \cdots, a_r; p_1, \cdots, p_r; p_{12}, \cdots, p_{r-1,r}) \\ &\qquad \cdot F_{n-r}(a_{r+1}, \cdots, a_n; p_{r+1}, \cdots, p_n; p_{r+1,r+2}, \cdots, p_{n-1,n}). \end{aligned}$$

This factorization holds in the most general case and is obvious from first principles, or it can be established by induction.

We now give an example illustrating the use of the above formulae. In this example, the condition

$$\sum a_i \geqq \sum p_i - \sum p_{ij} + \sum p_{ijk} \cdots,$$

which is necessary if the problem as stated is to have a meaning, is not satisfied. The result obtained is still useful for two reasons: (1) the problem may be made meaningful by embedding it in a larger problem, for example, by adding $\sum p_i - \sum p_{ij} + \sum p_{ijk} - \cdots - \sum a_{ij}$ blank cards with no restriction as to where these cards are to be put; (2) the coefficient of E^{n-k} in the polynomial $f(E)$ gives the number of ways of choosing k compatible conditions from the converse conditions and this is always meaningful. (The converse condition to "i is not jth" is defined as the condition "i is jth."

Example. Let $a_i=1$; $p_i=p$ $(i=1, 2, \cdots, n)$; $p_{12}=p_{23}= \cdots =p_{n-1,n}=r$; all remaining $p_{ij}=0$; all $p_{ijk}=0$. These conditions imply $r \leqq p/2$. Let the polynomial operator be U_n. By applying the recurrence formula of case III twice one obtains

$$U_n = (E - p)U_{n-1} - rU_{n-2}.$$

Also, $U_1 = E - p$; $U_2 = E^2 - 2pE + (p^2 - r)$. By induction it is easily shown that

$$\begin{aligned} U_n = \sum_{k=0}^{n} (-1)^k \{ [n, 0][n, k]p^k + \cdots \\ + (-1)^s[n - s, s][n - 2s, k - 2s]p^{k-2s}r^s + \cdots \} E^{n-k}. \end{aligned}$$

In the particular case where $p = 2$, $r = 1$, the solution can be reduced to the simpler form

$$U_n = \sum_{k=0}^{n} (-1)^k [2n - k - 1, k]E^{n-k}.$$

This polynomial is useful in the solution of the "problème des ménages"; cf. [3].

Generalization to the matching of several decks of cards. Battin [5] and Wilks [6] have discussed the following problem. Suppose there are k decks of cards of several suits $C_1, C_2, \cdots, C_r$, the decks not necessarily having the same number of cards in each suit, nor need the decks be of identical composition. One card at a time is dealt from each deck, and if all the cards at a drawing are of the same suit a k-ple match is said to occur. Amongst the questions of interest are: (1) the probability of h *k-ple* matches (simply called matches from here on), and (2) the probability of at most h matches.

A case which is very easily treated by symbolic methods is that of $(s+1)$ decks of cards, s of which are identical, the other deck being different. We also assume that this odd deck has less cards than the other decks. Let the identical decks have a_1 cards of suit 1, a_2 cards of suit 2, $\cdots$, a_n cards of suit n and let $a_1 + a_2 + \cdots + a_n = N$. Let the remaining deck have p_i cards of suit i $(i = 1, 2, \cdots, n)$, where $\sum p_i \leqq \sum a_i = N$. We can show that the probability of no matches is given by

$$P(0) = \frac{1}{(N!)^s} \left\{ \prod_{i=1}^{n} \Phi_1^{(s)}(a_i; p_i) \right\} (0!)^s,$$

the probability of exactly h matches is given by

$$P(h) = \frac{1}{(N!)^s} \left\{ \prod_{i=1}^{n} \Phi_1^{(s)}(a_i; p_i) \right\} (-1)^h [N - 0, h](0!)^s,$$

while the probability $Q(h)$ of at most h matches is given by

$$Q(k) = \frac{1}{(N!)^s}\left\{\prod_{i=1}^{n} \Phi_1^{(s)}(a_i; p_i)\right\}(-1)^k[N-0-1, k](0!)^s,$$

where $\Phi_1^{(s)}(a; p) = \sum(-1)^i(i!)^s[a, i]^s[p, i]E^{p-i}$, the summation being with respect to i which ranges from 0 to min (a, p). The proof of this can be obtained by induction on observing that $\Phi_1^{(s)}(a; p)$ satisfies the recurrence

$$\Phi_1^{(s)}(a; p) = \Phi_1^{(s)}(a; p-1) - a^s\Phi_1^{(s)}(a-1, p-1).$$

Note that the case $s=1$ (that is, the general two deck case) has first been given by Kaplansky and Fréchet [1 and 4], although a solution by Greville [7] is usually quoted as the first obtained.

We can even generalize the card matching problem as follows.

Suppose that there are s identical packs of cards, each pack having a_i cards, considered as distinct, marked i $(i=1, 2, \cdots, n)$. Suppose that the packs be arranged in a rectangular array, each row of the array consisting of all the cards of one pack. Suppose that to cards marked i, we prohibit p_i columns in the sense that a column forbidden to cards marked i must not contain s cards marked i (although less than s cards marked i is permissible). It is required to find the number of suitable arrangements.

Let p_{ij} be the number of columns forbidden simultaneously to cards marked i and j. Similarly p_{ijk}, p_{ijkl}, and so on, are defined. We consider only the analogue of case III of our original problem, extension to other cases being obvious. Let $p_{ijk}=0$ for all i, j, k, but at least one $p_{ij}\neq 0$. The number of suitable arrangements is

$$\Phi_n^{(s)}(a_1, \cdots, a_n; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n})(0!)^s,$$

where $\Phi_n^{(s)}$ satisfies the following recurrence: $\Phi_1^{(s)}(a, p)$ is identical with previously given definition;

$$\begin{aligned}\Phi_n^{(s)}&(a_1, \cdots, a_n; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n})\\ &= \sum(-1)^{k_1+\cdots+k_{n-1}}(k_1! \cdots k_{n-1}!)^s[a_1, k_1]^s[p_{1n}, k_1]\\ &\qquad\cdots[a_{n-1}, k_{n-1}]^s[p_{n-1,n}, k_{n-1}]BC,\end{aligned}$$

where the summation is carried out with respect to each of the indices $k_1, k_2, \cdots, k_{n-1}$, the index k_r ranging from 0 to min (a_r, p_{rn}). Here B and C represent the following expressions:

$$B = \Phi_{n-1}^{(s)}(a_r - k_r; p_r - p_{rn}; p_{rs}),$$

where r and s range from 1 to $n-1$; $C = \Phi_1^{(s)}(a_n; p_n - \{k_1+k_2+\cdots+k_{n-1}\})$.

The proof of this formula is by induction on $\sum p_{ij}$, where we make use of the relationship $\Phi_n^{(s)}(a_1, \cdots, a_n; p_1, \cdots, p_n; p_{12}, \cdots, p_{n-1,n}) = \Phi_n^{(s)}(a_1, \cdots, a_n; p_1, \cdots, p_{n-1}-1, p_n; p_{12}, \cdots, p_{n-1,n}-1) - a_{n-1}^s \Phi_n^{(s)}(a_1, \cdots, a_{n-1}-1, a_n; p_1, \cdots, p_{n-1}-1, p_n-1; p_{12}, \cdots, p_{n-1,n}-1)$.

The probability of (1) exactly k violations of the conditions and (2) at most k violations of the conditions can be written down in the usual way. If all the $a_i = 1$, then $\Phi_n^{(s)} = F_n$. It follows that if $f(E)0!$ is a solution of a one-pack problem then $f(E)(0!)^s$ is the solution of an s-pack problem provided $a_i = 1$.

For example, the problème des ménages asks for the number of ways in which n cards marked $1, 2, \cdots, n$, respectively, can be arranged so that the card marked 1 is not first or second, the card marked 2 is not second or third . . . the card marked n is not nth or first. The solution is $M_n(E)0!$ where

$$M_n(E) = \sum_{k=0}^{n} (-1)^k \left(\frac{2n}{2n-k}\right) [2n-k, k] E^{n-k}.$$

Since all the $a_i = 1$, the solution to a s-pack problème des ménages, in the sense of our theorem, is $M_n(E)(0!)^s$.

An alternative approach to the multiple matching problems, using symbolic methods, has recently been given by Kaplansky and Riordan in [8].

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SET PROPERTIES DETERMINED BY CONDITIONS ON LINEAR SECTIONS

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Let $\mathfrak{R}_n (n \geqq 2)$ be an n-dimensional Euclidean space, and let S be any set of points in $\mathfrak{R}_n$. There exist a number of instances in which the following question has an interesting answer. Suppose a property A holds on each $(n-1)$-dimensional linear section S_{n-1}^{i} of S. What additional property B assumed to hold on each section S_{n-1}^{i} will insure that property A holds on S?

The following terminology is used. A continuum is a compact connected set which may include the degenerate case of a single point. ? Also compactness includes closure. A generalized continuum is a set which is connected and closed. An $(n-r)$-dimensional *linear section* of a set S with an $(n-r)$-dimensional Euclidean hyperplane L_{n-r} is defined to be the set $S \cdot L_{n-r}$. A subscript will *always* designate the dimensionality of the set.

1. Theorems on closed, open and bounded sets.

The following theorem illustrates the theory, and plays an important role in a succeeding theorem. It is a case in which condition B is sufficient but not necessary. We shall always assume $n \geqq 2$.

THEOREM 1. *Let S be any set in $\mathfrak{R}_n$ $(n \geqq 2)$. If each $(n-1)$-dimensional linear section of S is connected and closed, then S is closed.*

PROOF. Suppose S is *not* closed. Then there exists a point $p \notin S$ which is a limit point of S. Let L_{n-1} be an $(n-1)$-dimensional hyperplane containing p, such that $S \cdot L_{n-1} \neq 0$. Since, by hypothesis, $S_{n-1} = S \cdot L_{n-1}$ is closed, there exists an $(n-1)$-dimensional closed cube $C_{n-1} \subset L_{n-1}$, which contains p in its interior, and for which $C_{n-1} \cdot S_{n-1} = 0$. Let P_n be an n-dimensional hyperprism passing through C_{n-1}, and perpendicular to L_{n-1}. Since p is a limit point of S which is not in S, and since S_{n-1} is closed, there exists a sequence of points $p^i \in S \cdot P_n$, such that $p^i \notin L_{n-1}$, and such that $p^i \rightarrow p$ as $i \rightarrow \infty$. Let L_{n-2} be any $(n-2)$-dimensional hyperplane contained in L_{n-1} such that $S \cdot L_{n-2} \neq 0$, and such that $L_{n-2} \cdot C_{n-1} = 0$. Then there exists a sequence of hyperplanes L_{n-1}^{i} determined by L_{n-2} and p^i. By hypothesis each set $S \cdot L_{n-1}^{i}$ is connected. Hence since $p^i \in S \cdot L_{n-1}^{i} \cdot P_n$, and since any point $q \in S \cdot L_{n-2} \cdot L_{n-1}^{i}$ is not in P_n, the connectedness of $S \cdot L_{n-1}^{i}$ im-

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plies that p^i and q can be joined by a connected subset of $S\cdot L_{n-1}^i$ which intersects the boundary of the prism, $B(P_n)$. Hence let $r^i\in S\cdot L_{n-1}^i\cdot B(P_n)$. Since the set $\{r^i\}$ is infinite, and since the prism P_n has a finite number of $(n-1)$-dimensional plane faces, there exist an infinite subset of $\{r^i\}$, namely $\{r^{i_j}\}$, which lie on *one* face of P_n. Designate this face by F_{n-1}^*, so that $r^{i_j}\in F_{n-1}^*\cdot L_{n-1}^{i_j}$. Furthermore since $p^{i_j}\rightarrow p$, as $i_j\rightarrow\infty$, the set $\{r^{i_j}\}$ lies on a bounded portion of F_{n-1}^*. Hence since $L_{n-1}^{i_j}\rightarrow L_{n-1}$ as $i_j\rightarrow\infty$, the set $\{r^{i_j}\}$ has a limit point r existing in L_{n-1}. Since $r^{i_j}\in F_{n-1}^*\cdot S$, and since by hypothesis $S\cdot F_{n-1}^*$ is closed, $r\in S\cdot F_{n-1}^*$. Hence $r\in S$. But $r\in F_{n-1}^*\cdot L_{n-1}\subset C_{n-1}$, which is a contradiction, since by construction $C_{n-1}\cdot S_{n-1}=0$. Thus the indirect proof is completed, and Theorem 1 is proved.

COROLLARY 1.1. *If each two-dimensional plane section of S is connected and closed, then S is closed.*

COROLLARY 1.2. *If each $(n-1)$-dimensional linear section of S is a generalized continuum, then S is a generalized continuum.*

In Corollary 1.2, the connectedness of S is well known [6, p. 64].[1] This second corollary is an illustration where no additional hypotheses B are needed on linear sections in order to guarantee property A on S.

THEOREM 2. *Let S be any set in $\mathfrak{R}_n$ $(n\geqq 2)$. Suppose that relative to each $(n-1)$-dimensional hyperplane L_{n-1}, the set $S\cdot L_{n-1}$ is an open one with a connected complement. Then S is open.*

PROOF. Let S_{n-1}^i be any linear section determined by L_{n-1}^i. Since S_{n-1}^i is open in L_{n-1}^i, the complement $C(S_{n-1}^i)$ is closed in L_{n-1}^i. Since each linear set $C(S_{n-1}^i)$ is then connected and closed, Theorem 1 implies that $C(S)$ is closed. Hence S is open.

COROLLARY 2.1. *Let S be any set in $\mathfrak{R}_n$ $(n\geqq 2)$. If relative to each two-dimensional plane L_2^i, the set $S\cdot L_2^i$ is an open one with a connected complement, then S is open.*

The following theorem is one in which boundedness is the principal property to be established. Here again connectedness plays an important role.

THEOREM 3. *Let S be any set in $\mathfrak{R}_n$ $(n\geqq 2)$. If each $(n-1)$-dimensional linear section of S is bounded and connected, then S is bounded and connected.*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

PROOF. Choose $p \in S$, and let L_{n-2} be a hyperplane containing the point p. Consider the family of hyperplanes L_{n-1}^{α} passing through L_{n-2}. Let $\mathfrak{S}_n^m$ ($m = 1, 2, \cdots$) be a set of spheres with centers at p of radii m.

Suppose that S is unbounded. Then there exists a sequence of points $x^i \in S$ ($i = 1, 2, \cdots$) such that the distance $\delta(p, x^i) \rightarrow \infty$ as $i \rightarrow \infty$. Let L_{n-1}^i designate the member of L_{n-1}^{α} for which $x^i \in L_{n-1}^i$. Since $p \in S \cdot L_{n-1}^i \cdot \mathfrak{S}_n^m$, and since, for any *fixed* value of m, $x^i \in S \cdot L_{n-1}^i - \mathfrak{S}_n^m$ (for sufficiently large values of i), the connectedness of $S \cdot L_{n-1}^i$ implies that p and x^i can be joined by a connected subset of $S \cdot L_{n-1}^i$ which intersects the boundary of $\mathfrak{S}_n^m$, $B(\mathfrak{S}_n^m)$. Choose $y^{m,i} \in S \cdot L_{n-1}^i \cdot B(\mathfrak{S}_n^m)$. Since $B(\mathfrak{S}_n^m)$ is compact, there exists a convergent subsequence $\{y^{m,i_j}\}$ which converges to a point $y^m \in \mathfrak{S}_n^m$, such that for the corresponding points x^{i_j}, $\delta(p, x^{i_j}) \rightarrow \infty$ as $i_j \rightarrow \infty$. Without loss of generality designate L_{n-1}^0 to be the member of L_{n-1}^{α} such that $y^m \in L_{n-1}^0$. There exists an integer N such that when $i_j > N$, $x^{i_j} \in S \cdot L_{n-1}^{i_j} - \mathfrak{S}_n^{m+1}$, and such that $\delta(p, x^{i_j}) \rightarrow \infty$ as $i_j \rightarrow \infty$. Hence by the connectedness of $S \cdot L_{n-1}^{i_j}$, there exist points $y^{m+1,i_j} \in S \cdot L_{n-1}^{i_j} \cdot B(\mathfrak{S}_n^{m+1})$. Since $y^{m,i_j} \rightarrow y^m$, and since $L_{n-1}^{i_j} \rightarrow L_{n-1}^0$ as $i_j \rightarrow \infty$, there exists a convergent subsequence of $\{y^{m+1,i_j}\}$ which converges to a point $y^{m+1} \in \overline{S} \cdot L_{n-1}^0 \cdot \mathfrak{S}_n^{m+1}$. Since the radius of $\mathfrak{S}_n^m$ is m, by induction it follows there exists a sequence $y^m \in \overline{S}\ L_{n-1}^0$, such that $\delta(p, y^m) \rightarrow \infty$ as $m \rightarrow \infty$.

If $S \subset L_{n-1}^0$, Theorem 3 is obviously true. Hence, suppose $q \in S - L_{n-1}^0$. Since L_{n-1}^0 divides $\mathfrak{R}_n$ into two half-spaces, namely $\mathfrak{R}_n^+$ and $\mathfrak{R}_n^-$, suppose without loss of generality that $q \in \mathfrak{R}_n^+$. Choose a hyperplane $L_{n-1}^+ \subset \mathfrak{R}_n^+$ so that L_{n-1}^+ is *parallel* to L_{n-1}^0, and *such that q is not on* L_{n-1}^+ *or between* L_{n-1}^+ *and* L_{n-1}^0. Since $y^m \in \overline{S} \cdot L_{n-1}^0$ there exists in any neighborhood of y^m a point $p^m \in S$, such that p^m and q are on *opposite sides* of L_{n-1}^+. Join p^m and q by a line L_1^*. Hence $L_1^* \cdot L_{n-1}^+ = r^m$ exists. Let $L_{n-2}^* \subset L_{n-1}^+$ be a hyperplane such that $r^m \in L_{n-2}^*$, and such that L_{n-2}^* is perpendicular to L_1^*. The line L_1^* and the subspace L_{n-2}^* determine a hyperplane L_{n-1}^*. Since $S \cdot L_{n-1}^*$ is connected, and since q and p^m lie on opposite sides of L_{n-2}^* in L_{n-1}^*, we have $S \cdot L_{n-2}^* \neq 0$. Let $s^m \in S \cdot L_{n-2}^*$. Since $L_{n-2}^* \subset L_{n-1}^+$, then $s^m \in L_{n-1}^+$. Since by construction $\delta(q, y^m) \rightarrow \infty$ as $m \rightarrow \infty$, p^m can be chosen so that $\delta(q, p^m) \rightarrow \infty$ as $m \rightarrow \infty$. Since as $p^m \rightarrow \infty$ the line L_1^* approaches parallelism to L_{n-1}^+, $\delta(q, r^m) \rightarrow \infty$ as $p^m \rightarrow \infty$. Since L_{n-2}^* is perpendicular to L_1^*, $\delta(q, s^m) \geqq \delta(q, r^m)$. Hence we have $\delta(q, s^m) \rightarrow \infty$ as $m \rightarrow \infty$. Since $s^m \in L_{n-2}^* \subset L_{n-1}^+$ for all m, the set $S \cdot L_{n-1}^+$ is *unbounded*. This is a contradiction of hypothesis. Thus S

is bounded. Since the connectedness of S is well known, Theorem 3 is proved.

2. **A characterization of star-like sets.** Aumann [1] has characterized compact convex sets by means of properties on linear sections. Also Liberman [4] has made another characterization by placing properties on the set itself and also on its supporting planes. The following theorem, while restricted to two-dimensional sections, yields, as far as it goes, a generalization of Aumann's result, for convexity is replaced by the weaker concept of star-likeness, and boundedness of the set is removed. Note that in Theorem 5 *no* hypotheses are placed on the set S itself. The following definition is a standard one. Refer to Brunn [2].

DEFINITION. *A set S is star-like with respect to a point $a\in S$ if each straight line through a intersects S in a connected set.*

In order to characterize star-like sets by linear sections the following definition of simply-connectedness in the plane is especially useful.

DEFINITION. *A connected plane set U is simply connected if each component of the complement of U is unbounded.*

THEOREM 4. *A closed set S in $\mathcal{R}_n$ $(n\geqq 3)$ is star-like with respect to a point $a\in S$ if and only if the following conditions hold.*

(1) *Each two-dimensional linear section of S through the point a is a simply connected, generalized continuum.*

(2) *For each point $q\in S$, there exists a constant $M>0$, such that each two-dimensional linear section containing a and q contains a continuum joining a and q of diameter less than M.*

The necessity is immediate. In particular for condition (2) note that M can be any number greater than the distance $\delta(a, q)$.

SUFFICIENCY PROOF. Suppose S is *not* a star with respect to the point a. Then since S is closed, there exist distinct points $b\in S$, $c\in S$, such that $\delta(a, c)=\delta(a, b)+\delta(b, c)$, and such that the open line segment L_1 between b and c is not in S. Consider any three-dimensional hyperplane L_3 such that $L_1\subset L_3$. Choose a coordinate system (x, y, z) in L_3 so that L_1 is contained in the x-axis. Let $L_2^{\theta+}\subset L_3$ be an open half-plane with the x-axis as an axis, whose directed normal makes a directed angle θ with the positive z-axis. Also suppose that $0\leqq\theta\leqq\pi$. Let L_2^θ be the plane containing $L_2^{\theta+}$, and define $L_2^{\theta-}\equiv L_2^\theta-\bar{L}_2^{\theta+}$.

Designate the component of the complement of $S_2^\theta\equiv S\cdot L_2^\theta$ which contains L_1 by C_2^θ. Since S_2^θ is a generalized continuum, the boundary of C_2^θ is a connected set [6, p. 117]. By a theorem in the plane [5, p. 203; 6, p. 108], the set $C_2^\theta-L_1$ is the sum of two mutually exclusive open connected sets $D_2^{\theta+}$ and $D_2^{\theta-}$, and L_1 is a subset of the boundary of each of these sets. The set $D_2^{\theta+}$ corresponds to $L_2^{\theta+}$ in the sense that for any point $r \in L_1$, there exists a circle $R_2 \subset L_2^{\theta}$ with center at r such that $D_2^{\theta+} \cdot R_2 \subset L_2^{\theta+}$ and $D_2^{\theta-} \cdot R_2 \subset L_2^{\theta-}$.

Hypotheses (1) and (2) imply that *one and only one* of the sets $D_2^{\theta+}$ and $D_2^{\theta-}$ is unbounded. Furthermore, the bounded set, say $D_2^{\theta+}$, is of diameter less than M. This is due to the fact that $D_2^{\theta+} \subset Q$, where Q is a set enclosed by the closed line segment (a, c) and by the subcontinuum in S_2^{θ} of diameter less than M which joins a and c. Clearly Q is of diameter less than M, whence $D_2^{\theta+}$ is of diameter less than M, when it is bounded.

Remark. The set of angles $\{\alpha\}$ for which $D_2^{\alpha+}$ is bounded is closed.

To prove this let $L_2^{\alpha_i} \rightarrow L_2^{\alpha}$ as $\alpha_i \rightarrow \alpha$, and suppose $D_2^{\alpha_i+}$ are bounded and that $D_2^{\alpha+}$ is unbounded. Choose points $r \in L_1$, and $s \in D_2^{\alpha+}$ such that the distance $\delta(r, s) > M$. Since $D_2^{\alpha+}$ is arcwise connected, let $A \subset D_2^{\alpha+}$ be a simple arc joining r and s, so that $A \cdot S_2^{\alpha} = 0$. Rotate A rigidly in L_3 about L_1 so that $A^{\alpha_i} \subset L_2^{\alpha_i}$ is a congruent image of A. By virtue of the preceding paragraph, $D_2^{\alpha_i+}$ are all of diameter less than M. Since $A^{\alpha_i} \cdot S_2^{\alpha_i} \neq 0$, since $A^{\alpha_i} \cdot S_2^{\alpha_i}$ are uniformly bounded, and since S is closed, we have $A \cdot S_2^{\alpha} \neq 0$. This is a contradiction; hence the remark holds. In exactly the same way, the set of angles $\{\beta\}$ for which $D_2^{\beta-}$ is bounded is closed. Since the two closed sets $\{\alpha\}$ and $\{\beta\}$ cover the continuum $0 \leqq \theta \leqq \pi$, they have a value in common. Hence there exists a plane L_2^{ϕ}, $0 \leqq \phi \leqq \pi$, such that each $D_2^{\phi-}$ and $D_2^{\phi+}$ is bounded. But in this case C_2^{ϕ} would be bounded, and S_2^{ϕ} would not be simply connected. Hence Theorem 4 is proved.

COROLLARY 4.1. *Let S be a compact set in $\mathfrak{R}_n$ $(n \geqq 3)$. The set S is a star with respect to a point $a \in S$ if and only if condition* (1) *in Theorem* 4 *holds.*

Compactness of S and condition (1) imply condition (2). Hence Corollary 4.1 follows from Theorem 4.

THEOREM 5. *Let S be any set in $\mathfrak{R}_n$ $(n \geqq 3)$. The set S is a closed convex set if and only if conditions* (1) *and* (2) *in Theorem* 4 *hold for all points $a \in S$.*

The necessity is obvious. To prove the sufficiency note that Theorem 1 implies that S is closed. Hence by Theorem 4, S is star-like with respect to *all* points of S. Thus by definition S is convex.

3. **A theorem in linear spaces.** The results of Theorem 3 can be generalized to hold in a normed, linear, metric space $\mathfrak{M}$. A hyperplane

L in $\mathfrak{M}$ is defined to be the set $\{x\}$ which satisfies an equation $f(x)=c$, where $f(x)$ is a linear functional, and where c is a real constant. A linear section of S with L is the set $S\cdot L$.

THEOREM 6. *Let S be any set in a normed linear metric space $\mathfrak{M}$. If each linear section of S is bounded and connected, then S is bounded and connected.*

PROOF. Consider two independent linear functionals $f_1(x)$ and $f_2(x)$ defined on $\mathfrak{M}$. Let T be a transformation of the type

$$T: \qquad \xi_1 = f_1(x), \qquad \xi_2 = f_2(x).$$

This transformation maps S in $\mathfrak{M}$ into a set S_2 in the plane $\mathcal{R}_2$. *Any* linear section $S_2\cdot L_1$ determined by the line $L_1, \alpha\xi_1+\beta\xi_2=\gamma$ corresponds by T to the section $S\cdot L$ where L is defined by $\alpha f_1(x)+\beta f_2(x)=\gamma$. Since T is linear (additive and continuous), and since by hypothesis $S\cdot L$ is connected and bounded, it follows that the linear section $S_2\cdot L_1$ is connected and bounded. Hence by Theorem 3 with $n=2$, the set S_2 is bounded. Thus each functional $f_1(x)$ and $f_2(x)$ is bounded for *all* x in S. Since $f_1(x)$ was an *arbitrary* linear functional, independent of $f_2(x)$, we have shown that *all* linear functionals defined on $\mathfrak{M}$ are bounded on S. Hence by a classical theorem of uniform boundedness [3], the set S is bounded. Since the connectedness is well known, Theorem 6 has been established. It should be noted that in light of Theorem 6 the proof for Theorem 3 need only have been given for $n=2$; however, since the proof for n dimensions was not appreciably longer, an elementary proof independent of the abstract boundedness theorem seemed desirable.

4. **Concluding remarks.** It should be observed that in Theorems 1–3 one cannot delete connectedness entirely, for then the theorems in general are no longer true. Theorem 5 has a preferred form since no hypotheses are placed on S itself. Theorem 4 needs to be formulated so as to hold for $(n-r)$-dimensional sections. This problem is still unsolved. It should be noticed in dealing with non-compact sets that the complement of an unbounded convex set or of an unbounded star need *not* be connected. Hence conditions on the complement *necessary* to yield a characterization take on a different form than those given by Aumann [1]. The author wishes to express his gratitude to his colleagues, Professor R. H. Sorgenfrey, Professor W. T. Puckett, and Professor M. Zorn who have made helpful suggestions.

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UNIVERSITY OF CALIFORNIA AT LOS ANGELES

THE SPACE L^ω AND CONVEX TOPOLOGICAL RINGS

RICHARD ARENS

1. **Introduction.** The motive for investigating the class L^ω of functions belonging to all L^p-classes has no measure-theoretic origin: it was our desire to discover whether or not in every convex metric ring[1] R one could find a system $\{U\}$ of convex neighborhoods of 0 having the property that $f, g \in U$ implies $fg \in U$. We show here that L^ω has no proper convex open set U containing 0 and satisfying the relation $UU \subset U$, thus supplying the desired counter-example.

The significance of neighborhood systems of the type $\{U\}$ described above is made somewhat clearer by a proof that they insure the existence and continuity of entire functions (for example, the exponential function) on the topological ring R.

Such neighborhood systems $\{U\}$ are always present in rings of continuous real-valued functions over any space, provided that convergence means uniform convergence on compact sets.

We also consider the relation of L^∞, L^ω, and the L^p-classes, since L^ω does not seem ever to have been discussed as a topological and algebraic entity.

2. **Notation and elementary facts.** Let us consider measurable functions defined on $[0, 1]$. For $p \geqq 1$ we shall consistently employ the usual notation

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[1] More precisely, metrizable, convex, complete topological linear algebra. For these one requires continuity in both ring operations and scalar multiplication. It will appear that L^ω has these properties.

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{1/p}$$

even when the right side is infinite.

Therefore L^p consists of all functions f for which $\|f\|_p$ is less than ∞.

L^ω evidently consists of all functions f for which $\|f\|_1, \|f\|_2, \cdots, \|f\|_p, \cdots$ are all finite.

Because of the relation[3]

$$\text{(H)} \qquad \|fg\|_p \leqq \|f\|_q \cdot \|g\|_r, \qquad 1/p = 1/q + 1/r,$$

one has

$$\|f\|_1 \leqq \|f\|_2 \leqq \cdots,$$

since the measure of $[0, 1]$ is 1. Therefore we may take the sets of functions f,

$$\|f\|_p < \epsilon$$

where $p \geqq 1$ and $\epsilon > 0$, as neighborhoods of 0 in L^ω. These neighborhoods are convex because

$$\|\lambda f + \mu g\|_p \leqq \lambda\|f\|_p + \mu\|g\|_p < \epsilon$$

when $\lambda, \mu \geqq 0, \lambda + \mu = 1$, and $\|f\|_p, \|g\|_p < \epsilon$. Therefore addition is continuous in L^ω and, by relation (H), multiplication is also.

Multiplication is not generally possible in L^p.

Now the inequalities above imply that the limit

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$$

always exists. (It may be infinite.) Those f's for which $\|f\|_\infty$ is finite form a set usually called L^∞, and $\|f\|_\infty$ is taken as a norm in L^∞. We shall employ the known fact that $\|f\|_\infty$ is also the least number h such that $|f(x)| > h$ only on a set of measure zero.

Multiplication in L^∞ is continuous, since

$$\|fg\|_\infty \leqq \|f\|_\infty \|g\|_\infty,$$

from which it follows that if U is any sphere about 0, contained in the unit sphere of L^∞, then $UU \subset U$.

3. **The relation of L^∞, L^ω, and L^p.** These spaces are related by successive proper inclusion.

THEOREM 1. *$L^\infty \subset L^\omega \subset L^p$ but $L^\infty \neq L^\omega \neq L^p$. The identity mappings*

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[3] Cf. E. J. McShane, *Integration*, Princeton, 1944, for most of the facts which we assume. A formula equivalent to (H) appears on p. 186.

$L^\infty \to L^\omega \to L^p$ are continuous, but their inverses are not. L^∞ is dense in L^ω, and L^ω is dense in each L^p.

PROOF. The inclusions and the continuity of the mappings are obvious.

If we define $l(x) = |\log x|$, then l does not belong to L^∞. Since $\|l\|_p = (p!)^{1/p}$, $l \in L^p$ for each $p \geqq 1$, and hence $l \in L^\omega$. Thus $L^\omega \neq L^\infty$.

Similarly, the function with values $x^{-1/2p}$ belongs to L^p, but not to L^{2p}, and hence not to L^ω.

Now let $l_n(x) = n^{-1}|\log x|$ or n, whichever is the smaller. Then $\|l_n - 0\|_p < n^{-1}\|l\|_p$ which tends to zero as $n \to \infty$; but $\|l_n - 0\|_\infty = n$, $n \to \infty$. Thus the inverse of the mapping $L^\infty \to L^\omega$ is not continuous.

A similar process applied to the function $x^{-1/2p}$ yields a sequence which converges to zero in L^p but not in L^{4p}, and thus not in L^ω.

Finally, suppose $f \in L^\omega$ be given. Define

$$f_n(x) = \begin{cases} -n & \text{when } f(x) < -n, \\ f(x) & \text{when } -n \leqq f(x) \leqq n, \\ n & \text{when } n < f(x). \end{cases}$$

Then $f_n \to f$ in each L^p and hence in L^ω. Since the f_n are taken from L^∞ the latter is dense in L^ω and in each L^p, which establishes the third sentence of the theorem.

L^ω can be metrized, so as to be complete, by

$$(f, g) = \sum_{p=1}^{\infty} \frac{2^{-p}\|f - g\|_p}{1 + \|f - g\|_p}.$$

4. **Multiplication in L^ω.** By relation (H), this is continuous. The following theorem shows the divergence between its properties and those of normed rings.

THEOREM 2. *L^ω is a convex metric commutative ring with the property that if U is a convex open set in L^ω containing* 0, *and if $UU \subset U$, then U coincides with the whole space L^ω.*

PROOF. There exists a $p \geqq 1$ and an $\epsilon > 0$ such that $\|f\|_p \leqq \epsilon$ implies $f \in U$. Therefore a function f having values not greater than h on a set of measure not greater than $(\epsilon/h)^p$, and vanishing elsewhere, must lie in U, together with all its powers $f^2, f^3, \cdots$.

Let $h = 2$, and set $m = (\epsilon/2)^p$, for brevity.

Consider any function g which has the value b on a set S of measure a, and vanishes elsewhere. Suppose k is any integer such that $a \leqq mk$. Select an integer n such that $bk \leqq 2^n$. Now we can cover S by k nonoverlapping subsets of measure not greater than m and define functions $f_1, \cdots, f_k$, where f_i has the value $(bk)^{1/n}$ on the ith subset of S, and vanishes elsewhere. Thus $f_1, \cdots, f_k \in U$, and also $f_1^n, \cdots, f_k^n \in U$. Since U is convex

$$g = \frac{1}{k} f_1^n + \cdots + \frac{1}{k} f_k^n$$

must belong to U.

Now any function g' assuming only a finite number of values is a linear combination, with positive constants *whose sum is* 1, of such functions as g. Therefore these functions lie in U.

Since these functions g' are known to be dense in L^∞ and thus in L^ω, we have U a dense, open convex set in L. Thus $U = L^\omega$.

COROLLARY. *The topology assigned to L^ω cannot be defined by any norm.*

5. **Entire functions in rings.** Of course Theorem 2 shows more about L^ω than is needed for a counter-example to the proposition mentioned in the introduction, as will appear from the following theorem, and the fact that $e^{|\log x|} = 1/x$ is not summable, while $|\log x|$, as we have seen, lies in L^ω.

THEOREM 3. *If R is a complete topological ring with a complete system $\{U\}$ of convex neighborhoods of zero each satisfying $UU \subset U$, and*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

is a power series representing an entire function, then, for each $f \in R$,

$$P(f) = a_0 + a_1 f + a_2 f^2 + \cdots$$

converges, and P is a continuous operation on R into itself.

In particular, for the exponential function, if U is convex, contains zero, and $UU \subset U$, then

$$e^U \subset 1 + 2U.$$

PROOF. Let us first show that $P(f)$ converges. Therefore, suppose U is any neighborhood of the system $\{U\}$. Let $f \in R$.

Then for some $t > 0$, $tf \in U$. Hence $(tf)^2$, $(tf)^3$, $\cdots$ will all lie in U. Further, let us find m_0 so large that for $m \geqq m_0$

$$|a_m t^{-m}| + |a_{m+1} t^{-m-1}| + \cdots$$

is less than 1. Then, since U is convex, we can deduce that for $n > m > m_0$,

$$a_m t^{-m}(tf)^m + \cdots + a_n t^{-n}(tf)^n$$

or its equivalent

$$a_m f^m + \cdots + a_n f^n$$

must lie in U.

Since R is assumed complete, $P(f)$ converges to a limit.

The continuity of P can be proved as follows:

$$D = P(f + h) - P(f) = \sum_{n=0}^{\infty} a_{n+1} g_{n+1}$$

where

$$g_n = (f + h)^{n+1} - f^{n+1}.$$

Let U be a neighborhood of the system $\{U\}$, and suppose $f/t \in U$ where $0 < t < \infty$. Select a real number a,

$$a > |a_1| (t + 1) + |a_2| (t + 1)^2 + \cdots, \qquad a \geqq 1,$$

and require h to be so close to zero that $ah \in U$.

There is no point in writing down the expansion of g_n since terms cannot be collected when R is not commutative. However, each term will contain h, and if g_n is written as a sum of products of powers of f/t and h, the coefficients will add up to $(t+1)^n - t^n$.

Since f/t and ah lie in U, and $UU \subset U$, we have

$$h_n = (t + 1)^{-n} a g_n \in U,$$

where, before dividing, we have replaced $(t+1)^n - t^n$ by $(t+1)^n$. Now D is a linear combination of $h_1, h_2, \cdots$ with coefficients whose absolute values add up to less than 1, and since U is convex we conclude $D \in U$.

Therefore P is continuous at f.

INSTITUTE FOR ADVANCED STUDY

ON UNIFORM SPACES AND TOPOLOGICAL ALGEBRA

G. K. KALISCH

There are several papers which deal with some generalization of metric spaces, for instance, Price [1],[1] Krull's general valuation theory [2], Hyers' pseudonorm [3], and so on. We show in the present paper that spaces possessing a generalized metric (that is, the distance is an element of a partially ordered abelian vector group) are uniform spaces, and that, conversely, any uniform space is a generalized metric space. We also show that topological groups possess right invariant generalized metrics, and, therefore, topological linear spaces, rings, and fields have generalized norms. Although it would be possible to construct a theory based on entirely arbitrary partially ordered abelian semigroups, in view of the construction used in the proof of Theorem 2, it is sufficient to consider only vector groups whose components are real numbers and which are ordered lexicographically. This restriction is not very serious, however, as can be seen from [4] and [5].

The elements of the vector groups we are considering are real numbers indexed according to a partially ordered directed set I. Let i, j, k be elements of I, then:

I (1) If $i \leqq j$, $j \leqq i$ then $i = j$.

(2) If $i \leqq j$, $j \leqq k$ then $i \leqq k$.

(3) Given i and j there exists k such that $k \geqq i$, $k \geqq j$.

We shall denote the elements of G, our vector group, by (r_i), r_i a real number, i an element of I. We define the order relation in G as follows: $g = (r_i) \leqq g' = (r_i')$ in case, if $r_j > r_j'$ for some j, there is some k such that $k < j$, $r_k < r_k'$, and $r_m \leqq r_m'$ for all $m < k$. A space will be called a ***generalized metric space*** in case there is given a function (s, t) defined on $S \times S$ with $s \in S$, $t \in S$, with values in a partially ordered vector group G as described above, and satisfying the following:

II (1) $(s, t) \geqq 0$; $(s, t) = 0$ if and only if $s = t$.

(2) $(s, t) = (t, s)$.

(3) $(s, t) \leqq (s, r) + (r, t)$.

A generalized metric space will be a topological space, and the generalized metric will define a uniform structure in S in case we define neighborhoods or convergence properly. Owing to the fact that G is partially ordered it is possible to topologize G by defining neighbor-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

hoods of 0 in G (weak case) by means of the vectorial structure of G; in terms of convergence it means that a set of vectors converges to a limit vector in case all components converge to the corresponding components of the limit vector (convergence of directed sequences). The corresponding neighborhoods of 0 are subsets of G depending on a finite set $i_1, \cdots, i_n$ of elements of I, and a positive real number ϵ as follows:

$$N_{i_1,\cdots,i_n;\,\epsilon}(0) = [c;\ |c_{i_k}| < \epsilon;\ k = 1, \cdots, n]$$

where c_i is the i-component of c. As soon as there is defined in G a topology it is possible to topologize and uniformize S: a neighborhood $U_N(s)$ for $s \in S$ and N a neighborhood of 0 in G is defined as

$$U_N(s) = [t;\ (s, t) \in N].$$

Since we wish to show that every uniform space is a generalized metric space, and since uniform spaces are subspaces of direct products of metric spaces (cf. [6]), it is necessary to use the weak topology of G. The topology and uniform structure of S can also be given in terms of convergence of directed sequences of elements of G: a directed sequence (s_i) of elements of S converges to $s \in S$ in case (s, s_i) converges to $0 \in G$. We have the following theorem.

THEOREM 1. *A generalized metric space is a uniform space.*

The most convenient formulation of uniform spaces for the purpose of this theorem is the one on page 7 of [6].

Conversely, let the uniform space S be isomorphic to a subset of the direct product $P = \prod P_\mu$ where P_μ are metric spaces and the μ's range over a set M. Let M_0 be the set of all finite subsets of M. The generalized distance between s and t for s and t in S is then defined by $(s, t) = (r_m)$ for $m \in M_0$, $r_m = \max_{\mu \in m} (\mathrm{dist}\,(s_\mu, t_\mu))$ where s_μ is the component of s in P_μ and $\mathrm{dist}\,(s_\mu, t_\mu)$ is the (ordinary) distance between s_μ and t_μ in P_μ and (r_m) is an element of the group G described above. This definition of (s, t) makes S a generalized metric space isomorphic with the original one. Thus we have the following theorem.

THEOREM 2. *Every uniform space is isomorphic to a generalized metric space.*

If P has only a finite number of noncompact direct factors then a strong topology can be used in G.

Topological groups are uniform spaces and hence generalized metric spaces as well. Kakutani's result on right invariant metrics in topological groups [7] suggests that the generalized metric in question can be chosen so as to be right (left) invariant, that is, $(xa, xb) = (a, b)$ for all x in the topological group S. We can apply Kakutani's proof to the individual components of the generalized metric of S each of which constitutes a pseudometric (nonseparated metric) of S. We can thus introduce a right (left) invariant generalized metric into S topologically equivalent to the one originally given, because the replacement of each component of a generalized metric by a topologically equivalent pseudometric does not change the topology of the generalized metric space. We therefore have the following theorem.

THEOREM 3. *Every topological group has a right (left) invariant generalized metric, and, hence, topological linear spaces, rings, and fields possess generalized norms.*

By a linear space we mean, of course, an abelian group with a topological field F as coefficient domain satisfying the obvious continuity conditions; and a generalized norm is a function $|t|$ defined for t in the algebraic structure T in question with values in a partially ordered abelian group G as above with the following properties:

III (1) $|a| \geqq 0$; $|a| = 0$ if and only if $a = o$.

(2) $|a+b| \leqq |a| + |b|$.

(3_L) $|\alpha a|$ is a continuous function of a and $\alpha \in F$.

(3_R) $|ab|$ is a continuous function of a and b.

Here 0 is the identity element of G, o the identity element of T; a and b are in T. III (3_L) is valid for topological linear spaces, while III (3_R) is valid for topological rings and fields. The results here mentioned are in a sense the best possible, as can be seen by the fact that Mahler's pseudovaluations [8] do not exhaust all possible topologies of, for instance, the rational number field; a topology which is the "direct sum" in Mahler's sense of infinitely many valuations would be an example of a topology in which we cannot have $|ab| \leqq |a|\,|b|$ (we define $|a| = \sum_\nu 2^{-\nu} |a|_\nu$).

In conclusion we wish to point out that the results contained in A. H. Frink's paper [9] can be extended to the present paper. The generalized distance function can be assumed to have values in any uniform space V and to satisfy the following conditions: II (1′) $(s, t) \neq u_0$ for some $u_0 \in V$; $(s, t) = u_0$ if and only if $s = t$. II (2) As above. II (3′) For every neighborhood N of u_0 and for every $s \in S$ there is a neighborhood M of u_0 such that $(s, r) \in M$, $(r, t) \in M$ imply $(s, t) \in N$. S is then topologized as above: $U_N(s) = [t;\ (s, t) \in N(u_0)]$. Using the generalized metric of V we can use its components as pseudometrics of S, and by the arguments of [9] we can prove S to be a generalized metric space. This allows us somewhat to modify Weil's postulates for uniform spaces, where (cf. [6, p. 7]) we replace U_{III} by requiring $p\in V(q)$ if and only if $q\in V(p)$; the index set I is ordered by inclusion of the corresponding neighborhoods; given $s\in S$, $i\in I$ there exists $j=j(s, i)\in I$ such that $s\in V_j(r)$, $r\in V_j(t)$ imply $s\in V_i(t)$. We prove S to be a uniform space by introducing into it a generalized distance function based on the set I and defining $(s, t)_i=1$ if s is not in $V_i(t)$ and 0 elsewhere.

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A REMARK ON LOCALLY COMPACT ABELIAN GROUPS

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It has recently been shown by Halmos [1][1] that there exists a compact topological group which is algebraically isomorphic to the additive group of the real line, an example being given by the character group of the discrete additive group of the rationals. Exploiting his argument a bit further it is easy to see that the most general such example is the direct sum of $\aleph$ replicas of the one already given where $\aleph$ is a cardinal such that $2^{\aleph} \leq C$. This having been observed it naturally occurs to one to ask for the most general locally compact topological group with the algebraic structure in question. It is the purpose of the present note to give a complete answer to this question. We shall do so by giving a proof of the following theorem.

THEOREM. *Let G be a locally compact topological group which is algebraically isomorphic to the additive group of a linear space over the rationals. Then G is isomorphic*[2] *to a direct sum of four groups G_1, G_2, G_3, and G_4 which may be described as follows. G_1 is the additive group of an n-dimensional ($n=0, 1, 2, \cdots$) real linear space with the customary Euclidean topology; that is, an n-dimensional vector group. G_2 is the direct sum of $\aleph$ replicas of the character group of the discrete additive group of the rationals. G_3 is a discrete group algebraically isomorphic to an $\aleph^-$-dimensional linear space over the rationals. G_4 is algebraically isomorphic to the additive group of a linear space over the rationals and contains a compact subgroup H whose quotient group is a discrete torsion group. H is a direct sum of finitely or infinitely many groups each of which is isomorphic to the additive topological group of the p-adic integers for some prime p. The groups G_1, G_2, G_3, and G_4 are unique up to an isomorphism. Furthermore (modulo isomorphisms) the groups G_4 and H determine one another. Thus the numbers n, $\aleph$, $\aleph^-$, and the function f from the primes to the cardinal numbers which gives the multiplicity of occurrence of each p-adic group in H form a complete set of invariants for the topological group G. There exists a G having any assigned set of invariants. Finally G is algebraically isomorphic to the additive group of the real numbers if and only if it has continuum many elements; that is, if and only if n, $\aleph$, $\aleph^-$ and f are chosen so that no G_i has more*

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[1] Numbers in brackers refer to the bibliography at the end of the paper.

[2] In this paper the word isomorphic without qualification means isomorphic as a topological group.

than continuum many elements and at least one G_i *has exactly continuum many elements.*

We begin the proof by observing as in [1] that an Abelian group J is algebraically isomorphic to a linear space over the rationals if and only if it is torsion free in the sense that the only element of finite order is zero and divisible in the sense that $nx=y$ is solvable for x in J for each y in J and each $n=1, 2, \cdots$; and that furthermore if J is provided with a discrete or compact topology under which it is a topological group then it is divisible (resp. torsion free) if and only if its character group[3] is torsion free (resp. divisible).[4]

In this and the following two paragraphs we subtract out the G_i one by one. First of all it follows at once from the structure theorem on page 110 of [3] that G is isomorphic to the direct sum of an n-dimensional vector group G_1 and a locally compact group J_1 where J_1 admits a compact subgroup J_2 with a discrete quotient group. Clearly J_1 is Abelian, divisible and torsion free.

Let G_2 be the component of zero in J_1. Since J_1/J_2 is discrete it follows that $G_2 \subseteq J_2$ and hence that G_2 is compact. We shall show first that G_2 is a direct summand of J_2 and then from this that it is a direct summand of J_1. Since G_2 is connected its character group is torsion free so that it itself is divisible. It follows at once from this and the fact that J_2 is torsion free that J_2/G_2 is torsion free. Let G_2' be the set of all elements of the character group C of J_2 which annihilate G_2. Then since G_2' is isomorphic to the character group of J_2/G_2 it follows that G_2' is divisible. Consider the family of all subgroups J of C such that $J \cap G_2' = 0$. It is a consequence of a well known lemma of Zorn that there exists a maximal such subgroup J_3 and it follows easily from the divisibility of G_2' that $G_2' + J_3 = C$. Since C is discrete it is the direct sum of J_3 and G_2'. Hence J_2 must be the direct sum of G_2 and the annihilator J_3' of J_3 in J_2. By applying the argument just used in obtaining J_3 to the set of all (not necessarily closed) subgroups of J_1 which contain J_3' and intersect G_2 in zero we can conclude the existence of a subgroup J_4 of J_1 which contains J_3' and is such that $G_2 \cap J_4 = 0$ and $G_2 + J_4 = J_1$. We shall show that J_1 is the direct sum of G_2 and J_4. It will follow in particular that J_4 is closed. Let O be an arbitrary open subset of J_1 which contains zero. Let $O' = O \cap J_2$. Since J_2 is the direct sum of G_2 and J_3' there exists an open subset O'' of J_2 which contains zero and is such that if $x+y \in O''$ where $x \in G_2$ and $y \in J_3'$ then $x \in O'$ and $y \in O'$. Now since J_1/J_2 is discrete J_2 is an open subset of J_1. It follows that O'' is an open subset of J_1. Suppose that $x+y \in O''$ where $x \in G_2$ and $y \in J_4$. Then since $x+y \in J_2$ we have $y \in J_3'$ and hence $x \in O' \subseteq O$ and $y \in O' \subseteq O$. Thus J_1 is indeed the direct sum of G_2 and J_4. That G_2 is of the character indicated in the statement of the theorem follows at once from the fact that its character group is divisible and torsion free and hence isomorphic to the additive group of a discrete linear space over the rationals.

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[3] The reader is referred to [2] or [3] for a discussion of the theory of characters. We shall make free and frequent use of the results of this theory in the course of this paper.

[4] The example indicated in the last paragraph of this paper shows that this relationship between divisibility and torsion freeness does not hold for general locally compact groups in spite of the assertion to the contrary in [1].

We now decompose J_4. By the structure theorem for Abelian locally compact groups referred to above J_4 contains a compact subgroup H with a discrete quotient group. (As a matter of fact J_3' is such a subgroup.) Clearly J_4 is a linear space over the rationals. Let G_4 be the smallest linear subspace containing H and let G_3 be a linear subspace such that $G_3 \cap G_4 = 0$ and $G_3 + G_4 = J_4$. That G_3 exists is an easy consequence of Zorn's lemma. The proof that J_4 is the direct sum of G_3 and G_4 is analogous to but easier than the proof that J_1 is the direct sum of G_2 and J_4 and will be left to the reader. G_3 obviously has the character required in the statement of the theorem and so has G_4 insofar as its relationship to H is concerned. To show that H has the required properties we observe that it is torsion free and totally disconnected as well as compact. Thus its character group is a discrete divisible torsion group. Now countable discrete Abelian torsion groups have been completely analyzed and the part of that analysis applying to divisible groups applies without essential change in the noncountable case (see for example [4]) and assures us that the character group of H is a weak[4] direct sum of a finite or infinite family of groups each of which is of the form R_p/N where p is a prime, R_p is the subgroup of the additive group of the rationals consisting of those numbers whose denominator may be taken a power of p, and N is the group of integers. It is readily verified that R_p/N is isomorphic to the character group of the additive topological group of the p-adic integers and hence that H has the required form.

It remains to show that G_1, G_2, G_3, and G_4 are unique to within an isomorphism and that to within an isomorphism G_4 and H determine each other uniquely, the other statements of the theorem being obvious. Now the uniqueness of the G's follows at once from the fact that they may all be defined in an invariant manner. In fact[6] G_1+G_2 is the component of zero in G and G_2+G_3 is the annihilator in G of the component of zero in the character group of G. G_2 then appears as the common part of G_1+G_2 and G_2+G_3 while G_4 is isomorphic to the quotient group of G modulo the sum of G_1+G_2 and G_2+G_3. Finally G_1 and G_3 are respectively isomorphic to the quotient groups of G_1+G_2 and G_2+G_3 modulo G_2.

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[4] The weak direct sum of a family of discrete groups is the subset of the ordinary direct sum consisting of those elements with only a finite number of nonzero components and retopologized so as to be discrete.

We consider finally the relationship between H and G_4. First of all suppose that H_1 and H_2 are two compact subgroups of G_4 having discrete quotient groups. Then H_1+H_2 is also a compact subgroup with a discrete quotient group. Thus in order to show that H_1 and H_2 are isomorphic we may confine our attention to the case in which $H_1 \subseteq H_2$. Since H_2/H_1 is both compact and discrete it must be finite. Thus if C_1 and C_2 are the respective character groups of H_1 and H_2 then C_1 is isomorphic to the quotient of C_2 modulo a finite subgroup. But it is obvious that factorization modulo a finite subgroup cannot change the number of summands of the form R_p/N in C_2. Thus C_1 and C_2 and hence H_1 and H_2 are isomorphic.

Suppose conversely that G_4 and G_4^- are torsion free divisible locally compact Abelian groups containing isomorphic compact subgroups H and H^- such that G_4/H and G_4^-/H^- are discrete torsion groups. Given $x \in G_4$ let r be the least integer such that $rx \in H$. Let $y \in H^-$ correspond to rx in H under the given isomorphism between H and H^-. Since G_4^- is divisible and torsion free there is a unique z in G_4^- with $rz=y$. Let $\phi(x)=z$. It is readily verified that the function ϕ so defined sets up an isomorphism between G_4 and G_4^-. This completes the proof of the theorem.

We should probably call attention here to the general phenomenon of which the relationship between H and G_4 is a special case. If T is an arbitrary torsion free Abelian group then it is easy to show that T is imbeddable in an essentially unique Abelian group S which is torsion free, divisible and such that S/T is a torsion group. One need only repeat the argument used in constructing the rational numbers from the integers. Furthermore if T is equipped with a topology under which it is a topological group then S may be topologized in only one way so that the relative topology in T coincides with the given one, so that T is closed and so that S/T is discrete. This topology is defined by taking the neighborhoods of zero in T as the neighborhoods of zero in S. It is clear that if T is locally compact then so is S. Calling S the divisible extension of T we see that G_4 is simply the divisible extension of H. It is not difficult to see that the additive topological group of the field of all p-adic numbers for a given p is the divisible extension of the additive topological group of p-adic integers for the same p. It follows from this last fact that whenever H has only a finite number of summands then G_4 is a direct sum of finitely many p-adic number groups.

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[6] We are supposing here that the original G_i have been replaced by isomorphic subgroups of G as we of course always may.

In closing we remark that if H is the direct sum of $\aleph$ p-adic integer groups for the same prime p where $\aleph \geq \aleph_0$ then the character group of G_4/H is the direct sum of $2^{\aleph}$ p-adic integer groups while that of H is the weak direct sum of only $\aleph$ groups of the form R_p/N. It follows at once that the character group of G_4 while torsion free cannot be divisible. On the other hand if $\aleph$ is finite then it is not difficult to see that G_4 and its character group are isomorphic. Furthermore an infinite number of summands for H does no harm provided that each prime appears only a finite number of times and one may show that G has a divisible character group if and only if $f(p)$ is finite for every prime p and is isomorphic to its character group if and only if $f(p)$ is finite for every prime p and $\aleph = \aleph^-$.

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IDEAL THEORY ON OPEN RIEMANN SURFACES

O. F. G. SCHILLING

Introduction. The theorems of the classical ideal theory in fields of algebraic numbers hold in rings of analytic functions on compact Riemann surfaces. The surfaces admitted in our discussion are closely related to algebraic surfaces; we deal either with compact surfaces from which a finite number of points are omitted or, more generally, with surfaces determined by an algebroid function. The local aspects of the resulting ideal theory are the same as those found in the theory of algebraic functions. However, the ideal theory in the large is quite different. We now cannot count on the simplifications which are implied by the ascending chain condition for ideals. The recent theories of functions on topological spaces provide the necessary tools for a simple theory in the large. We shall show that the topologization of rings of entire functions by means of the topology of the underlying surface furnishes a fruitful method. Thus, the closed maximal ideals will correspond to the points of the given surfaces. Finally, to quote another result, all closed ideals are principal.

1. **Some basic definitions.** Suppose that Σ is the complex number sphere whose points at finite distance $p = p_a$ are in 1-1 correspondence with the elements a of the complex number field C. We assume that each point p of Σ, the point at infinity included, has conformal neighborhoods. In general let t_p be a local uniformizing variable at p, that is, a function which affords the mapping of a conformal neighborhood of p in the unit circle $|t_p| < 1$. We associate with each point p a model Σ_p of Σ and consider functions $f(z)$ on Σ with values in $\{\Sigma_p\}$, in other words, $f(p) \in \Sigma_p$ for $p \in \Sigma$. As usual we say that a function $f(z)$ is meromorphic on Σ if there exists for each point $p \in \Sigma$ a convergent expansion

$$f(z) = \sum_{i=m}^{\infty} c_{p,i} t_p^i, \qquad c_{p,i} \in C, \tag{1}$$

where the first nonvanishing coefficient $c_{p,m}$ has a finite integral subscript. Then all meromorphic functions on Σ form a field $F(\Sigma)$ which contains C [2, 17].[1]

Next we associate to each function $f(z)$ the integer $m = V_p f(z)$ and

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[1] The numbers in brackets refer to the list of references.

define $V_p 0 = \infty$. The algebra of power series implies that the "*valuation*" V_p has the following properties

$$V_p(t_p) = 1, \qquad V_p(f+g) \geqq \min(V_p f, V_p g),$$
$$V_p(fg) = V_p(f) + V_p(g), \qquad V_p c = 0 \tag{2}$$

for each nonzero complex number [6, 12, 16]. We note that V_p measures the order of the zero or pole of f at the point p. Furthermore the value $V_p f$ does not depend on the special choice of the variable t_p, that is, if $t_p' = \sum_{i=1}^{\infty} d_i t_p^i$, $d_1 \neq 0$, $d_i \in C$, is analytic in a sufficiently small neighborhood of $t_p = 0$, and $f(z)$ is expanded with respect to t_p' as $\sum_{i=m'}^{\infty} c'_{p,i} t_p'^i$, then $m = m'$. We mention in this connection that the function theoretic value $f(p)$ of $f(z)$ at p is:

(i) the coefficient $c_{p,0}$ of (1) if $V_p f(z) = 0$,

(ii) the element 0 if $V_p f(z) > 0$, and

(iii) the element ∞ of Σ_p if $V_p f(z) < 0$, or if $f(p)^{-1} = 0$.

Then, for the functions f_1, f_2, with $f_1(p), f_2(p) \neq \infty$,

$$(f_1 + f_2)(p) = f_1(p) + f_2(p), \qquad (f_1 f_2)(p) = f_1(p) f_2(p). \tag{3}$$

Thus the functions f of $F(\Sigma)$ which are holomorphic at p form a ring R_p, the so-called valuation ring of p in $F(\Sigma)$. This ring contains a single prime ideal M_p^* which consists of all functions which vanish at p. In particular, the *homomorphism* $R_p \rightarrow R_p/M_p^* = \phi_p(R_p)$ *is the arithmetic description of the function theoretic value*, $f(p) = \phi_p(f(z))$.

We recall next the following consequences of the compactness of Σ:

(4) If $f(p) \neq 0$, ∞ for all $p \in \Sigma$ then f is a nonzero complex number,

(5) If $f \notin C$, then $f(p)$ is 0 or ∞ for only a finite set of points of Σ, that is, a meromorphic function has a finite number of zeros and poles.

After these preparations it is a simple matter to construct the field $F(\Sigma)$. We first note that $z - a$, $a \in C$, is a meromorphic function with $V_p(z-a) = 1$ with $p = p_a$, $V_p(z-a) = -1$ with $p = p_\infty$, $V_p(z-a) = 0$ for all other points p. Suppose that $f(z) \in F(\Sigma)$ has its poles and zeros at the points $p_{a_1}, \cdots, p_{a_r}$ where repetitions of the points and the point p_∞ are permitted. Then the function $\prod_{i=1}^{r}(z - a_i) = f^*(z)$ has the same poles and zeros as $f(z)$. Using the second of rules (2) and (4) we find that $f(z)/f^*(z)$ is a nonzero constant c of C. Thus

$$f(z) = c f^*(z), \tag{6}$$

or $F(\Sigma)$ is the field of all rational functions of the complex variable z.

2. **The field of meromorphic functions.** We omit now from Σ the

point p_∞ and obtain thus the open complex number plane Σ' whose points are in 1-1 correspondence with the numbers of C. The space Σ' is locally compact and the sum of the compact regions $|z| \leqq n$, $n \to \infty$. We define a meromorphic function $f(z)$ on Σ' by omitting in the definition of $F(\Sigma)$ the condition of rationality at p_∞. Then the field of meromorphic functions $F(\Sigma')$ contains $F(\Sigma)$ as proper subset. Statement (5) is now replaced by:

(7) A meromorphic function $f(z)$ has only a finite number of zeros and poles in a small neighborhood of each point $p \in \Sigma'$.

The field $F(\Sigma')$ contains as a subring $O(\Sigma')$ the integral domain of all entire functions $f(z)$ with $V_p f(z) \geqq 0$ for all $p \in \Sigma'$. Suppose that $f(z)$ is not a unit, that is, $f(z)$ has zeros a_j. Then the set of zeros $\{a_j\}$ is countable and (7) implies $|a_j| \to \infty$ as $j \to \infty$; moreover, only a finite number of a_j's have the same absolute value. The construction leading to (6) must now be replaced by *Weierstrass' theorem* [3]. We pick integers m_j such that sum $\sum_{j=1}^{\infty} z^{m_j}/a_j^{m_j+1}$, $a_j \neq 0$, is absolutely convergent for all z. Then we define $q_j(z) = \sum_{k=1}^{m_j}(1/k)^k(z/a_j)^k$ and set $f^*(z) = \prod_j (1 - z/a_j)e^{q_j(z)}$. The product $f^*(z)$ is absolutely convergent for all z and uniformly convergent in any finite region of Σ'. Therefore $f^*(z) \in F(\Sigma')$. Suppose $V_{p_0} f(z) = m$. Then $f(z)/(z^m f^*(z))$ has no zeros and poles on Σ'. Hence it is a unit function $e^{g(z)}$ of $O(\Sigma')$ and

$$f(z) = e^{g(z)} z^m \prod_j (1 - z/a_j) e^{q_j(z)}. \tag{8}$$

Conversely there always exists an entire function (8) with prescribed zeros. Moreover, an arbitrary function of $F(\Sigma')$ is always a quotient of entire functions, as follows by constructing entire functions according to (8) separately for its zeros and poles. Consequently $F(\Sigma')$ is the quotient field of $O(\Sigma')$.

3. **The ring of entire functions.** We note from the theory of polynomials that the polynomial ring $C[z]$ is equal to the intersection of the valuation rings R_p for all $p \in \Sigma'$. The definition of a function $f(z) \in F(\Sigma')$ implies that each point $p \in \Sigma'$ determines again a valuation V_p of $F(\Sigma')$ with a corresponding valuation ring R_p' in $F(\Sigma')$, for only local properties of functions are used. Moreover, the definition of an entire function shows that

$$O(\Sigma') = \bigcap_p R_p' \supset C[z]. \tag{9}$$

Furthermore, each ring R_p' has a unique prime ideal M_p' consisting of all functions vanishing at p such that, for $p = p_a$, $M_p' \cap O(\Sigma') = M_p = O(\Sigma')(z - a)$. Thus, each point $p \in \Sigma'$ determines uniquely the *prime ideal $M_p \subset O(\Sigma')$ which consists of all entire functions vanishing at p.*

We next define an ideal A of $O(\Sigma')$ to be an additive subgroup of $O(\Sigma')$ which admits the elements of $O(\Sigma')$ for multipliers. It has been noticed before that the (trivial) ideal theory of $C[z]$ has no direct analogue in $O(\Sigma')$ [7]. The ideal theory of $C[z]$ and its finite algebraic extensions has two characteristic properties. First there are no properly ascending chains of ideals and secondly there are no other prime ideals but the ones corresponding to points of the associated algebraic curve or Riemann surface. The ring $O(\Sigma')$ presents an entirely different arithmetic structure. Consider the entire function $\sin(\pi z)$ with the zeros $\pm n$ and set $f_1 = \sin(\pi z)$, $f_2 = \sin(\pi z)/z$, $f_3 = \sin(\pi z)/z(z^2-1^2), \cdots, f_{n+2}$ $\sin(\pi z)/z(z^2-1^2)\cdots(z^2-n^2)$. Then f_{n+2} has the zeros $\pm(n+1), \cdots$ and $f_1 = f_{n+2}z(z^2-1^2)\cdots(z^2-n^2)$. Then $O(\Sigma')f_1 \subset O(\Sigma')f_2 \subset \cdots$ is a properly ascending infinite chain of divisors. Moreover, admitting only finite sums, the join $\bigcup_i O(\Sigma')f_i = (f_1, f_2, \cdots) = A$ is not the unit ideal $O(\Sigma')$, for otherwise $1 = \sum_{i=1}^{N} g_i f_i$ with entire functions g_i. Such an equation cannot hold since the sum has infinitely many zeros.

The following remark indicates that a change of the definition of an ideal may be useful. Recalling the expansion $\pi/\sin(\pi z) = 1/z + 2\sum_{n=1}^{\infty}[(-1)^n z]/(z^2-n^2)$ we obtain $1 = (1/\pi)[f_1 - 2z^2 f_2 + 2z^2(z^2-1^2)f_3 + \cdots]$. Thus 1 is an infinite convergent sum of elements in A.

There are three obvious devices which may be investigated in order to develop an ideal theory in $O(\Sigma')$ such that the resulting theory resembles the ideal theory of fields of algebraic functions or algebraic numbers. The first method provides a *recasting of the definition of an ideal*, the second utilizes the technique of v-ideals and *quasi equality*, and the third requires a restriction to ideals with a *finite basis*. The last approach was used by Helmer [7]. We shall use in the sequel the first and second methods.

We now introduce a topology in $O(\Sigma')$ in order to carry out the first method.

DEFINITION 1. *Suppose that $\{f_n\}$ is an infinite sequence of entire functions. We define $f_n \to f$ as $n \to \infty$ to mean uniform convergence in any bounded region of Σ'.*

DEFINITION 2. *An ideal A of $F(\Sigma')$ with respect to $O(\Sigma')$ is called closed, if:*

(i) *A is closed under addition and under multiplication by elements of $O(\Sigma')$, that is, A is an ideal in the algebraic sense;*

(ii) *There exists a nonzero integral function f such that $fA \leqq O(\Sigma')$;*

(iii) *A is closed in the topology of Definition* 1.

We note that (iii) implies that A contains all convergent infinite sums $\sum_{i=1}^{\infty} a_i$ for which $a_i \in A$.

Let

$$\mathcal{V}A = \left\{ \min_{a\in A} V_p a = V_p A, \text{ all } p \in \Sigma' \right\}. \tag{10}$$

This vector $\mathcal{V}A$ has at most an enumerable infinitude of components $V_p A$ distinct from 0 since an element $a \in A$ has at most a countable set of zeros in Σ'.

LEMMA 1. *If* A *is closed and* $\mathcal{V}A = \{0\}$, *then* $A = O(\Sigma')$.

PROOF. It suffices to show that A contains an entire function without zeros. Let $g \in A$ and suppose $a_1, a_2, \cdots, a_n, \cdots$ are the zeros of g. Since $\mathcal{V}A = \{0\}$ there exists for each a_n a function $f_n \in A$ with $V_n f_n = 0$ or $f_n(a_n) \neq 0$, where V_n is the valuation for a_n. Applying (8) we can exhibit a function $f \in O(\Sigma')$ which has simple zeros at the points a_n. Suppose that $|(ff_n)/(z-a_n)| \leq M_n$ on $|z| \leq n$. Then the sum $\sum_{n=1}^{\infty} (ff_n)/[(z-a_n)2^n M_n] = h$ exists and represents an element of $O(\Sigma')$. By construction $h(a_n) \neq 0$. Consequently g and h are entire functions without common zeros. Hence there exist integral functions k and l such that $gk + hl = 1$ (see [7, pp. 351–352]). Therefore $1 \in A$ or $A = O(\Sigma')$.

Using a repeated diagonal process we can prove the following lemma.

LEMMA 2. *Let* A, B *be two closed ideals and define the product* $A \circ B$ *as the totality of all convergent sums* $\sum_{n=1}^{\infty} a_n b_n$, $a_n \in A$, $b_n \in B$. *Then* $A \circ B$ *is closed.*

LEMMA 3. *All closed ideals* A *are principal.*

PROOF. Weierstrass' theorem (8) together with (10) implies the existence of an element $a \in O(\Sigma')$ with $V_p a = V_p A$ for all $p \in \Sigma'$. Then the set of all quotients b/a, $b \in A$, is an ideal B in $O(\Sigma')$. The ideal B is closed for $b_n/a \to g$, $b_n \in B$, implies $g = h/a$ with $h \in O(\Sigma')$; but $b_n \to ag$ and therefore $ag \in A$ since A is closed. By construction $\mathcal{V}B = \{0\}$ and therefore, by Lemma 1, $B = O(\Sigma')$ or $A = aO(\Sigma')$.

In order to describe the ideal theory of the closed ideals of $F(\Sigma')$ with respect to $O(\Sigma')$ we introduce the following *vector group* $\mathcal{V}$ in which addition is defined by addition of the components [10]. We consider vectors of integers $\{m_p, p \in \Sigma'\}$ where at most a *denumerable infinitude of components* m_{p_n} *do not vanish* and where $|a_n| \to \infty$

as $n\to\infty$. The mapping $A\to\mathcal{V}A\in\mathcal{V}$ maps the group of closed ideals $\{A\}$ (with the multiplication of Lemma 2) into $\mathcal{V}$ in such fashion that $\mathcal{V}(A\circ B)=\mathcal{V}(A)+\mathcal{V}(B)$ and $\mathcal{V}(O(\Sigma'))=\{0\}$. Since all closed ideals are principal we have trivially $\mathcal{V}A^{-1}=-\mathcal{V}A$ where A^{-1} is the closed ideal of all $b\in F(\Sigma')$ with $bA\subseteq O(\Sigma')$. Conversely, Weierstrass' theorem (8) implies that each vector $\{m_p, p\in\Sigma'\}$ of $\mathcal{V}$ is the vector $\mathcal{V}A$ of a closed ideal A. Observe that $A=O(\Sigma')a$ for $a\in F(\Sigma')$ with $V_pa=m_p$ for all $p\in\Sigma'$. The ideal A is unique since $\mathcal{V}A_1=\mathcal{V}A$ for any other ideal A_1 with $\mathcal{V}A_1=\{m_p, p\in\Sigma'\}$ and since A is the totality of all functions c with $V_pc\geqq m_p$ for all $p\in\Sigma'$. We have proved:

THEOREM 1. *The closed ideals A of $F(\Sigma')$ with respect to $O(\Sigma')$ form a multiplicative group which is isomorphic to the vector group $\mathcal{V}$.*

We mention in passing that we may consider infinite products of ideals, provided obvious assumptions are made on the distribution of the zeros and poles.

The second method which utilizes the theory of quasi equality is closely related to the first, as the following observations show.

DEFINITION 3. *Two ideals, A_1, A_2, of $F(\Sigma')$ with respect to $O(\Sigma')$ are called quasi equal, $A_1\sim A_2$, if $\mathcal{V}A_1=\mathcal{V}A_2$* [10].

We now consider the classes of quasi equal ideals $[A]$ and define $[A][B]=[AB]$. This definition will be useful if we can show that quasi equality is an equivalence relation. For the proof we note:

(11) $\qquad \mathcal{V}(A)=\{0\}$ if and only if $A^{-1}=O(\Sigma')$.

We have $\mathcal{V}A^{-1}=\{0\}$ and thus $A^{-1}=O(\Sigma')$ since A^{-1} is closed. For the converse we observe that $\mathcal{V}A=-\mathcal{V}A^{-1}=\{0\}$.

(12) $\qquad (AB)^{-1}=A^{-1}$ for any ideal B with $\mathcal{V}B=\{0\}$.

Note that $AB\subseteq A$ implies $(AB)^{-1}\supseteq A^{-1}$. For the converse let $c\in(AB)^{-1}$, then $O(\Sigma')\supseteq c(AB)=(cA)B$. Consequently $cA\subseteq B^{-1}$. Therefore, by (11), $cA\subseteq O(\Sigma')$ or $c\in A^{-1}$. Hence $(AB)^{-1}\subseteq A^{-1}$.

(13) $A\sim B$ if and only if $A^{-1}=B^{-1}$, and there exist ideals C_1, C_2 with $\mathcal{V}C_1=\mathcal{V}C_2=\{0\}$ such that $AC_1=BC_2$.

Consider $AA^{-1}B=A(A^{-1}B)=(AA^{-1})B$ and put $A^{-1}B=C_1$, $AA^{-1}=C_2$. Then by (12), $A^{-1}=(AC_1)^{-1}=(BC_2)^{-1}=B^{-1}$, for $\mathcal{V}C_1=\mathcal{V}C_2=\{0\}$ The converse is obvious.

We can therefore assert that the group of classes of quasi equal ideals is isomorphic to the vector group $\mathcal{V}$ where the isomorphism is established by the mapping $[A]\to\mathcal{V}A$. This statement can be interpreted further.

LEMMA 4. *Let $\bar{A}$ be the closure of the ideal A, in the sense of the Definition 1, then $\mathcal{V}\bar{A}=\mathcal{V}A$.*

For the proof note that $\bar{A}\supseteq A$ implies $V_p\bar{A}\leqq V_p A$ for all $p\in\Sigma'$. For the converse we may assume without loss of generality that $A\subseteq O(\Sigma')$. Let $a=\lim_{n\to\infty}a_n$, $a_n\in A$, be an arbitrary element of $\bar{A}$. Suppose that $V_pA=m>0$, $b\in C$ with $p=p_b$. We consider the sequence $\{b_r=a_n(z-b)^{m-1}\}$ with the limit $f=a(z-b)^{m-1}$. We now pick a positive ϵ so small that $|b_n-f|<\epsilon/2$ in a sufficiently small neighborhood of p_b in Σ'. Since f is analytic we have $|f-f(b)|<\epsilon/2$ and consequently $|b_n-f(b)|<\epsilon$. If $f(b)$ were distinct from zero, then ϵ can be chosen such that $\epsilon<\big||b_n|-|f(b)|\big|<|b_n-f(b)|$. Hence, by contradiction, $f(b)=0$ and thus $V_pa\geqq V_pA$. Consequently $V_p\bar{A}\geqq V_pA$ and therefore $\mathcal{V}\bar{A}=\mathcal{V}A$. Consequently:

(14) All ideals in $[A]$ have the same closure $\bar{A}$, and $\bar{A}$ is the largest ideal in $[A]$.

Thus Theorem 1 states that the group theoretical structure of the set of all class of quasi equal ideals is the same as the structure of the group of all closed ideals.

4. The prime ideals of $O(\Sigma')$. We noticed in §2 that each point p of Σ' gives rise to a valuation of $F(\Sigma')$; moreover, p determines uniquely a maximal prime ideal M_p of $O(\Sigma')$. The results of §3 imply that the prime ideal M_p is closed since it consists of all entire functions which vanish at p. We now show that $O(\Sigma')$ contains infinitely many nonclosed prime ideals M. The example of §3, $A=(\sin(\pi z), \sin(\pi z)/z, \cdots)\subset O(\Sigma')$ proves the existence of a nonclosed ideal with $\mathcal{V}(A)=\{0\}$. Applying Zorn's lemma we can prove the existence of a prime ideal $M\supset A$, $M\neq O(\Sigma')$ with $\mathcal{V}M=\{0\}$. There are infinitely many such prime ideals. Pick two sequences of complex numbers a_n, b_n with $|a_n|$, $|b_n|\to\infty$ which are distinct. By (8) there exist functions f_n, g_n with zeros at $a_n, a_{n+1}, \cdots$ and $b_n, b_{n+1}, \cdots$, respectively. Let $B=(f_1, f_2, \cdots)$ and $C=(g_1, g_2, \cdots)$, then $\mathcal{V}B=\mathcal{V}C=\{0\}$. We assert that B and C determine distinct maximal prime ideal divisors. Suppose $B\subset M_1$, $C\subset M_2$ and $M_1=M_2$. Then f_1, $g_1\in M_1=M_2$. Hence there exist entire functions a, b such that $f_1a+g_1b=1\in M_1=M_2=O(\Sigma')$ because f_1, g_1 have distinct zeros. Therefore $M_1\neq M_2$.

LEMMA 5. *All prime ideals of $O(\Sigma')$ are maximal.*

Proof. If a prime ideal $M\subset O(\Sigma')$ were not maximal, there would exist a function $f\in O(\Sigma')$ for which $(M, f)\subset O(\Sigma')$. Let $g\in M$ and suppose that $\{a_n\}$ is the set of common zeros of f and g. By (8) there exists a function $h\in O(\Sigma')$ with precisely the zeros $\{a_n\}$. We write $g=hm$. Hence either $h\in M$ or $m\in M$. But m does not lie in M for m and f have no common zeros and thus generate $O(\Sigma')=(m,f)$, whence $(M,f)=O(\Sigma')$, contrary to hypothesis. Therefore $h\in M$ and $f\in M$ since f is a multiple of h. Thus M is maximal.

This lemma implies that each closed prime ideal M of $O(\Sigma')$ belongs to a point, that is, that there exists a unique p with $M=M_p$. Note that Lemma 1 implies $V_pM\neq 0$ for some p. Hence $M\subseteq M_p$ and thus $M=M_p$ by maximality. We note that $O(\Sigma')/M_p\cong C$.

The nonclosed prime ideals M can be characterized algebraically as follows.

LEMMA 6. *A maximal ideal M of $O(\Sigma')$ is nonclosed if and only if $O(\Sigma')/M$ is a proper extension of the complex number field C.*

PROOF. For a prime ideal M certainly $M\cap C[z]\neq C[z]$, since M would otherwise contain a nonzero constant and be equal to $O(\Sigma')$. Consequently either $M\cap C[z]\neq 0$, $C-0\not\subseteq M\cap C[z]$, or $M\cap C[z]=0$. In the first case $M\cap C[z]=(z-a)$ for some $a\in C$ because M is prime. Consequently $O(\Sigma')(z-a)=M_{p_a}\subseteq M$ and $M=M_{p_a}$ is closed. Hence $M\cap C[z]=0$ for a nonclosed prime ideal. But then $O(\Sigma')/M$ contains $C[z]/0\cong C[z]$ and therefore $O(\Sigma')/M\supset C$. Conversely $O(\Sigma')/M\supset C$ does not hold for closed prime ideals, hence M is not closed.

We consider next valuations V of $F(\Sigma')$ whose valuation rings contain $O(\Sigma')$. Assume that V does not arise from a point $p\in\Sigma'$. Then the prime ideal P of the valuation ring of V in $F(\Sigma')$ necessarily meets $O(\Sigma')$ in a nonclosed prime ideal $M=P\cap O(\Sigma')$. We note that then $\mathcal{V}M=\{0\}$, for otherwise there would exist a point p with $V_pM\neq 0$. Then $M\subseteq M_p$ or $V=V_p$ contrary to assumption. We assert:

LEMMA 7. *There exists no valuation V of rank one such that a nonclosed prime ideal M of $O(\Sigma')$ is equal to $P\cap O(\Sigma')$ where P is the prime ideal of V.*

PROOF. We have $\mathcal{V}[P\cap O(\Sigma')]=\{0\}$. Let $f\in P\cap O(\Sigma')$, then f must have infinitely many zeros; for if f were a unit then $P\cap O(\Sigma')=O(\Sigma')$, and if f had a finite number of zeros then V would be some V_p. Let $a_1, a_2, \cdots$, be the zeros of f, where we admit repetitions. We next construct by (8) a function h which has zeros at the a_n with the multiplicity n. Now let $s_n=f\cdot[f/(z-a_1)]\cdots[f/(z-a_1)\cdots(z-a_n)]$, then $s_n\in O(\Sigma')$ and $h/s_n=g_n\in O(\Sigma')$. Since $V(z-a_n)=0$, for otherwise $V=V_n$, V_n the valuation for a_n, we find $V(h)=V(s_n)+V(g_n)=nV(f)+V(g_n)$. This means that $V(h)$ is larger than any positive real number. Consequently V does not exist.

Combining the preceding observations, we have [5]:

THEOREM 2. *The points of* Σ' *are in* 1-1 *correspondence with the closed prime ideals of the ring* $O(\Sigma')$ *of analytic functions on* Σ'.

The statement of this theorem is closely related to some results dealing with continuous real or complex valued functions on topological spaces [5, 15]. Suppose that T is a topological space which can be represented as the sum $\bigcup_n C_n$ of compact sets, $C_n \subset C_{n+1}$, such that each compact subset of T is contained in a suitable C_n. We define in the ring $O^*(T)$ of real (or complex) valued continuous functions on T a compact open topology by agreeing that $f_n \to f$ in $O^*(T)$ if the sequence f_n approaches f uniformly on each compact subset of T. Then $O^*(T)$ is complete. We have the following lemma.

LEMMA 8. *Each closed ideal* A *of* $O^*(T)$ *without zeros on* T *is equal to* $O^*(T)$.

PROOF. We require of A that there exist for each point $p \in T$ a function $f \in A$ for which $f(p) \neq 0$. Take any C_i; there exists a point $p_{ij} \in C_i$ with a function $f_{ij} \in A$ such that $f_{ij}(p_{ij}) \neq 0$. By continuity there exists a neighborhood U_{ij} of p_{ij} on which f_{ij} does not vanish. Since C_i is compact it can be covered by a finite number n_i of such neighborhoods U_{ij} with associated functions $f_{ij} \in A$. Then $f_{ij}^2 > 0$ (or $f_{ij}\bar{f}_{ij} > 0$) on U_{ij}. Let $f_i = \sum_{j=1}^{n_i} f_{ij}^2 (\sum f_{ij}\bar{f}_{ij})$, then f_i does not vanish on C_i. Let $M_i = \max_{p \in C_i} |f_i(p)|$. Consider $\sum_{i=1}^{\infty} f_i/(M_i 2^i) = f$. The sum f is convergent in each C_i and hence on each compact subset of T. Consequently $f \in O^*(T)$ by the completeness of $O^*(T)$. Since A is closed and $f_i \in A$, we have $f \in A$. By construction f does not vanish on T and therefore $f^{-1} \in O^*(T)$, whence $A = O^*(T)$.

Let us assume further that T is a completely regular space. We state the following theorem.

THEOREM 3. *The closed maximal ideals* M *of* $O^*(T)$ *are in* 1-1 *correspondence with the points of* T.

PROOF. If p is a point of T then all $f \in O^*(T)$ with $f(p) = 0$ form a closed maximal ideal M_p. Conversely, the functions f of a maximal closed ideal M must have a common zero, p, for otherwise $M = O^*(T)$ by Lemma 8. Let A_p be the set of all functions vanishing at p. Then $M \subseteq A_p$ and consequently $M = A_p$. The functions of A_p cannot vanish at another point q since the complete regularity implies the existence of $f \in A_p$ with $f(p) = 0$, $f(q) \neq 0$.

The maximal ideals M^* of the ring $O^*(\Sigma')$ of all continuous complex valued functions on Σ' can be related to the maximal ideals

M of $O(\Sigma')$. For each maximal ideal M let the set $M^*\subset O^*(\Sigma')$ consist of all functions $f\in O^*(\Sigma')$ for which there exists an element $m\in M$ such that f vanishes at all zeros of m. This set M^* is an ideal. By the preceding definition certainly $fg\in M^*$ for any $g\in O^*(\Sigma')$. If f_1 is another function of M^* with a corresponding function $m_1\in M_1$, then f_1+f_2 vanishes at all zeros of a function h in M whose zeros are those common to m and m_1. (This h may be constructed by the method of Lemma 5.) Furthermore M^* is maximal. Suppose $(M^*, g)\subset O^*(\Sigma')$. We pick any $m\neq 0$ in M. Then m and g must have zeros in common. For otherwise $k=m\bar{m}+g\bar{g}$ is a function of (M^*, g) without zeros, so that $k^{-1}\in O^*(\Sigma')$ and $1=k^{-1}k\in(M^*, g)$, a contradiction. Let n be an entire function whose zeros are the common zeros of m and g. Then g vanishes at all zeros of n and $n\in M$. Hence $g\in M^*$.

Conversely each M^* determines a prime ideal $M^*\cap O(\Sigma')$ of $O(\Sigma')$; this ideal is maximal by Lemma 5. Using the original correspondence $M\rightarrow M^*$ we have $M^*\cap O(\Sigma')=M$.

5. **The ideal theory in finite extensions of** $O(\Sigma')$. Let K be a finite algebraic extension of degree n over $F(\Sigma')$. Then each element k of K is the root of an equation

$$x^n+f_1x^{n-1}+\cdots+f_n=0 \tag{15}$$

whose coefficients are meromorphic functions. Since the valuations V_p of $F(\Sigma')$ are determined by local properties of the functions in $F(\Sigma')$, we can employ a procedure of the theory of algebraic functions to determine all prolongations $W^{(1)}\cdots W^{(g)}$, $g=g(p)$, of V_p to K. We recall that the discrete rank one valuations $W^{(j)}$ can be determined as the unique valuations of the direct summands $K^{(j)}$ of the direct product $K\times F(\Sigma')_V$ where $F(\Sigma')_V$ is the completion of $F(\Sigma')$ with respect to $V=V_p$. Then $1\leqq g(p)\leqq n$. More precisely $W^{(j)}(k)$, $k\in K$, can be considered as $W^{(1)}(k^S)$ with a suitable conjugate k^S of k and a fixed prolongation $W^{(1)}$ of V_p [12].

Using the triangle inequality we find:

(16) A function $k\in K$ has at most a denumerable set of zeros and poles, W_i, in other words valuations W_i with $W_i(k)=0$, ∞. These valuations W_i applied to $F(\Sigma')$ contract to valuations V_i belonging to points a_i; if $\{W_i\}$ is an infinite set, then $|a_i|\rightarrow\infty$.[3] A function $k\in K$ never has an essential singularity at any W of K.

We can now construct the Riemann surface T of K and prove, as in the theory of algebraic functions, that the field $F(T)$ of meromorphic functions on T is equal to K. Let $O(T)$ be the ring of all entire functions on T and let R_W be the valuation ring of W in $K=F(T)$. We shall prove the following generalization of (9),

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[3] Or, $\{W_i\}$ has no infinite subsequence which is compact in T.

$$O(T) = \bigcap_W R_W. \tag{17}$$

We first prove [10] the following lemma.

LEMMA 9. *If* $W^{(1)}, \cdots, W^{(g)}$ *are the prolongations of* $V=V_p$ *to* K *then* $W^{(j)}(k)\geqq 0$, $1\leqq j\leqq g$, *implies* $V(f_i)\geqq 0$ *for the coefficients of the defining equation* (15). *In particular* $V(Nk)=0$ *for the norm* Nk *if all* $W^{(j)}(k)$ *are* 0.

PROOF. We imbed K in its least normal extension $L/F(\Sigma')$ with the Galois group $\{S\}$. Then $\overline{W}^{(\nu)}(k)\geqq 0$ for all prolongations $\overline{W}^{(\nu)}$ of V to L. Therefore $\prod_S(x-k^S)$ has coefficients in the valuation ring R_p. If $W^{(j)}(k)=0$, then $\overline{W}^{(\nu)}(k)=0$ and thus $\overline{W}^{(\nu)}(Nk)=0$ or $V(Nk)=0$.

As a consequence we note:

$$W^{(j)}[(Nk)k^{-1}] \geqq 0 \quad \text{if} \quad W^{(j)}(k) \geqq 0 \quad \text{for} \quad 1 \leqq j \leqq g. \tag{18}$$

Now (17) follows readily. Let $k\in\bigcap_W R_W$. Then by Lemma 9, $V_p(f_i)\geqq 0$ for the coefficients of (15). Consequently $f_i\in\bigcap_p R_p=O(\Sigma')$ and hence $\bigcap_W R_W\subseteq O(T)$. The converse inequality is obtained by applying V to (15).

In order to obtain the analogues to Theorems 1 and 2 we use the existence of special local uniformizing variables in $O(T)$ for the prolongations $W^{(1)}, \cdots, W^{(g)}$ of a valuation V_p.

LEMMA 10. *There exist elements* $t_1, \cdots, t_g$ *in* $O(T)$ *such that* $W^{(i)}t_i=1$, $W^{(j)}t_i=0$, $j\neq i$; $1\leqq i, j\leqq g$.

PROOF. A result of Ostrowski [12] asserts the existence of elements $k_1, \cdots, k_g\in K$ with $W^{(i)}k_i=1$, $W^{(j)}k_i=0$, $j\neq i$, $1\leqq i, j\leqq g$. Suppose that $W_\infty^{(1)}$ is a pole of k_i with the conjugates $W_\infty^{(2)}, \cdots, W_\infty^{(s)}$. Pick $m_p\geqq -\min_{1\leqq\nu\leqq s}W_\infty^{(\nu)}k_i$. Construct the integers m_p for all poles of k_i and take, by (8), a function $b_i\in O(\Sigma')$ which has zeros of orders m_p at the projections V_p of the sets of conjugates $W_\infty^{(\nu)}$. Then $b_ik_i=t_i$ has the required properties.

Consequently:

(19) For any distinct rank one valuations W_1, W_2 of K there exist elements $a_1, a_2\in O(T)$ with $W_1a_1>0$, $W_2a_1=0$ and $W_1a_2=0$, $W_2a_2>0$.

As in §3, equation (10), we define for an ideal A of K with respect to $O(T)$

$$\mathfrak{W}A = \left\{\min_{a\in A} Wa = WA, \text{all } W\right\}. \tag{20}$$

Then $WA \neq 0$ for at most a countable infinitude W of valuations, such that $|a_\nu| \to \infty$ for the projections V_ν of the W_i, if there are infinitely many. Let $\mathfrak{W}$ be the additive group of vectors $\{m_W, \text{ all } W\}$ with integers m_W, such that at most a countable infinitude m_{W_i} are distinct from zero, where $|a_i| \to \infty$ for the corresponding projections to $O(\Sigma')$.

A direct computation shows that:

$$\mathfrak{W}(AB) = \mathfrak{W}(A) + \mathfrak{W}(B) \text{ for any two ideals } A, B. \tag{21}$$

Now put all ideals $A, A', \cdots$ with $\mathfrak{W}A = \mathfrak{W}A' = \cdots$ in a class $[A]$. As in §3 the classes $[A]$ form a multiplicative semigroup with $[O(T)]$ for a unit if we define $\mathfrak{W}[A] = \mathfrak{W}A$. More precisely:

THEOREM 4. *The classes of ideals $[A]$ form a multiplicative group which is isomorphically mapped in the vector group $\mathfrak{W}$ by $[A] \to \mathfrak{W}[A]$.*

PROOF. We may assume without loss of generality that the representative A of $[A]$ lies in $O(T)$. Then the ideal quotient A^{-1} has $WA^{-1} = -WA = -m$ for each W. Since $AA^{-1} \subseteq O(T)$ it suffices to exhibit an element $c \in A^{-1}$ with $Wc = -m$. Let $t = t_1$ be a local uniformizing variable, according to Lemma 10, such that $Wt = W^{(1)}t = 1$ and $W^{(2)}t = \cdots = W^{(g)}t = 0$ for the conjugates $W^{(2)}, \cdots, W^{(g)}$ of W. Suppose that $\{W_i\}$ are the valuations with $W_i t > 0$ and call V_i the associated projections on $F(\Sigma')$. Let W_{ij} be the conjugates of W_i. Now pick an integer $m_i \geqq \max_j W_{ij} t$. By (8) there exists a function $f \in O(\Sigma')$ which has zeros at the V_i with the multiplicities m_i. Let $b = tf^{-1}$, then $W(b) = 1$, $W^{(2)}(b) = \cdots = W^{(g)}(b) = 0$, $W_{ij}(b) < 0$ for all i and j, and finally $W'(b) = 0$ for all other valuations W'. Now let a be a nonzero element of A, then $Wa \geqq m$. Form ab^{-m}, then $W(ab^{-m}) \geqq 0$, $W^{(\nu)}(ab^{-m}) = W^{(\nu)}a \geqq 0$, $W_{ij}(ab^{-m}) = W_{ij}a - mW_{ij}b > 0$, $W'(ab^{-m}) = W'a \geqq 0$. Hence $ab^{-m} \in O(T)$ and therefore $c = b^{-m} \in A^{-1}$. Therefore $[A]\ [A^{-1}] = [O(T)]$ by (22).

It remains to exhibit for given $\{m_W\}$ an ideal A with $\mathfrak{W}A = \{m_W\}$. Let P_W be the prime ideal of the valuation W in K. We shall exhibit in $A = \bigcap_W P_W^{m_W}$ an element a with $Wa = m_W$ and $W'a \geqq m_W$ for all other W'. By Lemma 10 we can form the product $\prod_{i=1}^{g} t_i^{m_i} = d \in O(T)$, $m_i = m_{W_i}$, which has the prescribed orders at $W = W^{(1)}$ and its conjugates $W^{(i)}$. Next we determine in $O(\Sigma')$ an element g such that $Vg = 0$ for the projection of W and $V'g$ is sufficiently large for the projections of the other W' with $m_{W'} \neq 0$. Then $a = dg \in A$ and therefore $\mathfrak{W}A = \{m_W\}$.

As in (13) any two ideals A_1, A_2 of $[A]$ are related by an equation $A_1B_1 = A_2B_2$ where $B_1, B_2 \in [O(T)]$. Let $\bar{A}$ be the join of all ideals in

$[A]$. Then $\mathfrak{W}\bar{A} = \mathfrak{W}[A]$ and $\bar{A} = \bigcap_W P_W^{m_W}$ if $\mathfrak{W}[A] = \{m_W\}$. Suppose now that we topologize the ring $O(T)$ by using the intrinsic topology of the Riemann surface. Then:

$$\text{(22)} \qquad \bar{A} \text{ is a closed ideal}$$

and more precisely the only closed ideal in $[A]$. Using the generalization of Lemma 2 we find that all closed ideals of K with respect to $O(T)$ form a multiplicative group which is isomorphic to the vector group $\mathfrak{W}$.

We now turn to the discussion of the prime ideals of $O(T)$ and note first that all prime ideals are maximal as a consequence of Lemma 5. Suppose now that $\mathfrak{W}A = \{0\}$ for an ideal A of $O(T)$. We assert:

$$\text{(23)} \qquad \mathfrak{V}[A \cap O(\Sigma')] = \{0\}.$$

For the proof note that for given V with the prolongations $W^{(1)}, \cdots, W^{(s)}$ there exist elements $a_1, \cdots, a_s$ in A with

$$\begin{gathered} W^{(1)}a_1 = 0,\ W^{(2)}a_1 \geqq 0, \cdots, W^{(s)}a_1 \geqq 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ W^{(s)}a_s \geqq 0,\ W^{(2)}a_s \geqq 0, \cdots, W^{(s)}a_s = 0. \end{gathered}$$

We apply Lemma 10 and consider the sum, which lies in A,

$$\text{(24)} \qquad a = a_1(t_2 \cdots t_s) + \cdots + a_s(t_1 \cdots t_{s-1}).$$

Then by construction and the triangle inequality $W^{(1)}a = \cdots = W^{(s)}a = 0$. Consequently, by Lemma 9, $V(Na) = 0$ and $V'(Na) \geqq 0$ for all remaining V'. Thus $Na \in O(\Sigma')$. Applying (18) we have $(Na)a^{-1} \in O(T)$. Consequently $a[(Na)a^{-1}] = Na \in A$, whence $V(A \cap O(\Sigma')) = 0$. Therefore $\mathfrak{V}(A \cap O(\Sigma')) = \{0\}$ since V was chosen arbitrarily.

We notice next that there exist maximal ideals M in $O(T)$ with $\mathfrak{W}M = \{0\}$. Pick any $B \subset O(\Sigma')$ with $\mathfrak{V}B = \{0\}$. Then $BO(T) \subset O(T)$. Hence there exists, by Zorn's Lemma, at least one $M \supset BO(T)$ with the desired properties. These maximal ideals are not closed for otherwise $M \cap O(\Sigma')$ would be closed and hence equal to $O(\Sigma')$ by Lemma 1. Because $O(T)/M \supseteq O(\Sigma')/M \cap O(\Sigma')$, Lemma 6 implies that $O(T)/M \supset C$ characterizes the nonclosed maximal ideals.

We remark that (23) implies in particular that the intersections $P_W \cap O(T)$ are the only closed maximal ideals of $O(T)$. On the other hand the points of the Riemann surface T of K determine the valuations W of K and vice versa. Consequently we have the generalization of Theorem 2:

The points of T are in 1-1 *correspondence with the closed maximal ideals of the ring $O(T)$ of all analytic functions on T.*

REMARK. In conclusion we point out that the methods of this section apply also to the ideal theory of rings of *quasi meromorphic functions* [9]. Such functions are meromorphic functions on Σ which admit essential singularities at a finite number of points $p_\infty, p_1, \cdots, p_k$; in other words, quasi meromorphic functions have locally the character of rational functions on Σ with the possible exception of the points $p_\infty, p_1, \cdots, p_k$. To obtain a reduction to the case of the ordinary entire functions suppose that S is an arbitrary (compact) algebraic Riemann surface with the valuations $\{W_S\}$. Pick any h valuations $W_{S,1}, \cdots, W_{S,h}$. By the Riemann-Roch theorem there exists a function u in the field $F(S)$ of meromorphic functions on S with poles at $W_{S,1}, \cdots, W_{S,h}$. We consider $F(S)$ as an algebraic extension of $C(u)$. The Riemann surface of $C(u)$ is the complex u-sphere and the field of quasi meromorphic functions on $S-\{W_{S,1}, \cdots, W_{S,h}\}$ contains $F(S)$ and is an algebraic extension of degree $[F(S):C(u)]$ over the field of meromorphic functions on the u-sphere from which the common projection of the $W_{S,\nu}$ is removed. To obtain the ideal theory of the ring of quasi analytic functions on $\Sigma-\{p_\infty, p_1, \cdots, p_k\}$ we apply the preceding arguments to $F(\Sigma)=C(z)$.[3] Finally, the principal ideal theorem (Lemma 3) for the closed ideals holds in this case. Similarly Theorem 2 can be extended.

It may be interesting to see which results of the theory of prolongations can be extended to relations between the ideals in the various rings of continuous functions on completely regular spaces and their finitely sheeted covering spaces. It appears that such problems have not been discussed to any extent.

6. The principal ideal theorem. We assume for the moment that S is a compact Riemann surface. Suppose that Ω_v is the ring of all functions in $F(S)$ which are integral over a polynomial ring $C[v]$, $F(S)\supset C[v]$. Then all ideals of $F(S)$ with respect to Ω_v are principal if and only if S has genus 0; this follows immediately from Abel's theorem and Jacobi's inversion theorem [2, 17].

We now show that, in contrast to the preceding compact case, the principal ideal theorem holds for the closed ideals on certain open Riemann surfaces. The existence of integral functions with prescribed zeros depends on a combination of the proof for (8) with the methods leading to Abel's theorem.

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[3] Quasi analytic functions on $\Sigma-\{p_\infty, p_1, \cdots, p_k\}$ are quasi meromorphic functions without poles.

We shall assume in the sequel that K is a field of algebroid functions as defined in §5. Thus, K is the set of all meromorphic functions on a finitely many sheeted covering surface T of Σ'. We agree that the elements of $F(\Sigma')$ are given as functions of the variable z.

Let $O(T)$ be the ring of integral functions over $O(\Sigma')$ as described in the preceding section. Weierstrass' formula (8) uses the existence of a uniformizing variable t_p in $O(\Sigma')$ for each point $p\in\Sigma'$ such that $t_p(q)\neq 0$ for all other points $q\in\Sigma'$. In the general case we first define a *differential* k_1dk for k, $k_1\in K$ as follows [6]:

(25) *k_1dk is the totality of all local derivatives k_1dk/dt_W for all local uniformizing variables t_W of W and all valuations W belonging to the points of T.*

If we use the completions K_W of K with respect to W it follows that $W(dz/dt_W)=W(dz)=e-1$ where e is the degree of ramification of W relative to its projection V on $F(\Sigma')$. In other words, dz is at W the local different of $K_W/F(\Sigma')_v$. As in the theory of algebraic functions we say that a differential ydz, $y\in K$, has a zero of order m at W if $W(ydz)=m$; a pole of order m is present if $W(ydz)=-m$, $m>0$. Moreover, the zeros and poles W_ι of a differential ydz of K have projections V_{a_ι} with $|a_\iota|\to\infty$ if there are infinitely many.

We now construct a *differential $d\omega_W$ of K which has a logarithmic singularity with the residue $+1$ at a given valuation W.* Later $d\omega_W$ shall be used to construct the analogue of the prime functions $z-a$, $a\in C$. Suppose that $a_1, \cdots, a_n$ are n linearly independent elements of $O(T)$ over $F(\Sigma')$; we may take $a_j=k^{j-1}$ where $k\in O(T)$ is a primitive element of $K/F(\Sigma')$. Let $a_j^{(h)}$, $1\leqq h\leqq n$, be the n conjugates of a_j. Then the system of linear equations

$$(26)\qquad \sum_{h=1}^{n} b_i^{(h)}a_j^{(h)}=\delta_{ij},\qquad \delta_{ij}\text{ the Kronecker symbols,}$$

has n conjugate sets of solutions $b_1^{(h)}, \cdots, b_n^{(h)}$, for the independence of the a_j's implies $\det|a_j^{(h)}|\neq 0$. Let $b_i^{(1)}=b_i$, $1\leqq i\leqq n$. We may assume without loss of generality that $z-c\neq 0 \pmod W$ for the given valuation W of K. Since W corresponds to a point of T we agree to denote the value of a function $f\in K$ at W by $f(W)$.[4] Consider the differential $d\omega_W^0=\sum_{i=1}^{n}b_i[a_i(W)]dz/(z-c)$. Then $W(d\omega_W^0)=-1$ is a consequence of the construction of the b_i's and the remarks on dz; we apply the homomorphism determined by W.[5] The differential $d\omega_W^0$

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[4] In other words, the function-theoretic value $f(W)$ is essentially the residue class of f modulo W.

[5] Note $\sum b_i(W)a_i(W)=1$ and $W(dz)=0$.

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may of course have other singularities. We first remove the potential poles at the conjugates $W^{(h)}$ of $W^{(1)}=W$. Suppose $W^{(h)}(d\omega_W^0)=-m_h$, $m_h\geqq 0$. By Lemma 10, the differential $d\omega_W^0\prod_{h\neq 1}t_h^{m_h}=d\omega_W^{00}$ has $W^{(h)}(d\omega_h^{00})=0$ for $h\neq 1$ and $W(d\omega_W^{00})=-1$. The possible poles W_i of $d\omega_W^{00}$ form an at most denumerable set with $|a_i|\to\infty$, if the V_{a_i} are the projections of the W_i. We next pick an integral function $g\in O(\Sigma')$ which does not vanish at W and its conjugates, that is, $V(g)=0$ for the contraction of W to $F(\Sigma')$. Moreover, g is to have zeros of sufficiently high orders at the V_{a_i}. This can be done according to Weierstrass' theorem. Then $W(gd\omega_W^{00})=-1$ and $W'(gd\omega_W^{00})\geqq 0$ for all other valuations. Finally, we multiply $gd\omega_W^{00}$ by a suitable constant and obtain a differential $d\omega_W$ with the desired properties.

Now let $I(W)$ be the indefinite integral $\int d\omega_W$. Then the exponential $e^{I(W)}$ is a multiplicative function on T which has locally the properties of a meromorphic function on T.[6] In particular $W(e^{I(W)})=-1$ and $W'(e^{I(W)})=0$ for all other valuations W'. However, in general, $e^{I(W)}$ will not be a function of K. We show next that $e^{I(W)}$ can be normalized so as to lie in K provided a certain hypothesis is valid for the vector space of everywhere finite integrals of K.

Let $H^1(K)=\{dv,\ W(dv/dt_W)\geqq 0$ for all $W\}$ be the *vector space of everywhere finite differentials* with complex coefficients. Suppose that $H_1(K)=\{S,\ \cdots\}$ is the *Betti group* of T. Then the integrals $\int_S dv=(S,\ dv)$ define for fixed dv a homomorphism of $H_1(K)$ into C and dually for fixed S a homomorphism of $H^1(K)$ into C since homologous closed curves determine the same definite integral. Let ϕ be a homomorphism of $H_1(K)$ into C. Then it is a well known fact that ϕ cannot always be realized as $(S,\ dv)$ for a suitable dv if integrals on an algebraic compact Riemann surface are considered [2].

We assume the following hypothesis:

(H) *Given a homorphism ϕ of $H_1(K)$ into C, then $\phi=(S,\ dv)$, $S\in H_1(K)$ for a suitable differential $dv\in H^1(K)$.*

Later we shall see that our hypothesis can be proved for certain open Riemann surfaces.

Let $\{\pi_S\}$, $S\in H_1(K)$, be the periods of the normalized integral $\int d\omega_W$ which was constructed at the beginning of this section. Then there exists by (H) an everywhere finite differential dv with $(S,dv)=\pi_S$ for all S. Hence $\int(d\omega_W-dv)=\tau_W$ has no periods on T and

$$e^{\tau_W}=t_W \tag{27}$$

is a uniformizing variable for W with $W'(t_W)=0$ for all other valuations W'. It is now easy to show that each closed ideal A of K with respect to $O(T)$ is principal, provided A has only a finite number of zeros and poles W_i with the multiplicities m_i. We form the function $a = \prod t_{W_i}^{m_i}$. Then $\mathfrak{W}a = \mathfrak{W}A$ and therefore $A = aO(T)$. If A has infinitely many zeros and poles then A can still be shown to be principal, but this time the functions t_{W_i} have to be modified so that their rate of growth is suitably bounded as $z \to \infty$; this can be achieved by a combination of the methods in [11] and [13].

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[6] A multiplicative function f on T reproduces itself but for a constant factor μ_S if it is continued analytically along a closed path S on T.

We now formulate the following theorem.

THEOREM 5. *Let T be the open Riemann surface which is obtained from a compact Riemann surface by omitting a point. Then all closed ideals of meromorphic functions on T with respect to the ring of analytic functions on T are principal. Each closed ideal is uniquely determined by its zeros and poles and their respective multiplicities. The set of zeros and poles of an ideal is at most countable and contains no infinite subsequence which is compact in T.*

REMARK. The statement of Theorem 5 can be extended to open Riemann surfaces which arise from algebraic Riemann surfaces by omitting more than one point. We note that an analogue to Weierstrass' theorem (8) holds.

We now return to hypothesis (H) and indicate briefly how it can be proved for surfaces T which arise from a compact algebraic Riemann surface T_0 by omitting a point $\overline{W}$, $T = T_0 - \{\overline{W}\}$ [12]. Let K_0 be the field of algebraic functions of T_0 and assume that g is the genus. Suppose that $S_1, \cdots, S_g, T_1, \cdots, T_g$ is a set of canonical retrosections of T_0. We may assume, after a suitable deformation of the cuts that W does not lie on any of them. The everywhere finite differentials du_i can now be selected such that, for $\det |\tau_{ki}| \neq 0$,

$$(28) \qquad (S_i, du_i) = \pi(-1)^{1/2}, \quad (S_j, du_i) = 0, \quad (T_k, du_i) = \tau_{ki}.$$

Using the gap theorem of Weierstrass [2], we remark that precisely g integers cannot occur as multiplicities of poles of functions (on K_0) at $\overline{W}$. Since there exist, however, differentials which are poles of any order exclusively at $\overline{W}$, we can pick precisely g differentials $du_{g+1}, \cdots, du_{2g}$ of K_0 whose multiplicities at $\overline{W}$ are the g missing orders for the functions. Using (28) we can subtract from $du_{g+1}, \cdots, du_{2g}$ suitable linear combinations of the differentials $du_1, \cdots, du_g$ and obtain differentials $du'_{g+1}, \cdots, du'_{2g}$ such that

(i) the multiplicities at $\overline{W}$ are preserved,

$$(29) \quad \text{(ii)} \qquad \begin{aligned} (S_k, du'_{g+j}) &= 0 \qquad \text{for } 1 \leq k, j \leq g, \\ (T_k, du'_{g+j}) &= \tau'_{kj}. \end{aligned}$$

The g differentials du'_{g+k} are independent over C, for otherwise there would exist a nontrivial linear combination $\int\sum_{j=1}^{g} du'_{g+j}c_j = f$, $c_j \in C$, without periods. Hence f would be a function of K_0 with a forbidden multiplicity at $\overline{W}$. Hence det $|\tau'_{kj}| \neq 0$. We can therefore replace du'_{g+j}, $1 \leqq j \leqq g$, by suitable linear combinations $dv_{g+1}, \cdots, dv_{2g}$ such that

$$(30) \qquad \begin{aligned} (S_k, dv_{g+j}) &= 0 \qquad\qquad \text{for } 1 \leqq k, j \leqq g, \\ (T_k, dv_{g+j}) &= \delta_{kj}\pi(-1)^{1/2}. \end{aligned}$$

Finally we add appropriate linear combinations of $dv_{g+1}, \cdots, dv_{2g}$ to the differentials $du_1, \cdots, du_g$ and obtain $2g$ independent differentials $dv_1, \cdots, dv_g, dv_{g+1}, \cdots, dv_{2g}$ with

$$(31) \qquad \begin{aligned} (S_k, dv_i) &= \delta_{ki}\pi(-1)^{1/2}, \ (T_k, dv_i) = 0, \qquad 1 \leqq h, i, k \leqq g, \\ (S_k, dv_{g+j}) &= 0, \ (T_k, dv_{g+j}) = \delta_{kj}\pi(-1)^{1/2}, \qquad 1 \leqq h, j, k \leqq g. \end{aligned}$$

These $2g$ differentials are now considered on $T = T_0 - \overline{W}$. Since we do not count $\overline{W}$, it turns out that our differentials are everywhere finite on T. We observe that the cuts S_i, T_i can be placed in a region of T whose projection on Σ' is given by $|z| \leqq \rho$. Therefore each $S \in H_1(K) = H_1(T)$ is an integral combination of the S_i, T_i. Suppose then that the homomorphism ϕ is given. Then it suffices to know $\phi(S_i)$, $\phi(T_i)$ for then $\phi(S) = \sum n_i\phi(S_i) + \sum n_i'\phi(T_i)$ if $S \sim \sum n_i S_i + \sum n_i' T_i$. Hence $\phi(S) = (S, dv)$ for a suitable dv as a consequence of (31) and hypothesis (H) is shown to hold.

A further instance of the validity of (H) was recently exhibited by Myrberg [11], who considered $K = F(\Sigma')(f(z))^{1/2}$ where $f(z)$ is an entire function with infinitely many zeros. We remark that Myrberg's device of approximating K by hyperelliptic fields can be generalized.

We note in conclusion that the methods used to prove Abel's theorem can be extended to prove a generalized Abel's theorem for meromorphic functions of finite order. The last restriction gives rise to congruences describing the necessary and sufficient conditions for a given set of zeros and poles to belong to a function of finite order [13].

7. Unramified extensions. The function theoretic methods of the preceding section can be further expanded to discuss the unramified finite algebraic extensions L/K in which each valuation W of K has $[L:K]$ distinct prolongations. It can be shown that the unramified abelian extensions are generated by radicals like $(k)^{1/n}$ where k has zeros and poles on T.

More generally, fields of quasi meromorphic functions $\overline{K}$ arising from $T_0 - \{\overline{W}, \overline{W}_1, \cdots, \overline{W}_s\}$ can be treated with our method.

Furthermore, the methods of the theory of algebraic functions can be applied directly to prove that a finite group G can be realized as the Galois group of a normal unramified extension L/K if and only if G is a homomorphic image of the Poincaré group $P(\overline{K}) = \{S_1, \cdots, S_g, T_1, \cdots, T_g, U_1, \cdots, U_s; \prod_{i=1}^{g} S_i T_i S_i^{-1} T_i^{-1} \prod_{j=1}^{s} U_j = 1\}$. We note that the roots $(t_j)^{1/n}$ are unramified functions over $\overline{K}$, where the t_j are suitable uniformizing variables for the $\overline{W}_j$.

Finally, the analogue of the theory of algebraic correspondences can be established in a variety of cases. It may be worthwhile to compare the results of H. Cartan [4] with the interpretation of meromorphic correspondences by infinite matrices arising in the fields considered by P. J. Myrberg [11].

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THE SUMMER MEETING IN ITHACA

The fifty-second Summer Meeting of the Society was held at Cornell University, Ithaca, New York, Tuesday through Friday, August 20–23, 1946. The Mathematical Association of America met on August 19–20 and the Institute of Mathematical Statistics on August 22–23. Over five hundred people attended the meeting among whom were the following three hundred ninety-nine members of the Society:

C. R. Adams, R. B. Adams, J. G. Adshead, R. P. Agnew, A. A. Albert, P. L. Alger, E. B. Allen, C. B. Allendoerfer, Warren Ambrose, T. W. Anderson, R. C. Archibald, R. F. Arens, Silvio Aurora, M. C. Ayer, W. L. Ayres, E. G. Baker, D. H. Ballou, E. W. Barankin, J. L. Barnes, P. T. Bateman, G. E. Bates, H. P. Beard, R. L. Beinert, Stefan Bergman, D. L. Bernstein, Felix Bernstein, Lipman Bers, Garrett Birkhoff, C. J. Blackall, David Blackwell, Gertrude Blanch, E. E. Blanche, R. P. Boas, Paul Boschan, S. G. Bourne, J. W. Bower, A. H. Bowker, C. C. Bramble, R. W. Brink, A. A. Brown, B. L. Brown, H. E. Buchanan, E. F. Buck, R. C. Buck, E. L. Buell, R. S. Burington, M. L. Burke, F. J. H. Burkett, Herbert Busemann, W. H. Bussey, K. E. Butcher, S. S. Cairns, W. D. Cairns, R. H. Cameron, C. C. Camp, H. H. Campaigne, W. B. Campbell, G. F. Carrier, I. S. Carroll, K. C. Cartwright, W. B. Carver, T. E. Caywood, W. F. Cheney, Herman Chernoff, C. E. Clark, J. A. Clarkson, M. D. Clement, G. R. Clements, I. S. Cohen, L. W. Cohen, Nancy Cole, R. H. Cole, J. B. Coleman, M. A. Coler, J. A. Cooley, Byron Cosby, N. A. Court, A. P. Cowgill, W. H. H. Cowles, V. F. Cowling, M. J. Cox, H. S. M. Coxeter, H. B. Curry, J. H. Curtiss, D. R. Davis, M. M. Day, F. F. Decker, L. S. Dederick, J. L. Doob, H. L. Dorwart, C. H. Dowker, Y. N. Dowker, Arnold Dresden, W. L. Duren, Jacques Dutka, P. S. Dwyer, L. A. Dye, E. D. Eaves, Samuel Eilenberg, Benjamin Epstein, Bernard Epstein, Paul Erdös, W. S. Erickson, Luis Esteban-Carrasco, H. S. Everett, Herbert Federer, Will Feller, A. D. Fialkow, F. A. Ficken, N. J. Fine, C. D. Firestone, K. W. Folley, L. R. Ford, W. B. Ford, R. M. Foster, A. H. Fox, J. S. Frame, M. R. Freundlich, Orrin Frink, R. E. Fullerton, M. G. Gaba, C. A. Garabedian, H. L. Garabedian, H. M. Gehman, Abe Gelbart, B. H. Gere, J. J. Gergen, J. H. Giese, B. P. Gill, M. A. Girshick, J. W. Givens, A. M. Gleason, H. E. Goheen, Michael Goldberg, Samuel Goldberg, Michael Golomb, M. O. González-Rodriguez, A. W. Goodman, R. O. Goodman, M. J. Gottlieb, S. H. Gould, F. G. Gravalos, Lewis Greenwald, V. G. Grove, N. G. Gunderson, W. W. Gutzman, Theodore Hailperin, P. R. Halmos, R. W. Hamming, W. J. Harrington, B. I. Hart, C. M. Hebbert, M. H. Heins, R. G. Helsel, J. G. Herriot, Fritz Herzog, M. R. Hestenes, E. H. C. Hildebrandt, T. H. Hildebrandt, Einar Hille, M. P. Hollcroft, T. R. Hollcroft, R. H. Hoskins, Harold Hotelling, S. E. Hotelling, R. C. Huffer, H. K. Hughes, E. M. Hull, Ralph Hull, Witold Hurewicz, W. A. Hurwitz, L. C. Hutchinson, T. J. Jaramillo, R. L. Jeffery, E. D. Jenkins, S. A. Jennings, Evan Johnson, R. E. Johnson, R. F. Johnson, A. W. Jones, B. W. Jones, Harris Jones, L. G. Jones, P. S. Jones, R. C. Jones, H. A. Jordan, Mark Kac, G. K. Kalisch, Irving Kaplansky, William Karush, J. L. Kelley, A. J. Kempner, D. E. Kibbey, W. M. Kincaid, J. R. Kline, T. L. Koehler, C. F. Kossack, H. L. Krall, G. A. Larew, C. G. Latimer, V. S. Lawrence, M. A. Lee, Solomon Lefschetz, Joseph Lehner, Walter Leighton, Howard Levene, W. J. LeVeque, Norman Levinson, Harry Levy, F. A. Lewis, C. C. Lin, Charles Loewner, Z. L.

Loflin, E. R. Lorch, Lee Lorch, A. N. Lowan, L. L. Lowenstein, C. I. Lubin, R. K. Luneburg, R. C. Lyndon, V. O. McBrien, Dorothy McCoy, N. H. McCoy, S. W. McCuskey, E. A. McDougle, A. W. McMillan, Brockway McMillan, E. J. McShane, J. K. L. MacDonald, C. C. MacDuffee, G. W. Mackey, G. R. MacLane, Saunders MacLane, M. S. Macphail, Dis Maly, Morris Marden, A. M. Mark, Ella Marth, M. H. Martin, D. C. May, A. E. Meder, L. E. Mehlenbacher, N. S. Mendelsohn, W. A. Mersman, A. N. Milgram, E. D. Miller, Norman Miller, Knox Millsaps, J. M. Mitchell, E. C. D. Molina, C. N. Moore, T. W. Moore, F. V. E. Morfoot, K. A. Morgan, Vladimir Morkovin, Richard Morris, A. P. Morse, D. S. Morse, Marston Morse, J. E. Morton, I. R. Moses, C. G. Mumford, M. E. Munroe, C. W. Munshower, W. R. Murray, J. R. Musselman, A. L. Nelson, E. D. Nering, Morris Newman, C. V. Newsom, R. E. O'Connor, E. G. Olds, L. F. Ollmann, F. W. Owens, H. B. Owens, S. T. Parker, W. V. Parker, W. H. Pell, P. M. Pepper, R. M. Peters, George Piranian, Everett Pitcher, Harry Pollard, Hillel Poritsky, William Prager, G. B. Price, A. L. Putnam, Tibor Radó, C. H. Rawlins, C. J. Rees, Mina Rees, Irving Reiner, Eric Reissner, C. N. Reynolds, R. G. D. Richardson, E. K. Ritter, R. B. Robbins, H. P. Robertson, J. B. Robinson, R. M. Robinson, S. L. Robinson, W. J. Robinson, P. C. Rosenbloom, Arthur Rosenthal, J. B. Rosser, P. J. Rulon, Hans Samelson, Arthur Sard, F. E. Satterthwaite, S. A. Schaaf, A. C. Schaeffer, M. A. Scheier, O. F. G. Schilling, E. R. Schneckenburger, Lowell Schoenfeld, K. C. Schraut, Abraham Schwartz, G. E. Schweigert, C. H. W. Sedgewick, R. W. Sedgewick, C. E. Seely, I. E. Segal, M. E. Shanks, I. M. Sheffer, Seymour Sherman, D. M. Smiley, M. F. Smiley, C. V. L. Smith, F. C. Smith, G. W. Smith, H. W. Smith, J. C. Smith, P. A. Smith, W. M. Smith, Andrew Sobczyk, Herbert Solomon, D. E. Spencer, V. E. Spencer, E. P. Starke, N. E. Steenrod, F. F. Stephan, F. M. Stewart, E. C. Stokes, R. W. Stokes, R. R. Stoll, M. H. Stone, R. C. Strodt, W. C. Strodt, A. C. Sugar, J. L. Synge, Otto Szász, M. E. Taylor, J. M. Thomas, R. M. Thrall, W. J. Thron, Leonard Tornheim, J. I. Tracey, W. R. Transue, W. J. Trjitzinsky, C. A. Truesdell, Y. W. Tschen, A. W. Tucker, J. W. Tukey, A. R. Turquette, F. E. Ulrich, H. E. Vansant, Abraham Wald, G. L. Walker, G. W. Walker, R. J. Walker, S. E. Walkley, Henry Wallman, J. L. Walsh, J. B. Walton, C. W. Watkeys, J. V. Wehausen, Alexander Weinstein, F. P. Welch, C. P. Wells, E. T. Welmers, G. W. Whitehead, E. A. Whitman, P. M. Whitman, W. F. Whitmore, Hassler Whitney, G. T. Whyburn, D. V. Widder, V. A. Widder, A. S. Wightman, J. E. Wilkins, S. S. Wilks, G. M. Wing, František Wolf, H. A. Wood, P. M. Young, J. W. T. Youngs, Oscar Zariski, J. A. Zilber, Antoni Zygmund.

On Tuesday afternoon, Wednesday, Thursday and Friday mornings, Professor Hassler Whitney of Harvard University gave the Colloquium Lectures on *Topology of smooth manifolds*. President T. H. Hildebrandt and Vice Presidents J. M. Thomas, L. R. Ford and Saunders MacLane presided in turn at these lectures.

Professor M. H. Stone of the University of Chicago on Wednesday morning gave his retiring presidential address entitled *Some general principles of functional representation*. President Hildebrandt presided.

On Thursday afternoon Professor J. L. Doob of the University of Illinois gave an address on *Probability in function space*. Professor S. S. Wilks presided.

Presiding officers for the sessions of contributed papers were: Topology, Tuesday afternoon, Professor G. T. Whyburn; Applied Mathematics, Tuesday afternoon, Dr. Hillel Poritsky; Statistics and Probability, Thursday morning, President W. G. Cochran of the Institute of Mathematical Statistics; Algebra and Theory of Numbers, Thursday morning, Professor Samuel Eilenberg; Analysis, Thursday afternoon, Professor M. R. Hestenes; Geometry, Thursday afternoon, Professor Oscar Zariski; Analysis, Friday morning, Professor G. B. Price; Algebra and Topology, Friday morning, Professor S. S. Cairns.

There was a business meeting Wednesday morning at which President T. H. Hildebrandt presided.

The general sessions were all held in 200 Baker Laboratory and the sections in this room and in 7 Baker Laboratory.

Registration headquarters were in Risley Hall. Balch and Risley dormitories were available to those attending the meetings. Meals were served in Willard Straight Hall and Martha Van Rensselaer Hall.

The ladies of the Cornell University Department of Mathematics served tea on Monday and Thursday afternoons.

On Tuesday afternoon the members of the mathematical organizations, their families and guests, were entertained at tea at the home of Professor Emeritus W. B. Ford of the University of Michigan and Mrs. Ford. The Fords have a beautiful home on the west shore of Cayuga Lake about twenty-five miles north of Ithaca. Tea was served on the porches and lawn overlooking the lake. Both the setting and the weather were ideal.

On Tuesday evening the Walden String Quartette entertained the mathematicians and their friends with a delightful program.

A group photograph was taken at 1:30 P.M., Wednesday.

For the Wednesday afternoon outing, a choice was given of either dining at The Krebs or going on a picnic. The Krebs is a famous restaurant at Skaneateles, New York. The picnic was held in Robert H. Treman State Park, Upper Enfield Glen.

The dinner for the three organizations was served in the Ivy Room and adjacent rooms of Willard Straight Hall. After dinner, the group adjourned to Memorial Hall for the program of the evening. The toastmaster, Professor C. B. Allendoerfer, introduced Provost Adams of Cornell University who gave an address of welcome, following which brief addresses were given by Professors P. A. Smith, L. R. Ford and Harold Hotelling. Professor C. C. Camp proposed a resolution, unanimously approved, expressing the appreciation of the members of the three organizations to the President and administration of Cornell

University, Professor and Mrs. W. B. Ford, the local committee and all who assisted them for their excellent arrangements and cordial hospitality.

The Council met at 8:15 P.M. on August 20, 1946, and at 2:15 P.M. on August 21, 1946, in Willard Straight Hall.

The Secretary announced the election of the following forty persons to membership in the Society:

Mr. Richard Davis Anderson, University of Texas;
Mr. Howard Clinton Arnold, Federal Enamel and Stamping Co., Pittsburgh, Pa.;
Mr. Albert Abraham Blank, Brown University;
Professor James Lawrence Botsford, Reed College;
Mrs. Joan Robinson Clark, Brown Instrument Co., Philadelphia, Pa.;
Mr. Francis Eugene Cothran, Los Angeles, Calif.;
Mr. William John Culmer, Radio Corporation of America, Indianapolis, Ind.;
Mr. William Carl Davis, Stockham Pipe Fittings Co., Birmingham, Ala.;
Mrs. Georgia Knox Del Franco, University of Miami;
Professor Flora Dinkines, University of South Carolina;
Professor Luis Esteban-Carrasco, University of Madrid;
Miss Charlotte Gladys Dragstedt, University of Chicago;
Mr. Bertram Morris Drucker, University of North Carolina;
Mr. Joseph H. Engel, New York, N. Y.;
Mrs. Norma MacLeod Gilbert, New York, N. Y.;
Mr. Oscar Goldman, Palmer Physical Laboratory, Princeton University;
Professor Edison Greer, Kansas State College;
Mr. Cornelius Groenewoud, Lieutenant, U.S.A.;
Mr. Norman Tyson Hamilton, Princeton University;
Mr. Alan Jerome Hoffman, New York, N. Y.;
Mr. Chuan Chih Hsiung, Michigan State College;
Mr. Ralph Bernard Johnson, University of Tennessee;
Mr. Rufus Alexander Koerting, University of Wisconsin;
Mr. Benjamin Lepson, Columbia University;
Mr. Edgar Linton, Kansas City, Mo.;
Professor Joe James Livers, Montana State College, Bozeman, Mont.;
Mrs. Chin Hua Liu Lu, Boston, Mass.;
Mr. Gregory Joseph Mann, Wright Junior College;
Mr. Bernard Mason, New York, N. Y.;
Dr. Eric M. Michalup, Caracas, Venezuela;
Professor Raymond David Mindlin, Department of Civil Engineering, Columbia University;
Miss Gabrielle Marcelle Morin, Columbia University;
Professor Rosa Catherine Paschal, Anderson College, Anderson, S. C.;
Dr. Roy Bly Saunders, University of Minnesota;
Mr. I. Richard Savage, Detroit, Mich.;
Dr. Henry Maksymiljan Schaerf, Montana State College;
Mr. William Raymond Scott, Ohio State University;
Mr. Robert Rex Seeber, Jr., Watson Scientific Computing Laboratory, New York, N. Y.;
Professor Kenichi Watanabe, University of Hawaii;
Mr. Albert Wilansky, Brown University.

The following appointments by President T. H. Hildebrandt were reported: Professors B. W. Jones (Chairman), T. R. Hollcroft, W. T. Martin, J. B. Rosser, and R. J. Walker as a Committee on Arrangements for the 1946 Summer Meeting at Cornell University; Professors Arnold Dresden (Chairman), H. W. Brinkmann, J. A. Clarkson, T. R. Hollcroft, B. W. Jones, C. O. Oakley, and J. C. Oxtoby as a Committee on Arrangements for the 1946 Annual Meeting at Swarthmore College; Professors A. W. Tucker (Chairman), R. H. Fox, T. R. Hollcroft, and Marston Morse as a Committee on Arrangements for the fall eastern meeting at Princeton University on November 2, 1946; Professor O. E. Neugebauer as the representative of the Society at a possible conference for discussion of an abstract journal in applied mechanics; Professor A. W. Tucker as a member of the Committee on Aid to Devastated Libraries (to replace Professor Solomon Lefschetz); Professor J. L. Synge as the representative of the Society on the Committee on American Participation for the Sixth International Congress for Applied Mechanics, to be held in Paris on September 22–29, 1946.

The appointment by the President of Professor Philip Franklin as the representative of the Society on the Council of the American Association for the Advancement of Science for the year 1946 was confirmed.

Times and places of meetings during 1947 were set as follows: February 22, April 25–26 and October 25 in New York City; April 25–26 in Chicago; April 26 at Stanford University.

It was reported that Professor D. C. Spencer had been invited to deliver an address at the November, 1946, meeting in Los Angeles, California, and that Professor S. E. Warschawski had been invited to deliver an address at the November, 1946, meeting in Ames, Iowa.

On recommendation of the Committee on Places of Meetings, the Council voted to hold the 1947 Summer Meeting at Yale University and the 1947 Annual Meeting at the University of Georgia.

On the recommendation of the Colloquium Editorial Committee, the Council voted to invite Professor T. Y. Thomas to deliver a series of Colloquium Lectures in 1948.

The resignation of Associate Secretary W. L. Ayres was accepted with regret. On recommendation of the Committee on Nominations, the Council elected Professor G. B. Price to fill the unexpired term of Professor Ayres, this term to expire December 31, 1947. (This election was later confirmed by the Board of Trustees.)

Professors Solomon Lefschetz and G. T. Whyburn were appointed representatives of the Society on the Council of the American Association for the Advancement of Science for the year 1947.

Professors R. V. Churchill and M. R. Hestenes were nominated as representatives of the Society in the Division of Physical Sciences of the National Research Council for the three-year period beginning July 1, 1947.

The committee which had been studying problems connected with the publication of the complete works of the late Professor G. D. Birkhoff reported to the Council. The committee was requested to re-study certain aspects of the question in the light of Council discussion.

The Council voted to invite Professor P. M. Morse of Massachusetts Institute of Technology to deliver the twenty-first Josiah Willard Gibbs Lecture at the 1947 Annual Meeting.

The Council voted to invite Professor Heinz Hopf of the Federal Institute of Technology in Zurich, Switzerland, as Visiting Lecturer for the academic year 1946–1947. (Professor Hopf found it impossible to accept this invitation.)

Professor M. H. Stone reported for the Policy Committee for Mathematics that the committee had discussed in detail a number of problems which face the mathematical profession in the post-war period. Prominent among these are: the various science bills before Congress; atomic research, particularly as it affects freedom of scientific investigation; the various changes in Selective Service regulations; international cooperation problems raised by the United Nations Educational, Scientific and Cultural Organization. A report of the activities of this committee for the period October 1, 1945–June 30, 1946, was submitted during the summer of 1946 to the Rockefeller Foundation, which has been supporting the work of the committee.

Professor Arnold Dresden reported for the Committee on Aid to Devastated Libraries that a request for members of the Society to contribute books, journals and cash to aid such libraries was to be inserted in a forthcoming issue of the Bulletin. The Council voted, on recommendation of this committee, to make certain recommendations to the Board of Trustees in connection with contributions of the Society to this project, to discharge the committee, and to appoint a committee of three to work out the proper distribution of the material collected in response to the Bulletin notice.

The Secretary was authorized to arrange a reciprocity agreement with the Société Mathématique de France similar to agreements with other societies abroad.

The Council authorized the President to appoint a committee to consult with the editors of the Annals of Mathematics regarding the publication problems of that journal.

The President was authorized by the Council to appoint a committee to consider the relation of the Society to the various groups of applied mathematicians.

The question of the United Nations Educational, Scientific and Cultural Organization, as it relates to international mathematical organization, was referred to the Policy Committee for Mathematics for discussion.

Titles and cross references to the abstracts of papers read are given below: papers 1–4 in the section for Topology, Tuesday afternoon; papers 5–9 in the section for Applied Mathematics, Tuesday afternoon; papers 10–16 in the section for Statistics and Probability, Thursday morning; papers 17–23 in the section for Algebra and Theory of Numbers, Thursday morning; papers 24–31 in the section for Analysis, Thursday afternoon; papers 32–39 in the section for Geometry, Thursday afternoon; papers 40–48 in the section for Analysis, Friday morning; papers 49–57 in the section for Algebra and Topology, Friday morning; papers 58–111, whose abstract numbers are followed by the letter *t*, were read by title. Paper 1 was read by Professor Kelley, paper 28 by Professor Hughes, paper 32 by Professor V. G. Grove for the author, paper 51 by Professor Birkhoff, paper 54 by Dr. Poritsky, paper 57 by Professor Eilenberg. Sister Petronia was introduced by Professor Karl Menger and Professor Ancochea was introduced by Professor Oscar Zariski.

1. R. F. Arens and J. L. Kelley: *Characterization of the space of continuous functions.* (Abstract 52-7-258.)

2. Herbert Federer: *Dimension and measure.* (Abstract 52-9-340.)

3. J. W. T. Youngs: *The representation problem for Fréchet surfaces.* Preliminary report. (Abstract 52-9-346.)

4. Felix Bernstein: *A new four-color problem.* (Abstract 52-7-259.)

5. A. C. Sugar: *On the relaxation-matrix method of solving boundary value problems.* (Abstract 52-7-248.)

6. Alexander Weinstein: *On the method of sources and sinks.* (Abstract 52-9-317.)

7. William Prager: *On the variational principles of plasticity.* (Abstract 52-9-313.)

8. C. A. Truesdell: *On Behrbohm and Pinl's linearization of the equation of two-dimensional steady flow of a compressible adiabatic fluid.* (Abstract 52-9-316.)

9. S. A. Schaaf: *On the superposition of a heat source and a contact resistance.* (Abstract 52-7-246.)

10. Howard Levene: *A test of randomness in two dimensions.* (Abstract 52-7-257.)

11. Herman Rubin: *Asymptotic distribution of moments from a system of linear stochastic difference equations.* (Abstract 52-9-332.)

12. David Blackwell: *Conditional expectation and unbiased sequential estimation.* (Abstract 52-9-327.)

13. Mark Kac: *A discussion of the Ehrenfest model.* Preliminary report. (Abstract 52-7-256.)

14. J. W. Tukey: *Sampling from contaminated distributions.* Preliminary report. (Abstract 52-9-334.)

15. Leo Katz: *On the class of functions defined by the difference equation* $(x+1)f(x+1)=(a+bx)f(x)$. (Abstract 52-9-330.)

16. F. E. Satterthwaite: *Retention of decimal places in matrix calculations.* (Abstract 52-9-333.)

17. R. C. Buck: *The measure-theoretic approach to density.* (Abstract 52-9-266.)

18. J. S. Frame: *On the reduction of the conjugating representation of a finite group.* (Abstract 52-9-268.)

19. Orrin Frink: *Complemented modular lattices and projective spaces of infinite dimension.* (Abstract 52-7-225.)

20. H. S. M. Coxeter: *Integral Cayley numbers.* (Abstract 52-7-223.)

21. C. N. Moore: *On the infinitude of prime pairs.* (Abstract 52-9-271.)

22. R. M. Robinson: *Primitive recursive functions.* (Abstract 52-9-274.)

23. P. T. Bateman: *On the representations of a number as the sum of three squares.* (Abstract 52-9-262.)

24. G. R. MacLane: *Concerning the uniformization of certain Riemann surfaces allied to the inverse-cosine and inverse-gamma surfaces.* (Abstract 52-7-236.)

25. J. G. Herriot: *On the restricted summability of multiple Fourier series.* Preliminary report. (Abstract 52-9-296.)

26. Y. W. Tschen: *Branch points and flat points of minimal surfaces in* R^3. (Abstract 52-9-310.)

27. J. A. Clarkson: *A characterization of C-spaces.* (Abstract 52-9-288.)

28. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of types of generalized Bessel functions.* (Abstract 52-9-297.)

29. A. W. Goodman: *On some determinants related to p-valent functions.* Preliminary report. (Abstract 52-9-292.)

30. S. H. Gould: *Lagrange multipliers and functional determinants.* (Abstract 52-9-293.)

31. Stefan Bergman: *The "extended class" in the theory of functions of two complex variables.* (Abstract 52-9-285.)

32. C. C. Hsiung: *Projective theory of surfaces and conjugate nets in four-dimensional space.* (Abstract 52-9-321.)

33. N. A. Court: *On the biratio of the altitudes of a tetrahedron.* (Abstract 52-7-252.)

34. V. G. Grove: *On congruences and conjugate nets.* (Abstract 52-9-320.)

35. Herbert Busemann: *Intrinsic area.* (Abstract 52-7-251.)

36. Tibor Radó: *The isoperimetric inequality and the Lebesgue definition of surface area.* I. (Abstract 52-7-253.)

37. L. A. Dye: *A Cremona transformation in S_3 defined by a pencil of quartic surfaces.* (Abstract 52-9-318.)

38. Michael Goldberg: *Tubular linkages.* (Abstract 52-9-319.)

39. G. W. Walker: *Games of the checkers family in line, plane, and space.* (Abstract 52-9-325.)

40. Bernard Epstein: *Integral representations of solutions of certain classes of partial differential equations.* (Abstract 52-9-290.)

41. William Karush: *A semi-strong minimum for a double integral problem in the calculus of variations.* (Abstract 52-7-234.)

42. Josephine M. Mitchell: *Value distribution of a function of two complex variables in a domain with distinguished boundary surface.* (Abstract 52-9-302.)

43. J. L. Synge: *Approximations in elasticity based on the concept of function space.* (Abstract 52-9-315.)

44. I. M. Sheffer: *On k-periodic systems of linear equations.* (Abstract 52-9-307.)

45. Otto Szász: *On the Möbius inversion formula and closed sets of functions.* (Abstract 52-7-240.)

46. R. P. Agnew: *Methods of summability which evaluate sequences of zeros and ones summable C_1.* (Abstract 52-9-281.)

47. George Piranian: *Infinite iteration of totally regular transformations.* Preliminary report. (Abstract 52-9-303.)

48. František Wolf: *On the continuation of analytic functions.* (Abstract 52-9-311.)

49. Irving Kaplansky: *Topological rings.* (Abstract 52-7-226.)

50. G. K. Kalisch: *Uniform spaces and topological groups as generalized metric spaces.* (Abstract 52-5-121.)

51. Garrett Birkhoff and P. M. Whitman: *Representation theory for certain non-associative algebras.* (Abstract 52-9-263.)

52. S. A. Jennings: *The group ring of a class of infinite nilpotent groups.* (Abstract 52-9-269.)

53. Irma R. Moses: *On the representation, in the ring of p-adic integers, of a quadratic form in n variables by one in m variables.* (Abstract 52-9-272.)

54. Hillel Poritsky and D. W. Dudley: *Conjugate action of involute helical gears with inclined axes.* (Abstract 52-9-312.)

55. Witold Hurewicz: *Algebraic and topological classification of mappings.* (Abstract 52-9-341.)

56. N. J. Fine: *On the Walsh functions.* Preliminary report. (Abstract 52-9-291.)

57. Samuel Eilenberg and Saunders MacLane: *Determination of the second homology and cohomology groups of a space by means of homotopy invariants.* (Abstract 52-11-399.)

58. R. P. Agnew: *Subseries of series which are not absolutely convergent.* (Abstract 52-9-282-*t*.)

59. Germán Ancochea: *Zariski's proof of the theorem of Bertini-Enriques in the case of an arbitrary ground field.* (Abstract 52-11-387-*t*.)

60. R. F. Arens: *Convex topological algebras.* Preliminary report. (Abstract 52-9-336-*t*.)

61. R. F. Arens: *Duality in topological linear spaces.* (Abstract 52-9-337-*t*.)

62. H. P. Atkins: *On fractional derivatives of univalent functions.* (Abstract 52-9-283-*t*.)

63. J. D. Bankier: *Modified regular continued fractions.* Preliminary report. (Abstract 52-9-284-*t*.)

64. H. W. Becker: *Stirling's numbers of the third kind.* (Abstract 52-9-326-*t*.)

65. Stefan Bergman: *On functions which satisfy certain systems of partial differential equations.* (Abstract 52-7-229-*t*.)

66. R. H. Bing: *Skew sets.* (Abstract 52-9-338-*t*.)

67. G. D. Birkhoff and D. C. Lewis: *Chromatic polynomials.* (Abstract 52-9-339-*t*.)

68. Paul Boschan: *The consolidated Doolittle technique.* (Abstract 52-9-328-*t*.)

69. A. T. Brauer: *Limits for the characteristic roots of a matrix.* (Abstract 52-9-264-*t*.)

70. Ellen F. Buck and R. C. Buck: *A note on finitely-additive measures.* (Abstract 52-9-265-*t*.)

71. C. C. Camp: *Integral equations with kernels having discontinuities along two diagonals.* (Abstract 52-9-287-*t*.)

72. H. S. M. Coxeter: *A simple proof of the eight square theorem.* (Abstract 52-7-222-*t*.)

73. H. V. Craig: *On the structure of intrinsic derivatives.* (Abstract 52-9-289-*t*.)

74. Roy Dubisch: *On the direct product of rational generalized quaternion algebras.* (Abstract 52-7-224-*t*.)

75. R. J. Duffin: *Nonlinear networks.* III. (Abstract 52-7-243-*t*.)

76. W. F. Eberlein: *Weak sequential compactness and regularity of Banach spaces.* (Abstract 52-7-231-*t*.)

77. C. J. Everett and H. J. Ryser: *Rational vector spaces.* (Abstract 52-9-267-*t*.)

78. Evelyn Frank: *The real parts of the zeros of a complex polynomial.* (Abstract 52-7-232-*t*.)

79. W. H. Gottschalk and G. A. Hedlund: *Recursive properties of transformation groups.* (Abstract 52-7-260-*t*.)

80. M. H. Heins: *On the Phragmén-Lindelöf principle.* (Abstract 52-9-295-*t*.)

81. P. G. Hoel: *The efficiency of the mean moving range.* (Abstract 52-9-329-*t*.)

82. B. F. Kimball: *Some basic theorems for developing tests of fit for the case of the non-parametric probability distribution function.* (Abstract 52-9-331-*t*.)

83. D. C. Lewis: *On polynomial interpolation and extrapolation by least squares with arbitrary weight function.* (Abstract 52-9-298-*t*.)

84. Szolem Mandelbrojt and G. R. MacLane: *On functions holomorphic in a strip region and an extension of Watson's problem.* (Abstract 52-9-299-*t*.)

85. P. R. Masani: *Laurent factorization of analytic functions in normed rings.* (Abstract 52-9-300-*t*.)

86. P. R. Masani: *Multiplicative Riemann integration in normed rings.* (Abstract 52-9-301-*t*.)

87. B. E. Meserve: *Inequalities of higher degree.* (Abstract 52-9-270-*t*.)

88. Harry Pollard: *The inversion of the transforms with reiterated Stieltjes kernels.* (Abstract 52-9-304-*t*.)

89. Harry Pollard: *The representation of* e^{-x^λ} *as a Laplace integral.* (Abstract 52-9-305-*t*.)

90. M. H. Protter: *Generalized spherical harmonics.* (Abstract 52-7-238-*t*.)

91. Tibor Radó: *The isoperimetric inequality and the Lebesgue definition of surface area.* II. (Abstract 52-7-254-*t*.)

92. W. T. Reid: *Solutions of linear differential equations.* (Abstract 52-9-306-*t*.)

93. C. E. Rickart: *The singular elements of a Banach algebra.* (Abstract 52-9-273-*t*.)

94. H. E. Salzer: *Alternative formulas for direct interpolation of a complex function tabulated along equidistant circular arcs.* (Abstract 52-7-245-*t*.)

95. R. D. Schafer: *Concerning automorphisms of non-associative algebras.* (Abstract 52-9-275-*t*.)

96. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* III. (Abstract 52-9-276-*t*.)

97. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra.* IV. (Abstract 52-9-277-*t*.)

98. I. E. Segal: *Irreducible representations of operator algebras.* (Abstract 52-7-239-*t*.)

99. Seymour Sherman: *Stability calculations and time lag.* (Abstract 52-5-179-*t*.)

100. S. S. Shü: *On Taylor and Maccoll's equation of a cone moving in the air with supersonic speed.* (Abstract 52-9-314-*t*.)

101. Abraham Spitzbart: *The minimum of a certain integral.* (Abstract 52-9-308-*t*.)

102. A. C. Sugar: *An elementary exposition of the relaxation method.* (Abstract 52-7-247-*t*.)

103. Fred Supnick: *A theorem on rectilinear deformation.* (Abstract 52-9-342-*t*.)

104. Fred Supnick: *On the packing of spheres.* I. (Abstract 52-9-323-*t*.)

105. Fred Supnick: *Topology of sphere clusters.* I. (Abstract 52-9-343-*t*.)

106. Fred Supnick: *Topology of sphere clusters.* II. *Analogue of Kuratowski's theorem.* (Abstract 52-9-344-*t*.)

107. W. J. Trjitzinsky: *Singular integral equations of the first kind and those related to permutability and iteration.* (Abstract 52-9-309-*t*.)

108. Sister Petronia Van Straten: *Toroidal and non-toroidal graphs.* (Abstract 52-9-345-*t*.)

109. M. L. Vest: *An involutorial space transformation associated with a $Q_{1,n}$ congruence.* (Abstract 52-9-324-*t*.)

110. Jacob Wolfowitz: *Confidence limits for the fraction of a normal population which lies between two given limits.* (Abstract 52-9-335-*t*.)

111. Oscar Zariski: *The concept of a simple point of an abstract algebraic variety.* (Abstract 52-11-393-*t*.)

T. R. Hollcroft,
Associate Secretary

BOOK REVIEWS

Étude des sommes d'exponentielles réelles. By Laurent Schwartz. (Actualités scientifiques et industrielles, no. 959.) Paris, Hermann, 1943. 89 pp. 85 fr.

The problems discussed here originate from Weierstrass' polynomial approximation theorem. An introductory section deals with fundamental concepts in vector spaces. Chapter I starts with a discussion of the theorem of Müntz and Szász: If $\lambda_0=0, \lambda_n \geqq \alpha > 0$ for $n \geqq 1$, then a necessary and sufficient condition for the closure of the sequence $\{x^{\lambda_n}\}$ in $C(0, 1)$ is the divergence of the series $\sum \lambda_n^{-1}$. (It is stated on page 24 that Kaczmarz and Steinhaus gave a new proof; however their proof is the same as Szász's proof.) The author then discusses in detail the case that $\sum \lambda_n^{-1}$ converges. One of the results in this case is: If $\lambda_n + 1/p > 0$ $(p>1)$, $\lambda_n \to \infty$, $\sum_{\lambda_n>0} \lambda_n^{-1}$ converges, and if the index of condensation of the sequence $\{\lambda_n\}$ is zero, then every function $f(z)$ belonging to the L_p span of the sequence $\{z^{\lambda_n}\}$ is analytic in $(0, 1)$, it can be continued into the region $|z| < 1$ of the Riemann surface of $\log z$, and it has an expansion $\sum a_n z^{\lambda_n}$ absolutely and uniformly convergent for $|z| < 1-\epsilon$, $\epsilon > 0$. For the case that the λ_n are integers a similar result appears in a paper by Clarkson and Erdős (Duke Math. J. vol. 10 (1943) pp. 5–11). The author also discusses the problem of completeness for the interval (a, b), where $0 < a < b$.

In Chapter II estimates for the coefficients a_n of generalized polynomials $g(x) = \sum a_n x^{\lambda_n}$ are given when $\int_0^1 |g(x)|^p dx \leqq 1$, $p \geqq 1$. For $\lambda_n = n$ and $p = \infty$ the exact bound was given by S. Bernstein. The author finds the exact bound for $p=2$, while for arbitrary p asymptotic estimates are given. (Hille, Szegő and Tamarkin gave asymptotic estimates for the exact bound of $|g'(x)|$; see Duke Math. J. vol. 3 (1937) pp. 729–739.)

The book has a bibliography and a table of contents. It would be interesting to consider the analogous problems in several variables.

Otto Szász

Mathematical theory of elasticity. By I. S. Sokolnikoff. New York, McGraw-Hill, 1946. 11+373 pp. $4.50.

This book contains approximately the first half of the material of a course given by the author in 1941 and 1942 in the Program of Advanced Instruction and Research in Mechanics, conducted by the

Graduate School of Brown University. In fact, except for the last chapter, the book is a revision of part of a set of lecture notes of this course, which notes have been circulated widely in mimeographed form. A second volume is contemplated which will contain the remainder of the material in these notes. The excellence of these notes has long been recognized, and appearance in printed form of the material contained therein is most welcome.

The first three chapters contain a very complete treatment of the fundamentals of the theory of elastic bodies, that is, the analysis of strain, the analysis of stress, and the stress-strain relations. In the classical treatments of these, many notations have been employed, all of which are cumbersome. In the present volume, a departure is made in this regard, and the tensor notation is used. This results in a great compactness and economy of effort. This volume should hasten the current trend towards an increased use of the powerful methods of tensor calculus in elasticity.

Chapter 4 deals with the application of the fundamental theory to the extension, torsion, and flexure of homogeneous beams. In this one chapter the tensor notation is not employed, in order that the contents of this chapter may be available to persons unfamiliar with the tensor notation but familiar with the fundamental theory. The treatment is quite thorough, and contains among other things a detailed account of the method usually associated with the name N. I. Muschelisvili. In this method, the problems of torsion and flexure, which are reduced to a number of Dirichlet problems, are solved by conformal mapping of the cross section of the beam onto the unit circle. The fundamental equations of elasticity are obtained in terms of general orthogonal curvilinear coordinates, and are used to solve a number of problems. There is also a section on the technical theory of beams, as used by the engineer.

Chapter 5, which is the last chapter, contains material which does not appear in the mimeographed notes on which this book is based. It deals with variational methods in elasticity. Various energy and reciprocity theorems are developed, and there is also a presentation of variational methods of obtaining approximate solutions of boundary value problems. Among these are the Rayleigh-Ritz method, Galerkin's method, and the method of Biezno and Koch. There is also a treatment of the method of finite differences as applied to the approximate solution of boundary value problems, with particular application to the problem of torsion as an example.

This book is written clearly and in a manner which is easy to follow. Also, it contains many interesting exercises. Because of this it is ad-

NOTES

The sixty-fifth meeting of the French Association for the Advancement of Science was held at Nice, September 9–14, 1946.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1947–1948. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, and are returnable by February 1, 1947.

The Mathematical Tables Project issues monthly progress reports. Those interested in the work of the Project will, upon request to the Mathematical Tables Project, 150 Nassau Street, New York City, be placed on the mailing list for copies of these reports.

The Naval Ordnance Development Award has been conferred upon the Applied Mathematics Group of the Division of War Research, Columbia University. Special individual certificates have been presented to the following mathematicians who were members of the group: M. R. Hestenes, Irving Kaplansky, Walter Leighton, D. P. Ling, Saunders MacLane, and Hassler Whitney.

Professor P. Alexandrov has been elected to membership in the American Philosophical Society.

Professor Lennie P. Copeland of Wellesley College has retired with the title emeritus.

Professor Charlotte I. Davison of Wilson College, Chambersburg, Pennsylvania, has retired with the title emeritus. She received the honorary degree of Doctor of Humane Letters from Wilson College in June, 1946.

Dr. D. M. Dribin of the Army Security Agency has been awarded the Legion of Merit.

Professor Peter Field of the University of Michigan has retired with the title emeritus.

Associate Professor Mayme I. Logsdon of the University of Chicago has retired with the title emeritus. During the academic year

1946–1947 she will be visiting professor at the University of Miami.

Dr. A. N. Lowan, Director of the Mathematical Tables Project, has received the Naval Ordnance Development Award.

Associate Professor R. D. Mindlin of Columbia University has been awarded the Medal for Merit.

Professor M. B. Porter of the University of Texas has retired with the title emeritus.

Professor Harlow Shapley of Harvard University has been elected correspondent for the astronomy section of the French Academy of Sciences.

Professor W. P. Milne of the University of Leeds has retired.

Sir Edmund T. Whittaker, professor of mathematics of the University of Edinburgh, has retired. He will be succeeded by Dr. A. C. Aitken.

Dr. M. Raziuddin Siddiqi, Director of Research Institute at Osmania University, has arrived in this country. He has lectured at Harvard and Princeton Universities.

Dr. J. C. Abbott has been appointed to an assistant professorship at the United States Naval Academy.

Associate Professor H. H. Alden of the University of Wyoming has been appointed to an assistant professorship at Ohio State University.

Associate Professor H. E. Arnold of Wesleyan University, Middletown, Connecticut, has been promoted to a professorship.

Mr. J. W. Arnold has accepted a position as engineer with International Telephone and Telegraph Corporation.

Professor Emil Artin of Indiana University has been appointed to a professorship at Princeton University.

Dr. Winifred A. Asprey of Vassar College has been promoted to an assistant professorship.

Assistant Professor A. A. Aucoin of the University of Houston has been promoted to an associate professorship.

Dr. S. P. Avann of Oregon State College has been appointed to an assistant professorship at the University of Washington.

Associate Professor H. W. Bailey of the University of Illinois has been appointed Associate Dean of the Undergraduate Division, Liberal Arts and Sciences, University of Illinois, Chicago, Illinois.

Assistant Professor Grace M. Bareis of Ohio State University has retired.

Dr. Joshua Barlaz of Ohio State University has been appointed to an assistant professorship at Rutgers University.

Mr. W. J. Barr has accepted a position with Experiment, Incorporated, of Richmond, Virginia.

Assistant Professor J. J. Barron of Marquette University has been promoted to an associate professorship.

Mr. M. S. Bartlett of Cambridge University has been appointed to a visiting professorship at the University of North Carolina.

Assistant Professor A. E. Basch of Amherst College has been appointed to an associate professorship at the Massachusetts State College at Fort Devens.

Mr. P. S. Bauer has been made president of the New England Electronics Corporation, Lynn, Massachusetts.

Mr. J. K. Baumgart has been appointed to an assistant professorship at Elmhurst College, Elmhurst, Illinois.

Associate Professor E. F. Beckenbach of the University of California at Los Angeles has been promoted to a professorship.

Dr. J. H. Bell has been appointed to an assistant professorship at Michigan State College.

Dr. T. J. Benac of the United States Naval Academy has been promoted to an assistant professorship.

Dr. P. G. Bergmann has been appointed assistant director of the Sonar Analysis Group, Bureau of Ships, Navy Department.

Dr. Dorothy L. Bernstein of the University of Rochester has been promoted to an assistant professorship.

Dr. E. E. Betz of the United States Naval Academy has been promoted to an assistant professorship.

Professor H. L. Black of Westminster College, New Wilmington, Pennsylvania, has been appointed to an assistant professorship at Michigan State College.

Dr. E. E. Blanche has accepted the position of principal administrative analyst, Management Branch, Plans and Policy Office, Service, Supply and Procurement Division of the War Department.

Associate Professor M. G. Boyce of Western Reserve University has been appointed to an associate professorship at Vanderbilt University.

Dr. C. P. Brady has been appointed to an associate professorship at the United States Naval Academy.

Mr. B. K. Brown of Syracuse University has been appointed to an associate professorship at the James Millikin University, Decatur, Illinois.

Dr. G. M. Brown of the Harrow School, England, has been appointed to an assistant professorship at Michigan State College.

Dr. G. W. Brown has been appointed to an associate professorship at the Iowa State College of Agriculture and Mechanical Arts.

Mr. R. H. Brown has been appointed lecturer at Columbia University.

Professor C. T. Bumer of Kenyon College has been appointed Head Mathematician, Assistant to Chief, Evaluation and Analysis Group, Bureau of Ordnance.

Assistant Professor H. E. Burns of Calumet Extension Center, Indiana University, has been appointed director.

Mr. K. A. Bush of the United States Naval Academy has been appointed to an assistant professorship at Mohawk College, Utica, New York.

Assistant Professor F. A. Butter, Jr., of Stanford University has been appointed mathematician at the Hughes Aircraft Company, Culver City, California.

Assistant Professor Ralph Byrne of the University of California at Los Angeles has been appointed to an associate professorship at the California Institute of Technology.

Assistant Professor S. S. Cairns of Queens College has been appointed to a professorship at Syracuse University.

Assistant Professor H. H. Campaigne of the University of Minnesota has accepted a position as mathematician with the Navy Department.

Assistant Professor E. W. Cannon of the University of Delaware has accepted a position as mathematician with the National Bureau of Standards.

Dr. Achille Capecelatro has been appointed supervisor of the Theory and Analysis Section, Underwater Sound Laboratory, New London, Connecticut.

Dr. B. G. Capó, on leave of absence from the Agricultural Experimental Station, Rio Piedras, Puerto Rico, has been appointed technical supervisor at the Ancram Paper Mills, Ancram, New York.

Dr. G. F. Carrier has been appointed to an assistant professorship at Brown University.

Dr. A. B. Carson has been appointed to an associate professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Mr. R. L. Caskey of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Associate Professor J. W. Cell of the State College of Agriculture and Engineering of the University of North Carolina has been promoted to a professorship.

Dr. Nicholas Chako has been appointed to an associate professorship in theoretical physics at the Kansas State College of Agriculture and Applied Science.

Assistant Professor Randolph Church of the Postgraduate School, United States Naval Academy, has been promoted to an associate professorship.

Mr. E. H. Clamons of the College of St. Thomas has accepted a position as design engineer with J. M. Dalglish and Company, St. Paul, Minnesota.

Assistant Professor C. L. Clark of Oregon State College has been promoted to an associate professorship.

Dr. W. W. S. Claytor has been appointed to a professorship at Hampton Institute.

Assistant Professor A. H. Clifford of the Massachusetts Institute of Technology has been appointed to an associate professorship at Johns Hopkins University.

Assistant Professor Nathaniel Coburn of the University of Texas has been appointed to an assistant professorship at the University of Michigan.

Professor W. G. Cochran of Iowa State College of Agriculture and Mechanical Arts has been appointed to a professorship at the University of North Carolina.

Dr. George Comenetz has accepted a position as research engineer with Westinghouse Electric Corporation.

Assistant Professor Byron Cosby of the United States Naval Academy has been appointed to an assistant professorship at the State University of Iowa.

Professor C. C. Craig of the University of Michigan has been appointed director of the Statistical Research Laboratory.

Associate Professor G. F. Cramer of Tulane University of Louisiana has accepted a position as mathematician with the Navy Department.

Miss Jane E. Crawford has accepted a position as engineer with General Electric Company.

Dr. W. J. R. Crosby of the University of Saskatchewan has been promoted to an assistant professorship.

Dr. E. L. Crow has accepted a position as mathematician at the United States Naval Ordnance Test Station, Inyokern, California.

Dr. A. B. Cunningham of Pennsylvania State College has been appointed to an assistant professorship at West Virginia University.

Dr. J. F. Daly has accepted a position as statistician with the Bureau of the Census.

Dr. Margaret R. Davis of Southeastern Louisiana College, Hammond, Louisiana, has been promoted to an associate professorship.

Associate Professor D. B. DeLury of Virginia Polytechnic Institute has been promoted to a professorship.

Dr. W. W. Denton has been appointed principal of the High School, Minden City, Michigan.

Dr. C. R. DePrima has been appointed to an assistant professorship in applied mechanics at the California Institute of Technology.

Associate Professor Douglas Derry of the University of Saskatchewan has been appointed to an associate professorship at the University of British Columbia.

Dr. J. B. Díaz of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Sister Agnes T. Dimond of the College of Mount Saint Joseph, Mount Saint Joseph, Ohio, has been promoted to a professorship.

Professor Emeritus L. L. Dines of Carnegie Institute of Technology has been appointed to a visiting professorship at Smith College.

Mrs. Nancy M. Dismuke has accepted a position as junior physicist with the Monsanto Chemical Company, Clinton Laboratories, Knoxville, Tennessee.

Dr. C. L. Dolph has been appointed lecturer at the University of Michigan.

Associate Professor J. L. Dorroh of Louisiana State University has been appointed to an associate professorship at the Illinois Institute of Technology.

Dr. T. L. Downs of Trinity College, Hartford, Connecticut, has been appointed to an associate professorship at the United States Naval Academy.

Dr. F. W. Dresch of the University of California has accepted a position as principal mathematician, Naval Proving Ground, Dahlgren, Virginia.

Mr. J. A. H. Duffie has been appointed to a professorship in physics at Assumption College, University of Western Ontario.

Dr. R. J. Duffin of the Carnegie Institution of Washington has been appointed to an associate professorship at the Carnegie Institute of Technology.

Mr. D. G. Duncan has been appointed lecturer at the University of British Columbia.

Assistant Professor Russell Dunholter of the College of Engineering and Commerce, University of Cincinnati, has been promoted to an associate professorship.

Dr. W. W. Durding has accepted a position as senior physicist with the Kellex Corporation, New York City.

Associate Professor P. S. Dwyer of the University of Michigan has been promoted to a professorship.

Associate Professor Margaret C. Eide of State Teachers College, River Falls, Wisconsin, has been promoted to a professorship.

Mr. Murray Ellis has accepted a position as mathematician at the Computing Laboratory, Harvard University.

Assistant Professor Howard Eves of the College of Puget Sound has been appointed to an assistant professorship at Oregon State College.

Assistant Professor F. G. Fender of Rutgers University has been promoted to a professorship.

Dr. A. D. Fialkow of Columbia University has accepted a position as mathematician with the Control Instrument Company, Brooklyn, New York.

Associate Professor E. J. Finan of Catholic University has been promoted to a professorship.

Dr. N. J. Fine has accepted a position with the Operations Evaluation Group, Navy Department.

Assistant Professor F. G. Fisher of the University of Santa Clara has been promoted to an associate professorship.

Mr. J. D. Fitzpatrick has been appointed to an assistant professorship at Marquette University.

Mr. N. S. Free of Mount Royal College, Calgary, Alberta, Canada, has been appointed to a lectureship at the University of British Columbia.

Dr. Bernard Friedman has been appointed to an assistant professorship at New York University.

Assistant Professor A. S. Galbraith of Colby College, Waterville, Maine, has accepted a position as mathematician at Aberdeen Proving Ground.

Associate Professor H. L. Garabedian of Northwestern University has been appointed principal physicist at the Monsanto Chemical Company, Knoxville, Tennessee.

Dr. Louis Garfin has been appointed actuary, Department of Insurance, State of Oregon.

Dr. G. N. Garrison of the College of the City of New York has been appointed to an assistant professorship at Lehigh University.

Dr. R. E. Gaskell of Brown University has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanical Arts.

Dr. B. E. Gatewood has accepted a position as consulting engineer with Beech Aircraft Corporation.

Associate Professor F. C. Gentry of the Louisiana Polytechnic Institute has been promoted to a professorship.

Mr. P. T. Gilbert has accepted a position as research chemist with the National Technical Laboratories, South Pasadena, California.

Mr. M. E. Gillis of the University of Tennessee has been appointed to an associate professorship at the University of Florida.

Dr. H. F. Gingerich of the Carnegie Institution of Washington has accepted a position as mathematician in the Navy Department.

Mr. H. T. Gittings has been appointed physicist at the Naval Ordnance Laboratory.

Assistant Professor Wallace Givens of Northwestern University has been appointed to an associate professorship at the Illinois Institute of Technology.

Mr. W. H. Glenn of Pasadena Junior College has been appointed head of the division of natural science of Muir Junior College, Pasadena, California.

Dr. Casper Goffman of Westinghouse Electric Corporation has been appointed to an assistant professorship at the University of Kentucky.

Dr. H. E. Goheen has accepted a position as mathematician with the Office of Research and Inventions, Navy Department.

Professor V. D. Gokhale of the University of the Philippines has been appointed to a visiting lectureship at Rutgers University.

Dr. H. H. Goldstine of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor Michael Golomb of Purdue University has been promoted to an associate professorship.

Dr. M. J. Gottlieb has been appointed to an assistant professorship at Washington University.

Assistant Professor S. H. Gould of Victoria College, University of Toronto, has been promoted to an associate professorship.

Assistant Professor H. S. Grant of Rutgers University has been promoted to an associate professorship.

Dr. A. A. Grau has been appointed to an assistant professorship at the University of Kentucky.

Dr. F. G. Gravalos of the De Laval Turbine Company, Trenton, New Jersey, has been appointed to an associate professorship at Rensselaer Polytechnic Institute.

Mr. Lewis Greenwald has been appointed physicist at Camp Evans Signal Laboratory, Belmar, New Jersey.

Dr. R. E. Greenwood of the University of Texas has been promoted to an assistant professorship.

Professor W. C. Griffith of Marion Institute, Marion, Alabama, has been appointed to an associate professorship at Centenary College, Shreveport, Louisiana.

Mr. N. D. Griffin of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. G. L. Gross has accepted a position as research engineer with the Grumman Aircraft Engineering Corporation, Bethpage, New York.

Assistant Professor C. C. Grove of the College of the City of New York has retired.

Dr. E. J. Gumbel of the New School for Social Research has accepted a position as a special lecturer in statistics at Newark College of Engineering.

Assistant Professor Marshall Hall of Yale University has been appointed to an associate professorship at Ohio State University.

Dr. N. A. Hall has been appointed head of the Analysis Section, Research Department, United Aircraft Corporation.

Assistant Professor P. C. Hammer of Oregon State College has been promoted to an associate professorship.

Dr. Duncan Harkin of Brooklyn College has accepted a position as mathematical consultant at the Naval Research Laboratory.

Assistant Professor F. S. Harper of the University of Nebraska has been appointed to a professorship at Drake University, Des Moines, Iowa.

Dr. W. J. Harrington of Pennsylvania State College has been promoted to an assistant professorship. He will be on leave on a visiting assistant professorship at Cornell University during the present academic year.

Mr. L. J. Harris has accepted a position as electrical engineer with the Aluminum Company of America.

Dr. Gerald Harrison of the Radiation Laboratory, Massachusetts Institute of Technology, has been appointed project engineer at the Sperry Gyroscope Company, Garden City, New York.

Assistant Professor O. G. Harrold of Pomona College has been appointed lecturer at Princeton University.

Dr. Philip Hartman has been appointed to an assistant professorship at Johns Hopkins University.

Mr. Cecil Hastings has accepted a position as research analyst with the Douglas Aircraft Company, Santa Monica, California.

Dr. Charles Hatfield has been appointed to an assistant professorship at the University of Minnesota.

Professor B. A. Hausmann of the University of Detroit has been appointed to a professorship at West Baden College, Loyola University, West Baden Springs, Indiana.

Professor E. K. Haviland of Lincoln University, Lincoln University, Pennsylvania, has been appointed to an assistant professorship at Johns Hopkins University.

Assistant Professor Euphemia L. Haynes of Miner Teachers College, Washington, D. C., has been promoted to an associate professorship.

Dr. Olaf Helmer has accepted a position as research mathematician with the Douglas Aircraft Company, Santa Monica, California.

Mr. Manuel Herschdorfer of Amherst College has been appointed to an assistant professorship at Seton Hall College, South Orange, New Jersey.

Assistant Professor A. D. Hestenes of Carnegie Institute of Technology has accepted a position as head of special testing equipment, Fire Control Department, United States Naval Air Test Center, Patuxent River, Maryland.

Mr. J. S. Hickman of Rochester Junior College has been appointed physicist at the United States Naval Electronics Laboratory, San Diego, California.

Assistant Professor E. H. C. Hildebrandt of Northwestern University has been promoted to an associate professorship.

Associate Professor E. L. Hill of the University of Minnesota has been promoted to a professorship of physics.

Assistant Professor P. G. Hoel of the University of California at Los Angeles has been promoted to an associate professorship.

Dr. V. A. Hoersch of the University of Illinois has been promoted to an assistant professorship.

Mr. R. H. Hoskins has accepted a position as actuarial clerk with John Hancock Mutual Life Insurance Company.

Associate Professor Aughtum S. Howard has been promoted to a professorship.

Mr. J. F. Hubbard has been appointed to an assistant professorship at Massachusetts State College at Fort Devens.

Assistant Professor P. M. Hummel of the University of Alabama has been promoted to an associate professorship.

Assistant Professor H. D. Huskey of the University of Oklahoma is on leave of absence. He will serve as mathematician at the National Physical Laboratory, Teddington, Middlesex, England.

Dr. L. C. Hutchinson of the Norden Laboratories has been appointed to an assistant professorship at the Polytechnic Institute of Brooklyn.

Mr. Seymour Jablon has accepted a position as price economist with the Bureau of Labor Statistics, United States Department of Labor.

Professor J. B. Jackson has been appointed Dean of Men at the University of South Carolina.

Mr. A. W. Jacobson of Wayne University has been promoted to an assistant professorship.

Assistant Professor W. C. Janes of the Kansas State College of

Agriculture and Applied Science has been promoted to an associate professorship.

Mr. J. B. Jeffries has been appointed to an associate professorship at the Agricultural and Technical College of North Carolina.

Dr. Herbert Jehle has accepted a position as research physicist with Franklin Institute.

Mr. Walter Jennings has been appointed to an assistant professorship at Virginia Polytechnic Institute.

Associate Professor H. S. Kaltenborn of the University of Idaho has been appointed to a professorship at Memphis State College.

Dr. Leo Katz of Wayne University has been appointed to an assistant professorship at Michigan State College.

Mr. J. R. F. Kent has been appointed to an assistant professorship at the University of British Columbia.

Dr. J. M. Kingston of the University of Washington has been promoted to an assistant professorship.

Dr. F. L. Kiokemeister of Purdue University has been appointed to an assistant professorship at Mount Holyoke College.

Mr. K. V. Knight has accepted a position as physicist with the Bureau of Ordnance, Navy Department, Washington, D. C.

Dr. E. A. Knobelauch of the University of Pennsylvania has been appointed lecturer at Rutgers University.

Dr. C. F. Kossack of the University of Oregon has accepted a position as mathematician with the Navy Department.

Associate Professor R. L. Krueger of Wittenberg College has been promoted to a professorship.

Dr. M. Z. Krzywoblocki has been appointed to an associate professorship of aeronautical engineering at the University of Illinois.

Dr. G. B. Lang of the University of Florida has been promoted to an assistant professorship.

Associate Professor H. D. Larsen of the University of New Mexico has been promoted to a professorship.

Miss Jane M. Lawler has accepted a position at the United States Naval Ordnance Test Station, Inyokern, California.

Associate Professor V. S. Lawrence of Cornell University has been appointed to an associate professorship at Virginia Polytechnic Institute.

Mr. H. R. Leifer has accepted a position as assistant chief, Registration and Research Section, Veterans Administration, Pittsburgh, Pennsylvania.

Mr. W. G. Leight has accepted a position as meteorologist with the United States Weather Bureau.

Dr. J. H. Levin has been appointed Chief of the Machines Section of the Computing Laboratory at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Assistant Professor P. E. Lewis of the Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Associate Professor Hans Lewy of the University of California has been promoted to a professorship.

Dr. F. W. Light of the Johns Hopkins University has been promoted to an assistant professorship.

Assistant Professor H. W. Linscheid of Eastern New Mexico College has been appointed to an associate professorship at the College of Emporia, Emporia, Kansas.

Associate Professor Charles Loewner of Syracuse University has been promoted to a professorship.

Dr. L. H. Loomis of Harvard University has been promoted to an associate professorship.

Mr. Jack Lorell has been appointed research engineer at the Jet Propulsion Laboratory, California Institute of Technology.

Mr. R. A. Lytle of the University of Virginia has been appointed to an adjunct professorship at the University of South Carolina.

Mr. W. S. McCulley of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Dr. Janet McDonald of Vassar College has been promoted to an assistant professorship.

Professor W. H. McEwen of Mount Allison University has been appointed to a professorship at the University of Manitoba.

Dr. A. W. McGaughey of the United States Naval Academy has been appointed to a professorship at Westminster College, New Wilmington, Pennsylvania.

Assistant Professor J. C. C. McKinsey of the University of Nevada has been appointed to an associate professorship at Oklahoma Agricultural and Mechanical College.

Dr. Brockway McMillan has been appointed a member of the technical staff of Bell Telephone Laboratories.

Professor H. M. MacNeille is on leave from Kenyon College and has been appointed scientific director, Office of Assistant Naval Attaché for Research, American Embassy, London.

Dr. W. G. Madow has been appointed to an associate professorship at the University of North Carolina.

Associate Professor J. D. Mancill of the University of Alabama has been promoted to a professorship.

Mr. Ralph Mansfield has accepted a position as chief engineer with Joseph Weidenhoff, Incorporated, Chicago, Illinois.

Associate Professor Morris Marden of the University of Wisconsin in Milwaukee has been promoted to a professorship.

Dr. A. J. Maria of Brooklyn College has been promoted to an assistant professorship.

Assistant Professor C. E. Marshall of Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Dr. R. M. Martin has been appointed to an assistant professorship in philosophy at Bryn Mawr College.

Dr. A. E. May of the University of Wisconsin at Kenosha has been promoted to an assistant professorship.

Mr. P. E. Meadows has been appointed to an assistant professorship at Washington and Lee University.

Professor Karl Menger of the University of Notre Dame has been appointed to a professorship at the Illinois Institute of Technology.

Dr. W. A. Mersman has accepted a position as research mathematician with Taylor Instrument Companies, Rochester, New York.

Professor H. A. Meyer of Indiana University has been appointed to an associate professorship at the University of Florida.

Associate Professor Herman Meyer of the University of Miami has been promoted to a professorship.

Dr. Abraham Miller has accepted a position as mathematician with the Bureau of Ordnance, Navy Department, Washington, D. C.

Mr. K. S. Miller has accepted a position as engineer with the Otis Elevator Company, New York City.

Assistant Professor W. I. Miller of Bucknell University has been promoted to a associate professorship.

Dr. H. J. Miser has been appointed to an assistant professorship at Williams College.

Associate Professor Josephine M. Mitchell of Winthrop College has been appointed to an associate professorship at Texas State College for Women.

Mr. E. C. Molina, formerly with Bell Telephone Laboratories, has been appointed a special lecturer at Newark College of Engineering.

Dr. L. J. Monville of John Carroll University has been promoted to an assistant professorship.

Assistant Professor T. W. Moore of the United States Naval Academy has been promoted to an associate professorship.

Dr. C. W. Moran of Illinois Institute of Technology has accepted a position at Wright Junior College.

Professor Eugenie M. Morenus of Sweet Briar College has retired.

Dr. Vladimir Morkovin has accepted a position as mathematician in the Planning Division, Office of Naval Research, Navy Department.

Mr. R. C. Morrow of the United States Naval Academy has been promoted to an assistant professorship.

Associate Professor J. E. Morton of Cornell University has been promoted to a professorship.

Dr. Frederick Mosteller has been appointed lecturer and research associate in social relations at Harvard University.

Sister M. Joanne Muggli of Saint Benedict's College, Saint Joseph, Minnesota, has been promoted to an assistant professorship.

Associate Professor Ernest Nagel of Columbia University has been promoted to a professorship of philosophy.

Assistant Professor David Nelson of Amherst College has been appointed to an assistant professorship at George Washington University.

Assistant Professor C. J. Nesbitt of the University of Michigan has been promoted to an associate professorship.

Dr. Albert Newhouse of the University of Houston has been promoted to an assistant professorship.

Associate Professor G. D. Nichols of the University of Arkansas has been appointed acting dean of the College of Arts and Sciences.

Professor R. E. Norris of State Teachers College, Milwaukee, Wisconsin, has been appointed dean of instruction.

Associate Professor E. P. Northrop of the University of Chicago has been appointed associate dean of the college.

Assistant Professor Emma J. Olson of Kent State University, Kent, Ohio, has been promoted to an associate professorship.

Dr. H. L. Olson has been appointed to a professorship at Indiana Technical College, Fort Wayne, Indiana.

Dr. Isaac Opatowski has been appointed to an assistant professorship at the University of Michigan.

Dr. W. K. H. Panofsky has been appointed to an assistant professorship of physics at the University of California.

Mr. R. I. Pepper has been appointed to an assistant professorship at Winthrop College, Rock Hill, South Carolina.

Dr. A. S. Peters of New York University has been promoted to an assistant professorship.

Associate Professor H. R. Phalen of the College of William and Mary has been promoted to a professorship.

Dr. Edmund Pinney of Oregon State College has been appointed lecturer at the University of California.

Associate Professor R. J. Pitts of Fort Valley State College, Fort Valley, Georgia, has been promoted to a professorship.

Associate Professor H. S. Pollard of Miami University has been promoted to a professorship.

Associate Professor George Polya of Stanford University has been promoted to a professorship.

Assistant Professor A. R. Poole of Montana State College has been appointed to an assistant professorship at Oregon State College.

Assistant Professor J. W. Porter of Bluefield State College, Bluefield, West Virginia, has accepted a position as contact representative with the Veterans Administration, Philadelphia, Pennsylvania.

Assistant Professor E. S. Quade of the University of Florida has been promoted to an associate professorship.

Mr. L. J. Quaid of the University of Minnesota has been promoted to an assistant professorship.

Assistant Professor E. D. Rainville of the University ofMichigan has been promoted to an associate professorship.

Assistant Professor H. W. Raudenbush of Queens College, Flushing, New York, has been promoted to an associate professorship.

Reverend John Raymond of Saint Martin's College, Lacey, Washington, has been promoted to a professorshiip.

Assistant Professor M. O. Reade of Purdue University has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor L. M. Reagan of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at the University of Wichita.

Professor L. J. Reed of Johns Hopkins University has been appointed vice president.

Professor P. K. Rees of Southwestern Louisiana Institute, Lafayette, Louisiana, has been appointed to an associate professorship at Louisiana State University.

Assistant Professor Eric Reissner of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. H. J. Riblet has accepted a position as radio engineer with Submarine Signal Company, Boston, Massachusetts.

Dr. C. E. Rickart of Yale University has been promoted to an assistant professorship.

Dr. F. D. Rigby has been appointed mathematician at the Office of

Naval Research, Navy Department.

Mr. R. K. Ritt has been appointed a lecturer at Columbia University.

Assistant Professor R. M. Robinson of the University of California has been promoted to an associate professorship. He will be a visiting lecturer at Princeton University this academic year.

Dr. P. C. Rosenbloom of Brown University has been appointed to an assistant professorship at Syracuse University.

Associate Professor A. E. Ross of Saint Louis University has been appointed to a professorship at the University of Notre Dame.

Mr. S. G. Roth of Cooper Union has been appointed to an assistant professorship at New York University.

Assistant Professor Raphael Salem of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Mr. John Salerno has been appointed mathematician with the United States Coast and Geodetic Survey in New York City.

Assistant Professor Hans Samelson of Syracuse University has been appointed to an assistant professorship at the University of Michigan.

Dean R. G. Sanger of the University of Chicago has been appointed to a professorship at Kansas State College of Agriculture and Applied Science.

Associate Professor A. C. Schaeffer of Stanford University has been promoted to a professorship.

Dr. Alice T. Schafer has been appointed lecturer at New Jersey College for Women, Rutgers University.

Dr. W. T. Scott of Northwestern University has been promoted to an assistant professorship.

Assistant Professor C. E. Sealander of the University of South Dakota has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanical Arts.

Dr. C. W. Seekins has been appointed to an assistant professorship at the United States Naval Academy.

Miss M. Anice Seybold of the University of Illinois has been appointed to a professorship at North Central College, Naperville, Illinois.

Dr. M. E. Shanks of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Mr. R. L. deShelly has been appointed chief of the statistical division, French Supply Council in the United States of America.

Assistant Professor R. W. Shephard of Purdue University has been appointed a lecturer at the University of California.

Mr. W. L. Shepherd has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Dr. D. T. Sigley has been appointed research engineer, Institute of Cooperative Research, Johns Hopkins University.

Professor Heinz Simon of Southern Union College has been appointed to a professorship at William Penn College, Oskaloosa, Iowa.

Mr. Harold E. Smith of Junior College of Connecticut has been promoted to an assistant professorship.

Dr. T. L. Smith has been appointed chief engineer at the Supersonic Wind Tunnel, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Dr. A. F. Smullyan of Williams College has been appointed to an assistant professorship of philosophy at the University of Washington.

Associate Professor D. C. Spencer of Stanford University has been promoted to a professorship.

Dr. Abraham Spitzbart of the University of Wisconsin has been appointed to an assistant professorship at the University of Wisconsin in Milwaukee.

Assistant Professor D. W. Starr of Southern Methodist University has been promoted to an associate professorship.

Mr. Fritz Steinhardt has been appointed lecturer at Columbia University.

Dr. J. C. Stewart of Louisiana State University has been appointed to an assistant professorship at Lawrence College, Appleton, Wisconsin.

Assistant Professor R. R. Stoll of Williams College has been appointed to an assistant professorship at Lehigh University.

Assistant Professor I. L. Stright of Baldwin-Wallace College, Berea, Ohio, has been appointed to an associate professorship at Northern Michigan College of Education, Marquette, Michigan.

Mr. Irving Sussman of Johns Hopkins University has been appointed to an assistant professorship at the University of Dayton.

Associate Professor Alfred Tarski of the University of California has been promoted to a professorship.

Associate Professor H. S. Thurston of the University of Alabama has been promoted to a professorship.

Associate Professor Gerhard Tintner of the Iowa State College of Agriculture and Mechanical Arts has been promoted to a professorship.

Dr. C. B. Tompkins has been appointed director of research, Engineering Research Associates, Incorporated, Saint Paul, Minnesota.

Associate Professor C. C. Torrance of Case School of Applied Science has been appointed to an associate professorship at the Postgraduate School, United States Naval Academy.

Dr. C. A. Truesdell has been appointed mathematician at the Naval Ordnance Laboratory, Washington, D.C.

Dr. G. B. Van Schaack has been appointed to an assistant professorship at Union College, Schenectady, New York.

Dr. Andrew Vazsonyi has accepted a position as engineer with North American Aviation, Los Angeles, California.

Dr. V. J. Varineau of the United States Naval Academy has been appointed to an assistant professorship at the University of Wyoming.

Dr. C. W. Vickery has been appointed to an associate professorship at the Wright Field Graduate Center of Ohio State University.

Dr. T. C. G. Wagner has been appointed to an associate professorship at the University of Maryland.

Associate Professor O. E. Walder of South Dakota State College has been promoted to a professorship.

Dr. P. R. Wallace has been appointed to an associate professorship at McGill University.

Dr. Henry Wallman of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor R. M. Walter of New Jersey College for Women, Rutgers University, has been promoted to an associate professorship.

Assistant Professor C. T. Wang of the University of Akron has been appointed to an assistant professorship of aeronautical engineering at New York University.

Mr. Joseph Wannemacher has accepted a position with the New Jersey Bell Telephone Company, Newark, New Jersey.

Associate Professor L. E. Ward of the State University of Iowa has been appointed mathematician with the Aircraft Fire Control Section, Research, Development, and Test Organization, Naval Ordnance Test Station, Inyokern, California.

Associate Professor W. G. Warnock of the University of Alabama has been appointed to an associate professorship at Rensselaer Polytechnic Institute.

Dr. J. V. Wehausen has been appointed mathematician at the David Taylor Model Basin.

Associate Professor Alexander Weinstein of the University of Toronto has been appointed to an associate professorship at Carnegie Institute of Technology.

Assistant Professor C. P. Wells of Michigan State College has been promoted to an associate professorship.

Dr. R. L. Westhafer has been appointed to an assistant professorship at New Mexico College of Agriculture and Mechanical Arts.

Professor A. H. Wheeler of Clark University has retired.

Dr. A. L. Whiteman of Purdue University has been appointed mathematician in the Office of the Chief of Naval Operations, Navy Department.

Dr. W. F. Whitmore of Massachusetts Institute of Technology has accepted a position as operations analyst with the Navy Department.

Dr. J. E. Wilkins has accepted a position as mathematician with the American Optical Company, Buffalo, New York.

Dr. Wilfred Wilson of the University of Illinois has been promoted to an assistant professorship.

Dr. Y. K. Wong of the University of North Carolina has been promoted to an associate professorship.

Associate Professor C. B. Wright of East Texas State Teachers College, Commerce, Texas, has been promoted to a professorship.

Assistant Professor C. R. Wylie of Ohio State University has been appointed to a professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Assistant Professor S. D. Zeldin of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Sister M. Claudia Zeller of the College of Saint Francis has been promoted to an assistant professorship.

The following appointments to instructorships are announced: University of Arizona: Miss Gloria Olive; Bay City Junior College, Bay City, Michigan: Miss Meta M. Ewing; Brooklyn College: Dr. H. S. Kieval, Mr. Ira Rosenbaum; Brown University: Dr. Bjarni Jónsson, Mr. Albert Wilansky; University of California: Dr. C. A. Hayes, Dr. Alfred Horn, Dr. E. L. Lehmann; University of California at Los Angeles: Dr. Leonard Greenstone; University of Chicago: Mr. R. S. Fouch, Dr. Daniel Zelinsky; University of Cincinnati: Mr. H. D. Lipsich; College of the City of New York: Mr. J. H. Blau, Dr. Solomon Hurwitz, Mr. Herbert Solomon; Clark Junior College, Vancouver, Washington: Mr. R. H. Stair; Cornell University: Dr. Y. H. Kuo; Creighton University: Miss Helen E. Clarkson; DePaul University: Rev. F. J. Fischer; Emory University: Mr. E. R. Rohrer; Franklin Technical Institute, Boston, Massachusetts: Mr. Robert Kates; Harvard University: Dr. G. P. Hochschild; Hollins College, Hollins College, Virginia: Miss Azelle Brown; University of Illinois: Dr. Theodore Bedrick, Dr. W. A. Ferguson, Dr. Joseph Landin, Dr. E. J. Scott; University of Illinois, Chicago, Illinois: Miss Alice J. Froseth, Mr. R. G. Stoneham; Illinois Institute of Technology: Dr. E. G. H. Comfort, Dr. M. H. M. Esser, Dr. Margaret S. Matchett; Johns Hopkins University: Mr. Emmanuel Mehr; University of Kentucky: Mr. D. F. Atkins; Lafayette College, Easton, Pennsylvania: Dr. J. C. Smith; University of Maryland: Mrs. Ruth A. Bari; University of Miami: Mrs. Georgia K. DelFranco; Michigan State College: Miss Laura J. Bullard, Miss Frances E. Davis, Mr. Emerson Grindall, Mr. H. H. Holloway, Mr. C. A. Jacokes; University of Minnesota: Mr. E. J. Specht; Montana State College: Dr. H. M. Schaerf; Newark College of Arts and Sciences, Rutgers University: Mr. J. J. McCarthy; New York University: Mr.

S. D. Bernardi; University of North Carolina: Mr. Edward Paulson; Northwestern University: Mr. V. C. Harris, Mr. R. E. von Holdt, Mr. D. H. Potts, Dr. R. H. Stark; University of Notre Dame: Rev. H. F. DeBaggis; University of Oklahoma: Mr. R. L. Wine; Oklahoma Agricultural and Mechanical College: Mr. Milton Berg, Mr. J. L. Cawood, Miss Helen Dayton, Mr. R. D. Morrison, Mr. P. C. Stanger; University of Pennsylvania: Dr. I. S. Cohen; Pennsylvania State College: Miss Helen F. Story; Purdue University: Dr. Lee Byrne; Rice Institute: Dr. H. D. Brunk; University of Rochester: Dr. Horace Komm; Rutgers University: Dr. L. M. Court, Mr. Aaron Galuten, Mr. Luis Nanni, Dr. H. J. Zimmerberg; Saint Louis University: Mr. A. J. Lorenz; Stanford University: Mr. A. V. Baez; State Teachers College, Superior, Wisconsin: Mr. J. O. Danielson; Syracuse University: Mr. B. K. Dickerson, Dr. P. W. Gilbert, Miss Eleanor A. Ludwig; University of Tennessee: Mr. R. B. Johnson; University of Texas: Mr. J. M. Hurt; Agricultural and Mechanical College of Texas: Mr. W. E. Beeman, Dr. J. A. Daum, Mr. N. W. Wells; United States Naval Academy: Mr. A. R. Craw, Mr. G. J. Mann, Dr. R. C. Rand, Dr. H. K. Sohl, Mr. W. J. Strange, Mr. E. G. Swafford; Wabash College, Crawfordsville, Indiana: Mr. P. T. Mielke, Mr. V. G. Robinson; Washington University: Mr. Marlow Sholander, Dr. W. J. Thron; University of Wisconsin in Milwaukee: Miss Mildred J. Brannon.

The death on June 14, 1946, of Federigo Enriques, formerly of the University of Rome, has been reported. A group of his students are instituting a "Grant Federigo Enriques," with the object of helping some of his pupils who are in financial difficulties in the continuation of their scientific careers.

Word has been received of the death of Professor Friedrich Hartogs, formerly of the University of Munich, in 1943 and of Professor Richard Baldus of the Technische Hochschule of Munich in 1945.

The death of Professor Basilio Mania of the University of Milan has been reported.

Mr. E. B. Escott of Oak Park, Illinois, died May 30, 1946, at the age of seventy-eight years. He had been a member of the Society since 1898.

Mr. A. M. Freeman, director of mathematical research at Boston Fiduciary and Research Associates, died May 20, 1946, at the age of forty-eight years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

347. A. T. Brauer: *Limits for the characteristic roots of a matrix.* II.

Let $A=(a_{\kappa\lambda})$ be a square matrix of order n. Set $P_\kappa=\sum_{\lambda=1,\lambda\neq\kappa}^{n}|a_{\kappa\lambda}|$. In an earlier paper (Duke Math. J. vol. 13 (1946)) it was proved that each characteristic root ω_ν of A must lie in at least one of the n circles $|z-a_{\kappa\kappa}|\leq P_\kappa$. This result will be improved as follows. Each characteristic root must lie in at least one of the $C_{n,2}$ ovals of Cassini $|z-a_{\kappa\kappa}||z-a_{\lambda\lambda}|\leq P_\kappa P_\lambda$. It follows in particular that, for $\nu=1, 2, \cdots, n$, $|\omega_\nu|\leq\max_{\kappa,\lambda=1,2,\cdots,n}\{|a_{\kappa\kappa}|+(P_\kappa P_\lambda)^{1/2}\}$. Received September 25, 1946.)

348. H. W. Brinkmann: *On the prime divisors of a polynomial with integral coefficients.*

Let $f(x)$ and $g(x)$ be polynomials with rational integral coefficients. It is shown in this paper that there exist infinitely many primes p that are divisors of both polynomials, that is, there exist integers a, b so that $f(a)\equiv g(b)\equiv 0 \pmod{p}$. The proof is not elementary. As a corollary it follows that it is impossible for all but a finite number of prime divisors of a polynomial to belong to the same residue class $kx+m$ if $m\not\equiv 1 \pmod{k}$. This answers a question raised by A. T. Brauer (Duke Math. J. vol. 13 (1946) pp. 235–238). The problem of deciding when there are infinitely many prime divisors of $f(x)$ that are *not* divisors of $g(x)$ is also attacked and as a consequence it is possible to generalize Brauer's main theorem in the paper just referred to. (Received August 26, 1946.)

349. E. R. Kolchin: *Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations.*

This is a modern algebraic development of the Galois theory of homogeneous linear ordinary differential equations of Picard and Vessiot. Based on a theory of algebraic groups of matrices (developed here, for any algebraically closed field of arbitrary characteristic, without recourse to Lie theory) and on the Ritt theory of algebraic differential equations, the paper extends the Picard-Vessiot theory, fills in its main gaps, and brings it up to present day standards of rigor. The coefficient domain is an arbitrary differential field of characteristic 0 with an algebraically closed field of constants. Among the results obtained are the analogue to the fundamental theorem of Galois theory (one-to-one correspondence between subgroups and intermediate fields), and an extension of Vessiot's big theorem on solvability "by quadratures." The first part of the development is quite general and contains the rudiments of a general Galois theory of differential fields (ordinary or partial) of characteristic 0. (Received September 16, 1946.)

350. E. R. Kolchin: *Extensions of differential fields*. III.

The purpose of the present note is to show how the point of view of a preceding paper (*Extensions of differential fields*. I, Ann. of Math. vol. 43 (1942) pp. 724–729) can be used in developing the concepts of resolvent, dimension and order introduced by J. F. Ritt in his theory of algebraic differential equations. (Received September 16, 1946.)

351. R. M. Robinson: *Unsymmetrical approximation of irrational numbers*.

As an application of a theorem about lattice points, B. Segre (Duke Math. J. vol. 12 (1945) pp. 337–365) has shown that for any $\tau \geqq 0$ every irrational number ξ admits infinitely many rational approximations A/B such that $-1/(1+4\tau)^{1/2}B^2 < A/B-\xi < \tau/(1+4\tau)^{1/2}B^2$. For $\tau=1$, this reduces to the inequality $-1/5^{1/2}B^2 < A/B-\xi < 1/5^{1/2}B^2$ of Hurwitz. For other values of τ, one side of the inequality is strengthened, the other weakened. In this paper, a proof of Segre's theorem is given using continued fractions. Some sharper results are also obtained; in particular, it is shown that for $\epsilon>0$ the inequality $-1/(5^{1/2}-\epsilon)B^2 < A/B-\xi < 1/(5^{1/2}+1)B^2$ has infinitely many solutions. This result is interesting since it shows that one side of Hurwitz's inequality can be strengthened without essentially weakening the other. (Received September 20, 1946.)

352. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra*. V.

Postulates for a group of ordered dyads are constructed in terms of elements of a set $S(\alpha)$ and a relation of equality (equivalence) between dyads $T(\alpha\beta)$ of these elements, as follows: 1. α, β in S imply $\alpha\beta$ in T, and conversely. 2. $\alpha\alpha$, $\beta\beta$ in T imply $\alpha\alpha=\beta\beta$. 3. $\alpha\beta$, $\gamma\delta$ in T imply the existence of ξ, η in S such that $\beta\xi=\gamma\delta$ and $\eta\gamma=\alpha\beta$. 4. $\alpha\beta=\alpha\xi$ implies $\xi=\beta$ and $\alpha\beta=\eta\beta$ implies $\eta=\alpha$. Definitions. 1. $\alpha\gamma$ is the "product" of $\alpha\beta$ and $\beta\gamma$. 2. $\lambda\mu$ is the product of $\alpha\beta$ and $\gamma\delta$, $\alpha\beta\times\gamma\delta=\lambda\mu$, means: There exists ξ such that $\beta\xi=\gamma\delta$ and $\alpha\xi=\lambda\mu$ or there exists η such that $\alpha\beta=\eta\gamma$ and $\eta\delta=\lambda\mu$. Relatively to the preceding postulates and definitions, any complete set of mutually nonequivalent dyads constitutes a group with a self-conjugate dyad ($\alpha\alpha$) as the identical element. Examples: (1) Any regular group of configurational sets of dyads: two dyads are equivalent if and only if they occur in the same configurational set. (2) The set of segments of points on a descriptive line (including null segments): $\alpha\beta=\gamma\delta$ if and only if $\alpha\beta$ and $\gamma\delta$ are congruent and have the same orientation. In the latter example $\alpha\gamma$ is usually designated as the "sum" of $\alpha\beta$ and $\beta\gamma$. (Received September 27, 1946.)

353. A. R. Schweitzer: *Sums and products of ordered dyads in the foundations of algebra*. VI.

The construction of postulates for a field of dyads is based on set S of elements (α) and a set T of ordered dyads (α/β): A_1. There exists ω uniquely in S such that for any α in S, α/ω is not in T. A_2. α, β in S, $\beta\neq\omega$, imply α/β in T and conversely. A_3. ω/ξ, ω/η in T, $\xi\neq\eta$, imply $\omega/\xi=\omega/\eta$. B_1. Excluding ω/ξ, the dyads of the set T form a group under multiplication relatively to the postulates and definitions of the preceding abstract (with $\alpha\beta$ replaced by α/β). B_2. Multiplication is commutative. C. The dyads of T, including ω/ξ, form a commutative group under the undefined composition "addition." Under C postulates are first stated (C_1) in terms of addition of dyads with the same posterior elements (denominators); it is then assumed (C_2) that α/β, γ/δ in T imply the existence of ξ such that $\gamma/\delta=\xi/\beta$. Definition: $\alpha/\beta+\gamma/\delta=\lambda/\mu$ means: There exist ξ, η such that $\gamma/\delta=\xi/\beta$, $\eta/\beta=\lambda/\mu$, where $\alpha/\beta+\xi/\beta=\eta/\beta$. D. Multiplication over addition to the right (left) is distributive. The independence of C_2 is discussed. Application is made to the author's *Foundations of Grassmann's extensive algebra* (Amer. J. Math. vol. 35 (1913) pp. 39–49). (Received September 27, 1946.)

354. J. D. Swift: *Periodic functions over a finite field.*

A function of period a over the Galois field, $GF(p^n)$, is defined as a function over the elements of the field to the elements of the field such that $f(x+na)=f(x)$, where n is an integer. Multiply periodic functions are defined in an analogous manner. A $GF(p^n)$ admits functions with a number of independent periods not greater than $n-1$. The basic function of period a is: $f(a;x)=x^p-a^{p-1}x$. The basic function of periods $a_1, \cdots, a_n$ may be defined recursively from the above as: $f(f(a_1, \cdots, a_{n-1}; a_n); f(a_1, \cdots, a_{n-1}; x))$. These basic functions are odd and additive, and $f(a_1, \cdots, a_n; cx)$ is periodic with periods $a_1/c, \cdots, a_n/c$. The principal result is: *Any periodic function of periods* $a_1, \cdots, a_k$ *over the* $GF(p^n)$, $k\leq n-1$, *may be expressed by*: $g(x)=\sum_1^l \alpha_i f^i(a_1, \cdots, a_k; x)+\alpha_0$), where $l=p^{n-k}-1$, *and the* α_i *are a set of elements of the field.* (Received September 16, 1946.)

355. P. M. Whitman: *Finite groups with a cyclic group as lattice-homomorph.*

It is shown that if G and H are groups, $L(G)$ and $L(H)$ their lattices of subgroups, $L(G)$ is finite, $L(H)$ is a lattice-homomorphic image of $L(G)$, and H is cyclic, then G contains a cyclic subgroup which is mapped onto H by the homomorphism. (Received September 26, 1946.)

ANALYSIS

356. Warren Ambrose: *Direct sum theorem for Haar measures.*

A variation of a theorem of A. Weil (*L'intégration dans les groupes topologiques*, Paris, 1940, pp. 42–45) is proved. (Received September 19, 1946.)

357. R. F. Arens: *Location of spectra in Banach *-algebras.*

A Banach *-algebra A is a Banach space with a continuous multiplication and a *-operation satisfying $(\lambda f+\mu g)^*=\bar{\lambda}f^*+\bar{\mu}g^*$, $f^{**}=f$, $(fg)^*=g^*f^*$, and $k|f|\,|f^*|\leq|ff^*|$, $k>0$. One can renorm A such that $\|fg\|\leq\|f\|\,\|g\|$, and there will exist $k'>0$ such that $k'\|f\|\,\|f^*\|\leq\|ff^*\|$. It is proved that, if $f=u+iv$, $u^*=u$, $v^*=v$, $uv=vu$, then $|x\cos\theta+y\sin\theta|\leq\|u\cos\theta+v\sin\theta\|$ for any complex number $x+iy$ in the spectrum of f, and $0\leq\theta\leq2\pi$. Thus if $f=f^*$, the spectrum is real. The case $k'=1$ studied by I. Gelfand and M. Neumark, Rec. Math. (Mat. Sbornik) N. S. vol. 12 (1943) pp. 197–213, is a special case. (Received August 8, 1946.)

358. Lipman Bers: *A property of bounded analytic functions.*

Let $f(z)$ be bounded and analytic for $|z|<1$. If $\{z_n\}$ is a sequence of points such that $z_n \rightarrow 1$, $\tan \arg (z_n - 1) = O(1)$, $f(z_n) \rightarrow \alpha$, then either $f = \alpha$ at infinitely many points, or a similar sequence exists on any nontangential path terminating at $z = 1$. If in addition $|z_n| \leqq |z_{n+1}|$, $|z_n - z_{n+1}| = O(1 - |z_{n+1}|)$, then either $f = \alpha$ at infinitely many points, or f possesses at $z = 1$ the sectorial limit α. (Received September 12, 1946.)

359. Lipman Bers: *On rings of analytic functions.*

Let D be a plane domain. The regular one-valued analytic functions defined in D form a topological ring $R(D)$. Theorem 1. Two rings $R(D)$ and $R(D')$ are isomorphic if and only if D' can be mapped onto D by a conformal transformation, or by a conformal transformation followed by a reflection. Theorem 2. If D is not the whole Riemann sphere, then any isomorphism between $R(D)$ and $R(D')$ is a homeomorphism due to either a conformal or an anticonformal transformation of D onto D'. (Received September 30, 1946.)

360. R. C. Buck: *Interpolation series.*

Carlson (Nova Acta Regiae Societatis Scientiarum Upsaliensis (4) vol. 4 (1915)) has given conditions on $h(\theta)$, the growth function of the entire function $f(z)$, under which the Newton series $\sum_{n=0}^{\infty} z(z-1) \cdots (z-n+1)\Delta^n f(0)/n!$ converges to $f(z)$. Gontcharoff (Rec. Math. (Mat. Sbornik) vol. 42 (1935) pp. 473–483) has achieved similar results for the Abel series $\sum_{n=0}^{\infty} z(z-n)^{n-1} f^{(n)}(n)/n!$. In the present paper, summability of these and other expansions is considered. In particular, the Newton series is Mittag-Leffler summable for all functions of the class $K(a, c)$, $c < \pi$, thus providing another proof of the familiar Carlson uniqueness theorem. These results are compared with those contained in a paper of Gelfond (Rec. Math. (Mat. Sbornik) N. S. vol. 4 (1938) pp. 115–148). A simple consequence is that if $f(z) \in K(a, c)$, $c < \pi$, and $(-1)^n \Delta^n f(0) \geqq 0$, then for any p, $\lim n^p \Delta^n f(0) = 0$; thus, if $f(n)$ is integral, $f(z)$ is a polynomial. (Received September 27, 1946.)

361. R. C. Buck: *On some properties of arithmetic density.* Preliminary report.

In a previous paper it was shown that if $\{A_k\}$ is a monotone sequence of sets of $\mathfrak{D}$, the class of all sets of integers having a density, a definition of $\lim A_k$ can be given, unique to within sets of density zero, and for which density is continuous. In the present paper, this is applied to yield a wide variety of results. A simple example is the following: if $f(z)$ is an entire function of type zero, and $|f(n+1)/f(n)| \geqq \delta > 0$, then $\lim n^{-1} \log |f(n)| = 0$. (Received September 27, 1946.)

362. V. F. Cowling: *A generalization of a theorem of LeRoy and Lindelöf.*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence unity. Let the coefficients a_n be the values taken on by an analytic function $a(z)$ at $z = 0, 1, \cdots$. Suppose $a(z)$ is analytic and single-valued with the possible exception of a pole of order K at infinity in an angle with vertex $k > 0$ (non-integral) on the axis of reals and including the axis of positive reals in its interior. Let the sides of this angle make angles ψ_1 and ψ_2 with the axis of positive reals. Then, if $\gamma > 0$ but otherwise arbitrarily small and $0 < \psi_1 < \pi/2$, $0 < \psi_2 < \pi/2$, $f(z)$ is regular in the domain common to $r \leqq \exp[\theta \tan \psi_1] - \gamma$ and $r \leqq \exp[(\theta - 2\pi) \tan \psi_2] - \gamma$ for $0 \leqq \theta < 2\pi$. A similar theorem is proved in the case that $a(z)$ has an essential singularity at infinity. If the angles ψ_1 and ψ_2 are both greater than 90° in magnitude, some theorems are proved which are closely related to the sufficient condition that a function defined by a Taylor series have exactly one singularity in the entire complex plane. (Received September 14, 1946.)

363. V. F. Cowling: *On the form of certain entire functions.*

Let $F(z)=\sum_{n=0}^{\infty}(a_n/n!)z^n$, where the a_n are integers. If $F(z)$ is an entire function not a polynomial and $\limsup_{r\to\infty} \log |F(re^{i\theta})|/r=\rho(\theta)<1+\cos\theta$ for $0\leqq\theta\leqq 2\pi$ then $F(z)=Q(z)e^z$, where $Q(z)$ is a polynomial in z. Other theorems of a similar nature are proved. (Received September 14, 1946.)

364. C. L. Dolph: *Nonlinear integral equations of the Hammerstein type.*

The author generalizes certain results of A. Hammerstein (Acta Math. vol. 54 (1930)), R. Iglisch (Math. Ann. vol. 108 (1933)), and M. Golomb (Math. Zeit. vol. 39 (1935)) on the existence of solutions to the nonlinear integral equation $\psi(x)=\int_a^b k(x, y)f[y, \psi(y)]dy$. A solution will exist in L_2 if the linear operator is Hermitian and if $f(x, y)$ satisfies $\mu_N y+A\leqq f(x, y)\leqq \mu_{N+1}y+A$ for large $|y|$, $\lambda_N<\mu_N<\mu_{N+1}<\lambda_{N+1}$, λ_N and λ_{N+1} being the Nth and $(N+1)$th characteristic values of the kernel. The proof is by the method of Leray-Schauder. The solution is unique and a constructive process is given for it when $\partial f(x, y)/\partial y$ satisfies the derivative of the above inequality. When the kernel is positive-definite and $f(x, y)$ satisfies $\mu_N y^2-C_N\leqq 2\int_0^y f(x, v)dv\leqq \mu_{N+1}y^2+C_{N+1}$ the equation is equivalent to an Euler-Lagrange equation of a functional over L_2. Although under these conditions no a priori estimate can exist solely for topological reasons, the calculus of variations in the large establishes the existence of a solution when the associated functional possesses only one maximum on each linear manifold in L_2 parallel to the one spanning the origin and the first N characteristic functions of the kernel. (Received August 23, 1946.)

365. W. F. Eberlein: *Weak compactness in Banach spaces.*

Let E be any Banach space and denote by E_T the linear topological space formed by the elements of E under the weak neighborhood topology. A set S in E_T is said to be compact if every infinite subset of S possesses at least one limit point in E_T. (As shown by Šmulian, this type of compactness is equivalent to weak sequential compactness.) It is proved that a set S in E_T is bicompact if and only if it is closed and compact. This result contains a theorem previously announced by the author (cf. Bull. Amer. Math. Soc. Abstract 52-7-231), as well as a demonstration of the equivalence of various notions of "weak" and "topological" completeness introduced by others. (Received August 5, 1946.)

366. Alfred Horn: *The asymptotic behavior of solutions of systems of Volterra integral equations.*

Previously announced results (Bull. Amer. Math. Soc. Abstract 52-5-154) are extended and generalized to systems. Using capitals for n-rowed square matrices and lower case for vectors, consider the system $u(x, y, \lambda)=\lambda\int_y^x K(x, t, \lambda)u(t, y, \lambda)dt+f(x, y, \lambda)$, where K and f have asymptotic developments of the form $\sum_{i=0}^{\infty}\lambda^{-i}K^i$, $\sum_{i=0}^{\infty}\lambda^{-i}f^i$. Suppose the characteristic roots $\gamma_i(x)$ of $K^0(x, x)$ are distinct and not 0 for each x in $[\alpha, \beta]$. Let Z be the region in which $Rl(\lambda\gamma_1(x))\geqq Rl(\lambda\gamma_i(x))$ for $\alpha\leqq x\leqq\beta$, and let Z_1, Z_2 be the subregions in which $Rl(\lambda\gamma_1(x))\leqq 0$, and $Rl(\lambda\gamma_1(x))\geqq 0$ respectively. Then u has an asymptotic expansion $\sum_{i=0}^{\infty}\lambda^{-i}F^i(x, y)e(x, y)+\sum_{i=0}^{\infty}\lambda^{-i}g^i(x, y)$ in Z, where $e_1(x, y)=\exp(\lambda\int_y^x\gamma_1(s)ds)$, the remainder being bounded in Z_1 and of order $\exp(\lambda\int_y^x\gamma_1(s)ds)$ in Z_2. The assumption $\gamma_1(x)\neq 0$ is essential here and for this case the above theorem constitutes a generalization of G. D. Birkhoff's theorem for systems of differential equations. Better bounds are obtained for the subregions in which $|\pi-\arg\lambda\gamma_1|\leq\pi/2-\delta$, $|\arg\lambda\gamma_1|\leq\pi/2-\delta$. When f is the pth column of K, u becomes the pth column of the resolvent S of K. In this case, $g^1=0$, and $g^2=\cdots=g^r=0$ for all the columns of S if and only if there exist matrices A^j, B^j such that $K^i(x, y)=\sum_{j=0}^{i}A^j(x)B^{i-j}(y)$, $0\leq i\leq r-2$. Applications are given. (Received July 29, 1946.)

367. W. N. Huff: *On the type of the polynomials generated by $f(x\tau)\phi(\tau)$.*

The polynomials $y_n(x)$ generated by $g(x,\tau)=f(x\tau)\phi(\tau)=\sum_0^\infty y_n(x)\tau^n$ with $f(x\tau)=\sum_0^\infty a_n(x\tau)^n/n!$ and $\phi(\tau)=\sum_0^\infty b_n\tau^n/n!$ are considered. Necessary and sufficient conditions for the polynomials to be of A, B and C type k and of infinite type in accordance with the definition of Sheffer are obtained along with the explicit relations satisfied by the polynomials in each case. The Hermite polynomials are the only orthogonal polynomials of type zero among the $y_n(x)$. For polynomials of A type 1 the differential equation and recurrence relation are found. The $y_n(x)$ include no orthogonal polynomials of A type 2 and the orthogonal polynomials of A type 1 are those of Laguerre. (Received September 19, 1946.)

368. R. C. James: *Inner products in normed linear spaces.*

Let $x\perp y$ be defined as meaning $\|x+ky\|\geq\|x\|$ for all k. If $x\perp y$ implies $y\perp x$, or if $y\perp x$ and $z\perp x$ imply $y+z\perp x$, then the norm can be defined by an inner product if the space is of three or more dimensions. However, $x\perp y$ and $x\perp z$ imply $x\perp y+z$ if and only if the unit sphere has a tangent hyperplane at each point. It then follows from the relations between orthogonality and linear functionals that the norm can be defined by an inner product if the space is of three or more dimensions and one of the following conditions is satisfied: (1) Whenever x and y are such that there is a nonzero linear functional f with $f(x)=\|f\|\cdot\|x\|$ and $f(y)=0$, then there is a nonzero linear functional g with $g(y)=\|g\|\cdot\|y\|$ and $g(x)=0$. (2) Whenever $f(x)=\|f\|\cdot\|x\|$ and $g(y)=\|g\|\cdot\|y\|$, there are numbers a and b (not both zero) such that $f(ax+by)+g(ax+by)=\|f+g\|\cdot\|ax+by\|$. (3) Normal projection is a linear operation if defined, where the normal projection of x on a linear subset L is the element $u\in L$ for which $\|x-u\|$ is minimum. (Received September 25, 1946.)

369. M. Z. Krzywoblocki: *A local maximum property of the fourth coefficient of schlicht functions.*

Let the power series $w=f(Z)=Z+a_2Z^2+a_3Z^3+a_4Z^4+\cdots$ define a one-to-one conformal mapping of the unit circle $|Z|<1$ onto some region in the w-plane. As is well known, Faber and Bieberbach proved that $|a_2|\leq 2$. Löwner showed that $|a_3|\leq 3$. Schaeffer and Spencer gave a second proof that $|a_3|\leq 3$. In the present paper it is shown that $|a_4|$ has a local maximum for the function $f(Z)=Z/(1-e^{i\phi}Z)^2=Z+2e^{i\phi}Z^2+3e^{2i\phi}Z^3+4e^{3i\phi}Z^4+\cdots$, thus completing certain calculations indicated by Yoh. More precisely, the following theorem was proved: If Löwner's K-function has the form $K(t)=e^{i[\alpha_0+\alpha(t)]}$ where α_0 is an arbitrary real constant and

$\alpha(t)$ is a real function satisfying the condition $|\alpha(t)| \leq 1/18000$, then for the corresponding fourth coefficient b_4, $|b_4| \leq 4$. No importance should be attached to the value 1/18000; it can easily be replaced by a somewhat larger number merely by refinement of the estimates given below. It would be desirable, however, to prove the theorem under the weaker restriction that $\int_0^\infty \alpha(t)^2 dt < \infty$. (Received September 12, 1946.)

370. G. W. Mackey: *On the domains of closed linear transformations in Hilbert space.* Preliminary report.

The principal result of this paper is the following. Let $\mathfrak{M}$ be the family of all closed linear subspaces of Hilbert space. Let $\overline{\mathfrak{M}}$ be the smallest family of linear subspaces of Hilbert space which includes $\mathfrak{M}$ and contains with each two subspaces A and B their intersection $A \cap B$ and their linear union $A + B$. Then a subspace is in $\overline{\mathfrak{M}}$ if and only if it is the domain of definition of a closed linear transformation. Furthermore, every member of $\overline{\mathfrak{M}}$ may be represented in the form $(A+B)\cap(C+D)$ where A, B, C and D are members of $\mathfrak{M}$. (Received September 30, 1946.)

371. A. P. Morse: *Perfect blankets.*

In this paper, which is to appear in Trans. Amer. Math. Soc., it is shown that a rather wide class of Euclidean closed sets can be so associated with a point that if f and ϕ are *any* two measures of a quite general sort then the Euclidean closed sets can be used in forming difference quotients $f(\beta)/\phi(\beta)$ which, in the limit, determine a derivative almost everywhere in the sense of ϕ. (Received August 30, 1946.)

372. P. V. Reichelderfer: *On the definition of the essential multiplicity for continuous transformations in the plane.*

Let $T: z = t(w)$, $w \in S$, be a bounded continuous transformation from the set S in the w-plane into the z-plane. In the paper by Radó and Reichelderfer entitled *A theory of absolutely continuous transformations in the plane* (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307) different definitions are used for the essential multiplicity of a point z under T in S, according as S is a bounded finitely connected Jordan region or a bounded domain. The main purpose of this note is to show that their definitions differ merely in form by proving the following theorem: Given a bounded continuous transformation $T: z = t(w)$, $w \in D$, defined on a bounded domain D in the w-plane, a positive number ϵ, and a point z in the z-plane, there exists a bounded continuous transformation $T_\epsilon: z = t_\epsilon(w)$, $w \in D$, whose distance from T on D is less than ϵ, and such that the number of models of z under T_ϵ in D is equal to the essential multiplicity of z under T in D. (Received August 7, 1946.)

373. P. C. Rosenbloom: *Entire associative functions.*

If $f(z_1, z_2)$ is an entire function of two complex variables and $f(z_1, f(z_2, z_3)) = f(f(z_1, z_2), z_3)$, then f is one of the functions c, z_1, z_2, z_1+z_2-a, $(z_1-a)(z_2-a)/b$ where a, b, and c are constants and $b \neq 0$. The proof depends on repeated use of Picard's theorem for functions of one complex variable. (Received September 30, 1946.)

374. W. T. Scott and H. S. Wall: *On the convergence and divergence of continued fractions.*

The authors show that a necessary condition for the convergence of the continued fraction $K_1^\infty(1/b_p)$ is that at least one of the following three statements holds: (a) $\sum |b_{2p+1}|$ diverges, (b) $\sum |b_{2p+1}s_p^2|$ diverges, where $s_p = b_2 + b_4 + \cdots + b_{2p}$, (c) $\lim_{p\to\infty} s_p = \infty$. This condition, which first appeared in a theorem of Hamburger, is called condition (H). In particular, they show that the continued fraction diverges if $\sum b_{2p}$, $\sum b_{2p+1}$ converge, at least one absolutely, thus extending a result of Stern and von Koch. The condition (H) is sufficient for convergence in the case where $b_{2p-1} = k_{2p-1}s_p$, $b_{2p} = k_{2p}$, $k_1 > 0$, $k_{2p+1} \geqq 0$, $R(k_{2p}) \geqq 0$, $R(s_p) \geqq \delta$, $|s_p| < M$ ($\delta > 0$, $M > 0$, $p = 1, 2, \cdots$). This result includes theorems of Stieltjes, Van Vleck, Hamburger and Mall. (Received August 19, 1946.)

375. I. E. Segal: *The group algebra of a locally compact group.*

Earlier results of the author (see Bull. Amer. Math. Soc. Abstract 46-7-366 and Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1940) pp. 348–352) are extended and refined. (Received August 10, 1946.)

376. J. E. Wilkins: *The converse of a theorem of Tchaplygin on differential inequalities.*

If $y(x)$ is a solution of the equation $L[y] \equiv y'' - p_1y' - p_2y - q = 0$, such that $y(x_0) = y_0$, $y'(x_0) = y_0'$, and if $v(x)$ is such that $L[v] > 0$, $v(x_0) = y_0$, $v'(x_0) = y_0'$, then $v(x) > y(x)$ when $x_0 < x \leqq x_1$ provided that x_1 is the first zero to the right of x_0 of the solution $u(x)$ of the equations $u'' - p_1u' - p_2u = 0$, $u(x_0) = 0$, $u'(x_0) = 1$. This is a best possible result in the sense that either x_1 does not exist or there exists a function $v(x)$ satisfying the above requirements for which $v(x) - y(x)$ vanishes at a point arbitrarily close to x_1. (Received September 27, 1946.)

APPLIED MATHEMATICS

377. H. W. Becker: *Circuit algebra.*

By means of the symbols $+$, $\|$, and $\perp$, any passive electrical network is representable on the linotype. They denote series, parallel, and bridge connections respectively, and have inverses $-$, $\_$, and $\top$. The definition $R = a\|b = ab/(a+b)$ generates a system parallel to ordinary arithmetic, except that infinity and zero exchange roles, and so on, hence called paraarithmetic. The number of integer solutions of this equation depends only on the prime factor structure, not magnitude, of R. If $R = p_1^{n_1} \cdots p_m^{n_m}$, this number is $\Psi_m(C+1)$, where $C_0 = 1$, $C_v = 2^{v-1}$, and Ψ is an expansion with generalized binomial coefficients (m, v), the sum of the products of the n's v at a time. Where all the n's equal 1, this reduces to $(3^m+1)/2$. Considered as a static structure, a *SP* network is collapsible. Rigidity is imparted by bridge connections, or trusses. The elementary model is the Wheatstone bridge $(a_1+a_2)\|(b_1+b_2)\perp\beta = [(a_1+a_2)\cdot(b_1+b_2)\beta + a_1a_2(b_1+b_2) + (a_1+a_2)b_1b_2]/[(a_1+a_2+b_1+b_2)\beta + (a_1+b_1)(a_2+b_2)]$. In this algebra, shorting or opening an operand effects remarkable transformations amongst the operators; as, $(a_1+a_2)\|(b_1+b_2)\perp 0 = a_1\|b_1 + a_2\|b_2$. In general, $\perp$ connections are specified by subscripts at the pluses involved. Thus the network whose components are the edges of a cubic lattice energized at two opposite vertices is $[(a_1 +_1 a_2)\|(b_1 +_2 b_2) + a_3]\|\,[c_1 + (c_2 +_1 c_3)\|(d_1 +_2 d_2)]\perp\beta_1\perp\beta_2$. By a method of combinatory synthesis, the total and transfer conductances are then formulated, alternatively to the Kirchhoff method. (Received September 9, 1946.)

378. Herman Chernoff: *A note on the inversion of power series.*

The author presents a method of calculating the coefficients of the inverse of a given power series. This method has the advantage of being compact, requiring only one page for calculations, of being systematic, and of not requiring the substitution in complicated formulae. In fact the only operations are cumulative multiplications and only occasional divisions. The method can be extended to many problems of calculations in power series. (Received August 10, 1946.)

379. R. E. Gaskell: *An extension of the finite Fourier transformation.*

Applications of the finite Fourier transformation (see R. V. Churchill, ***Modern operational mathematics in engineering***, p. 267) are limited to problems involving special linear differential operators, $\sum a_k \partial^k/\partial x^k$, where the coefficients, a_k, do not involve the independent variable, x. Further, the boundary conditions used must be of a special form. A more general transformation which can be used in problems involving any linear differential operator, L, of order r, is introduced. The transformation is defined by $T\{F\} = \int_a^b \psi_n F dt$, where ψ_n is a solution of the equation $\bar{L}(\psi) = \lambda_n^2 \psi$, involving the adjoint operator. The transformation can be inverted with the help of ϕ_n, a solution of $L(\phi) = \lambda_n^2 \phi$. The homogeneous end conditions $\Phi_i = 0$ for ϕ_n are taken from the boundary conditions of the problem, those for ψ_n are determined so that the bilinear concomitant has the special form $\sum \Phi_i \Psi_{r-i+1}$. An elementary example requiring the steady state temperature in a thick-walled cylindrical tube of finite length, with radiation at one end, serves to illustrate the more general transformation. (Received September 3, 1946.)

380. H. E. Goheen: *A bound for the error in computing the Bessel functions of the first kind by recurrence.*

A method is determined for obtaining a close upper bound to the absolute value of the error in the computation of $J_n(x)$, the Bessel function of the first kind, the method of computation assumed being the obvious one of using the recurrence relation on approximate values of $J_0(x)$ and $J_1(x)$. It is shown that while in general the absolute value of the error increases without limit, it is less than an easily determined function of n, x and an upper bound to the absolute values of the errors in $J_0(x)$ and $J_1(x)$. (Received September 16, 1946.)

381. Harry Polachek and R. J. Seeger: *On the existence of solutions for three-shock Prandtl-Meyer configurations in the case of weak reflected shocks.*

Simple analytical expressions are obtained for three-shock Prandtl-Meyer configurations for the case where the pressure ratio across the reflected shock ξ' approaches unity. For the presence of Prandtl-Meyer waves in the regions between (a) the reflected shock and the density discontinuity and (b) the incident and reflected shocks, these expressions are identical. By means of this expression, a theorem concerning the existence of Prandtl-Meyer waves in the above regions is derived. The ω-ξ plane (defined as the angle between the direction of material flow and the incident shock, and the pressure ratio across the incident shock, respectively) is divided into four regions by the curves $f = 0$ and $g - f = 0$, where g and f are known functions of ω and ξ. In one of these regions no solutions are possible, while two of these regions contain solutions only for case (a) above, and one region contains solutions only for case (b). (Received September 16, 1946.)

382. R. C. Roberts: *On the lift of a triangular wing at supersonic speeds.*

In a recent paper (Quarterly of Applied Mathematics vol. 4 (1946)) H. J. Stewart has found the lift on a flat plate wing, where the plan form of the wing is an isosceles triangle whose base is perpendicular to the free stream flow, and the wing lies entirely within the Mach cone through the vertex opposite to the base. In the present paper, the author has extended the method to the case where the plan form of the wing is any triangle under similar conditions. The result obtained shows that the slope $dC_L/d\alpha$ of the lift coefficient curve is $\{2\pi k/[E(k')(M^2-1)^{1/2}\}\{(k_1+k_2)/2k\}^{1/2}$. In this formula, M is the Mach number of the free stream, $E(k')$ is the complete elliptic integral of the second kind with modulus $k'=(1-k^2)^{1/2}$, k_1 and k_2 are related to the angles ω_1, ω_2 between the perpendicular to the base and the sides of the triangle by the formulae $k_1=(M^2-1)^{1/2}\tan\omega_1$, $k_2=(M^2-1)^{1/2}\tan\omega_2$, and $k=[1+k_1k_2-((1-k_1^2)(1-k_2^2))^{1/2}]/(k_1+k_2)$. When $k_1=k_2$ the result reduces to the formula $dC_L/d\alpha=2\pi\tan\omega_1/E(k')$, obtained by Stewart. (Received September 18, 1946.)

383. H. E. Salzer: *An alternative definition of reciprocal differences.*

In place of Thiele's reciprocal difference $\rho_n(x_1\cdots x_nx_{n+1})$ which has the advantage of symmetry in the x_i's, a new definition for the nth reciprocal difference is employed, namely $P_n(x_{n+1}x_n\cdots x_1)=\rho_n(x_1\cdots x_nx_{n+1})-\rho_{n-2}(x_1\cdots x_{n-2}x_{n-1})$. Although nonsymmetric in the x_i's, $P_n(x_{n+1}x_n\cdots x_1)$ is generated by a simpler recursion formula than the $\rho_n(x_1\cdots x_nx_{n+1})$ and enables Thiele's continued fraction interpolation formula to be expressed in a more concise and natural form. Both these properties lead to a saving of numerical work. The proof that the reciprocal difference (new definition) of a certain order of a rational function is constant is shown to be more direct than the proof of the same statement for Thiele's reciprocal differences. This new definition is also applied to the completely confluent case, where all the x_i's approach x_1. (Received September 7, 1946.)

384. H. E. Salzer: *Tables for facilitating the use of Chebyshev's quadrature formula.*

Chebyshev's n-point quadrature formula $\int_{-1}^{+1}f(z)dz=(2/n)\sum_{i=1}^{n}f(z_i)+R_n$, where z_i are the zeros of $T_n(z)=$ polynomial part of $z^n\exp(-n/2\cdot 3z^2-n/4\cdot 5z^4-\cdots)$, has the advantage of being equally weighted, but the disadvantage that the z_i's are irregularly spaced. For a partial check on all the $f(z_i)$ taken together, coefficients $D_i^{(n)}$ are given to obtain the $(n-1)$th divided difference as $\sum_{i=1}^{n}D_i^{(n)}f(z_i)$, for $n=3(1)7$, 9, $i=1, 2, \cdots, n$, mostly to $9S$. The calculations on $D_i^{(n)}$ were checked by the relations $D_i^{(n)}=D_{n-i+1}^{(n)}$, n odd, and $D_i^{(n)}=-D_{n-i+1}^{(n)}$, n even, and by use in examples. Use of the checking coefficients $D_i^{(n)}$, especially for the larger n's, is more likely to detect an error nearer to the central $f(z_i)$. Also for the same range of n and i, the zeros of the polynomials $T_n(z)$ are given to $10D$, extending all previous calculations. Since Chebyshev's formula is of practical use only when the zeros of $T_n(z)$ are all real, an investigation showed that $T_n(z)$ does not have all real roots for $n=11$ through 15 ($n=8$ and 10 already noted in the literature) and raises the interesting question whether $T_n(z)$ has all real roots for any $n>15$. The exact expressions for $T_n(z)$ are given for $n=1(1)12$. (Received September 12, 1946.)

385. S. A. Schaaf: *A cylinder cooling problem.*

The temperature distribution is obtained for a heat conducting region consisting of an infinitely long cylinder $0 \leq r < a$ initially at temperature T_0, immersed in an infinite medium $r > a$ composed of a different substance initially at zero temperature, with a contact resistance condition at the interface $r = a$. The Laplace transform is used and, in inverting, it is necessary to show that $D(s) = I_0'(\alpha s)K_0(\beta s) - \lambda I_0(\alpha s)K_0'(\beta s) - \mu s I_0'(\alpha s)K_0'(\beta s)$, where $I_0(s)$ and $K_0(s)$ are Bessel functions and α, β, λ and μ are positive real numbers, does not vanish for $|\arg s| \leq \pi/2$. This is done by considering the integral of $D'(s)/D(s)$ around a contour consisting of two semicircular arcs $|s| = R_1, R_2$ and the segments of the imaginary axis joining them. (Received September 26, 1946.)

386. Fred Supnick: *Cooperative phenomena.* I. *Structure of the linear Ising model.*

The partition function $f(T)$ (the physical term) plays an important part in the theory of crystal statistics (cf. C. H. Wannier, Review of Modern Physics (1945) pp. 50–60). Let the set of spins $u_1, \cdots, u_n$ each capable of two orientations be characterized by $u_i = +1$ or $u_i = -1$, and arranged in cyclic order. It is assumed that only adjacent elements interact. To evaluate $f(T)$ the interaction energy E must be found. E involves the calculation of $\Sigma = \sum_{i=1}^{n} u_i u_{i+1}$ where $u_{n+1} = u_1$. All spin distributions are considered in evaluating $f(T)$. The author calls Σ the interaction constant of the spin distribution. Now, let Σ_i be any integer with $|\Sigma_i| \leq n$. In this paper the set of all possible spin distributions with interaction constants equal to Σ_i is determined. A method is given for constructing each spin distribution with $\Sigma = \Sigma_i$. Results involving the number of spin distributions with the same interaction constant are obtained. Both cyclic and non-cyclic cases are considered. (Received September 28, 1946.)

GEOMETRY

387. Germán Ancochea: *Zariski's proof of the theorem of Bertini-Enriques in the case of an arbitrary ground field.*

Zariski (Trans. Amer. Math. Soc. vol. 50 (1941)) gave a new proof of the theorem of Bertini-Enriques on reducible linear systems of V_{r-1}'s on an algebraic V_r, by considering this theorem in a larger sense than the customary since irrational pencils are also included. The proof, given for the case of ground fields of characteristic zero, is based on several lemmas concerning the behavior, with respect to irreducibility, of an algebraic variety under ground field extensions. Most of these lemmas have been extended by Chevalley to the case of arbitrary ground fields (Trans. Amer. Math. Soc. vol. 55 (1944)). In the present paper the theorem of Bertini-Enriques, in the sense of Zariski, is extended to ground fields of characteristic $p \neq 0$. The auxiliary lemmas are reconsidered from a different standpoint than that of Chevalley, by using the Chow-van der Waerden concept of an associated form of an algebraic variety. It also has been found necessary to incorporate in Zariski's definition of an irreducible pencil on V_r the extra requirement that the field of functions on V be separably generated over the ground field. With these changes the theorem of Bertini-Enriques is proved essentially as in Zariski's paper, provided that the ground field be an infinite field for the case of linear systems. (Received August 2, 1946.)

388. John DeCicco: *Constrained motion upon a surface under a generalized field of force.*

Some theorems are given concerning the trajectories of a particle moving in a given smooth surface Σ when acted upon only by a generalized field of force. Kasner has developed the differential geometry of the dynamical trajectories of general positional fields of force on a plane, in space, and on a surface, in his Princeton Colloquium Lectures. Recently Kasner and DeCicco have introduced the concept of generalized (anistropic) fields of force which depend not only upon the position of the point but also upon the direction through the point. After discussing the differential equation of third order defining generalized dynamical trajectories on the surface Σ, the ∞^1 lines of force and the ∞^2 rest trajectories are compared. A formula for the ratio ρ of the departure of the rest trajectory to that of the line of force at the initial point is obtained. In the positional case, Kasner proved that $\rho=1/3$. The new formula involves the angular rate and is similar to the one obtained for the plane in the Transactions paper of 1943. (Received September 27, 1946.)

389. John DeCicco: *Union-preserving transformations of higher surface elements*

Transformations in space from differential surface-elements of order n: $(x, y, z, p_{10}, p_{01}, \cdots, p_{j,m-j}, \cdots, p_{0n})$ where $p_{j,m-j}=\partial^m z/\partial x^j\partial y^{m-j}$ for $j=0, 1, 2, \cdots, m$ and $m=1, 2, \cdots, n$, into planar-elements (X, Y, Z, P, Q) are studied. It is shown that the union-preserving transformations are defined by a single directrix equation of the form $\Omega(X, Y, Z, x, y, z, p_{j,m-j})=0$, where $j=0, 1, 2, \cdots, m$ and $m=1, 2, \cdots, n-1$, or by a pair or triplet of such directrix equations. It is noticed that directrix equations contain partial derivatives of z with respect to x and y, of orders up to and including $n-1$, but not n. For $n=1$, this result coincides with the theorem of Lie concerning contact transformations of planar-elements. Finally it is proved that the only available union-preserving transformations in the whole domain of surface-elements are, first, Lie's group of contact transformations and, second, the author's set of union-preserving transformations from surface-elements of order n into planar-elements, together with the extensions of these two types. Union-preserving transformations of curve-elements of higher order have been studied by Kasner and the author. (Received August 15, 1946.)

390. Edison Greer and P. O. Bell: *A study of analytic surfaces by means of a projective theory of envelopes.*

A new formulation of a projective theory of envelopes for the study of algebraic surfaces which are covariantly related to an analytic surface S at a point P_x has been recently developed (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 22–50). The present paper applies this theory of envelopes to investigate relationships of covariantly determined quadrics having contact of the second order with S at P_x with systems of curves on the surface. A characteristic curve is determined on each quadric Q at each point P_x of a curve C_λ of S. The cone which contains this characteristic curve and has as its vertex the point P_x is introduced as the *characteristic cone* $\mathfrak{C}_\lambda$ *of* Q *with respect to* C_λ *at* P_x. The geometric interrelations of the cone $\mathfrak{C}_\lambda$, the quadric Q, and the curve C_λ with respect to the surface S are investigated. By subjecting two of these entities to certain geometric restrictions, properties of the third are determined. The principal results thus obtained are geometric characterizations of significant families of quadrics, systems of hypergeodesics, the canonical pencil, projective curvatures of a curve C_λ, the general transformation of Čech, the curves of Darboux and Segre, and a class of generalized pangeodesics. (Received September 9, 1946.)

391. Walter Prenowitz: *Characterization of the lattice of convex sets of a descriptive geometry.*

L_1, the lattice of convex sets of a descriptive (ordered linear) geometry, and L_2, that of the linear spaces of a projective geometry, are characterized simultaneously. The geometries are of arbitrary dimension greater than 1. A new lattice concept *linear dependence of a point on a sequence of points* is introduced and used to define *closed* element as one which contains, with any sequence of points, all linearly dependent points. A generalization of modularity involving "closed element" is a common property of L_1 and L_2. Garrett Birkhoff's characterization of L_2 in the finite-dimensional case is deducible from the results. The simultaneous treatment of L_1 and L_2 is based on axiomatizations of descriptive and projective geometry in terms of point and "order" which differ only in a single postulate. (Received September 28, 1946.)

392. Alice T. Schafer: *The neighborhood of an undulation point on a space curve.*

This paper employs the methods of projective differential geometry to study the neighborhood of an undulation point on an analytic space curve. By a suitable choice of the projective coordinate system, canonical power-series expansions representing the curve in the neighborhood of the undulation point are deduced. These expansions are then used to study properties of the curve in the neighborhood of the point, with particular emphasis placed on osculants of the curve, projections of the curve onto the faces of the tetrahedron of reference, and sections of the tangent developable of the curve made by faces of the tetrahedron of reference. (Received September 28, 1946.)

393. Oscar Zariski: *The concept of a simple point of an abstract algebraic variety.*

Let V and W be irreducible algebraic varieties over an arbitrary ground field κ, of dimension respectively r and ρ, $W \subseteq V$. If $\mathfrak{m}$ is the maximal ideal in the quotient ring $\mathfrak{o}$ of W then the ring $\mathfrak{m}/\mathfrak{m}^2$, regarded as a vector space over the field $\mathfrak{o}/\mathfrak{m}$, is of dimension not less than $r-\rho$. If the dimension is exactly $r-\rho$, W is said to be simple for V. In the first part of the paper this general concept of a simple W is studied by purely local methods. In the second part the global theory is developed. Here the main result is a general Jacobian criterion for simple loci, which reduces to the classical one whenever κ is either of characteristic zero or is a perfect field of characteristic $p \neq 0$. This general criterion implies that the singular manifold of V is always an algebraic proper subvariety of V. An absolutely simple W is defined by the condition that it remain simple under any extension of the ground field. Criteria for an absolutely simple W are: (1) the ordinary Jacobian criterion of the classical case; (2) V is locally, at W, analytically eqivalent to a linear S_r. (Received August 22, 1946.)

LOGIC AND FOUNDATIONS

394. E. L. Post: *Recursive unsolvability of a problem of Thue.*

Thue's problem (Skrifter utgit av Videnskapsselskapet i Kristiania 1914. I.

Matematisk-Naturvidenskabelig Klasse, no. 10) may be restated as follows. Given an arbitrary finite set of symbols, with A's and B's arbitrarily given strings (zeichenreihen) involving no other symbols than those in the given set, P and Q operational variables, to determine whether B is an assertion in the system with initial assertion A and operations $P\ A_iQ$ *produces* $P\ B_iQ$, $P\ B_iQ$ *produces* $P\ A_iQ$, $i=1, 2, \cdots, \mu$. Through the intermediary of the Turing machine, a known recursively unsolvable decision problem is reduced to the decision problem of a system with initial assertion A' and operations $P\ A_i'\ Q$ *produces* $P\ B_i'\ Q$, $i=1, 2, \cdots, \mu'$, having the property that the set of assertions of the system is unchanged when the system is transformed into Thue type by adding the inverse operations $P\ B_i'\ Q$ *produces* $P\ A_i'\ Q$. The recursive unsolvability of the problem of Thue easily follows. (Received September 20, 1946.)

STATISTICS AND PROBABILITY

395. H. W. Becker: *Rooks and rhymes.*

Kaplansky and Riordan have shown that ${}_cR_{r-1}$, or the number of ways of putting c non-attacking rooks on a right-angled isosceles triangle of side $r-1$, is the Stirling number $\Delta^{r-c}0^r/(r-c)!$. This is the number of selections of c points on such a chess board, such that none have any row or column index in common, an idea incidental to various statistical problems. The point sets are well ordered, in 1-to-1 correspondence with the sequations (rhyme schemes) and distribution cycles. Further classifications ${}^{(i)}R_r$, ${}^{i}R_r$, ${}_{c'}R_r$, and ${}_{(c)}R_r$ are formulated in terms of rhyme functions according to: row location of topmost rook; number of rooks in the principal diagonal; column vacancies; and column location of the bottom rook. A typical isomorphism is ${}_{c'}R_r$ is equal to the R_r with cth column empty is equal to the number of distributions of $r+1$ men into crews, such that one man is incompatible with, and must be segregated from, $r-c$ other men. (Received July 24, 1946.)

396. Nilan Norris: *An extension of an equality among averages.*

A classic theorem of algbra states that if A, G, and H are respectively the arithmetic, geometric, and harmonic means of two positive real numbers, then $G^2=AH$. In this paper proof is given that a sufficient condition for the extension of this equality to any n positive numbers is that the logarithms of the variates be symmetrically distributed about an axis of ordinates at $\log G$ of the n numbers. For samples and populations obeying the symmetry condition with respect to $\log G$, the theorem is extended to an unlimited number of averages as yielded by certain sample and integral forms of general means (generalized means value functions). (Cf. an unpublished manuscript of J. B. Canning, *A theorem concerning a certain family of averages of a certain type of frequency distribution.*) (Received August 15, 1946.)

TOPOLOGY

397. Salomon Bochner and Deane Montgomery: *Groups on analytic manifolds.*

This paper studies the nature of complex and real Lie groups acting on certain complex or real manifolds in the large. For example, it proves that the group of all complex analytic homeomorphisms of a compact complex manifold is a complex Lie group. (Received September 20, 1946.)

398. C. H. Dowker: *An extension of Alexandroff's mapping theorem.*

A continuous mapping of a space into the nerve of an open covering is called canonical if the inverse image of the star of each vertex of the nerve is contained in the open set which corresponds to that vertex. An existence theorem for such mappings was proved by P. Alexandroff (Ann. of Math. (2) vol. 30 (1928) p. 121) and extended by S. Lefschetz (*Topics in topology*, Princeton, 1942, pp. 35–49). Another extension is the following: A normal space has a canonical mapping into the naturally topologized nerve of a covering if and only if the covering has a neighborhood-finite refinement. Consequently, a space has a canonical mapping into the naturally topologized nerve of every covering if and only if the space is paracompact and normal. The proof depends on showing that every complex with natural topology is a paracompact space. (Received October 1, 1946.)

399. Samuel Eilenberg and Saunders MacLane: *Determination of the second homology and cohomology groups of a space by means of homotopy invariants.*

The homotopy invariants of a topological space X include the fundamental group π_1, the second homotopy group π_2, and the fashion in which π_1 operates on π_2. A new invariant k^3 of the space X is introduced as a 3-dimensional (algebraic) cohomology class of π_1 with coefficients in π_2. By using this additional invariant, the second cohomology groups of the space can be constructed algebraically from π_1 and π_2. This includes the known determination of the subgroup of spherical annihilators as the second (algebraic) cohomology group of π_1. The homology groups are obtained by duality. If $k^3=0$, then the spherical subgroup of the homology group is a direct summand of the whole group. (Received August 23, 1946.)

400. Ky Fan: *On partially ordered additive groups of continuous functions.*

The totality $C(\Omega)$ of all real bounded continuous functions defined on a topological space Ω may be regarded as a *partially ordered additive group* (p.o.a.g.). An "*ideal space*" Ξ is constructed from the p.o.a.g. $C(\Omega)$. If Ω is a compact Hausdorff space, Ω and Ξ are homeomorphic. This provides a simple proof of the following theorem due to S. Kakutani (Ann. of Math. vol. 42 (1941) p. 1008), M. Krein and S. Krein (Rec. Math. (Mat. Sbornik) N.S. vol. 13 (1943) p. 25): If Ω and Ω^* are two compact Hausdorff spaces (or if they are completely regular and satisfy the first countability axiom), Ω and Ω^* are homeomorphic if and only if $C(\Omega)$ and $C(\Omega^*)$ are isomorphic as p.o.a.g. More precisely, the general form of the isomorphisms (in the sense of p.o.a.g.) between $C(\Omega)$ and $C(\Omega^*)$ is obtained. For any topological space Ω, to every "*order-preserving direct sum decomposition*" of the p.o.a.g. $C(\Omega)$ into two proper subgroups, there corresponds a decomposition of Ω into two disjoint closed proper subsets. A characterization of the dimension of a compact metric space Ω in terms of properties of the p.o.a.g. $C(\Omega)$ is also given. (Received September 16, 1946.)

401. R. H. Fox: *On a problem of S. Ulam concerning Cartesian products.*

Let A be the Cartesian product of a line segment with the bounded 3-dimensional manifold which is obtained from the lens-space (5, 1) by removing a small 3-cell, and denote by B the bounded 4-dimensional manifold which is similarly obtained from the lens-space (5, 2). It can be shown that A and B are not homeomorphic and that $A\times A$ is homeomorphic to $B\times B$. This answers in the negative a question proposed by S. Ulam (Fund. Math. vol. 20 (1933)). (Received September 24, 1946.)

402. S. T. Hu: *On extension of homotopy.*

A closed subset X_0 of a topological space X is said to have the absolute homotopy extension property AHEP if, for an arbitrary topological space Y, every partial homotopy $f_t(X_0)\subset Y$ of an arbitrary mapping $f_0(X)\subset Y$ has an extension $f_t^*(X)\subset Y$ such that $f_0^*=f_0$. It is well known that X_0 has the AHEP in X if X is a polyhedron and X_0 a subpolyhedron (Alexandroff-Hopf, p. 501). The author gives several generalizations of this well known fact. (1) If X_0 and X are compact ANR, then X_0 has the AHEP in X. (2) If $T=(X\times 0)+(X_0\times I)$ is an ANR, then X_0 has the AHEP in X. (3) If X is a compact ANR, then X_0 has the AHEP in X if and only if T is an ANR. The span $s(f_t, X)$ of a homotopy $f_t(X)\subset Y$ over X is defined to be the least upper bound of $\rho(f_{t_1}(x), f_{t_2}(x))$ for all $x\subset X$ and $0\leq t_1<t_2\leq 1$. For each $\epsilon>0$, every partial homotopy $f_t(X_0)\subset Y$ of an arbitrary mapping $f_0(X)\subset Y$ has an extension $f_t^*(X)\subset Y$ such that $f_0^*=f_0$ and $s(f_t^*, X)\leq s(f_t, X_0)+\epsilon$, provided that either of the following conditions is satisfied: (i) Y is an ANR and X a normal space; (ii) Y is a metric space and X, X_0 are compact ANR. (Received August 29, 1946.)

403. Deane Montgomery: *A theorem on locally euclidean groups.*

It is proved that if G is a locally euclidean, connected, simply connected topological group of dimension n greater than one, then G contains a closed subgroup of dimension greater than zero and less than n. (Received August 10, 1946.)

404. S. B. Myers: *Banach spaces of continuous functions.*

If B is a Banach space, let T denote any maximal subset of B with the property that, for any finite subset $(t_1, \cdots, t_n)$, $\|\sum t_i\|=\sum\|t_i\|$, and let F_T be the functional $F_T(b)=\inf_{t\in T}(\|b+t\|-\|t\|)$. If X is a topological space, $B(X)$ will denote the space of all real bounded continuous functions on X. A subset $E\subset B(X)$ is *completely regular* over X if for each closed set $C\subset X$ and point $x_0\in X-C$, there exists $b\in E$ such that $b(x_0)=\|b\|$, $\sup_{x\in C}|b(x)|<\|b\|$. The following two theorems are the main results of this paper, the first being a generalization of the Banach-Stone theorem, and the second a characterization of the space of continuous functions. I. If X_1, X_2 are compact, and if a closed, completely regular, linear subspace of $B(X_1)$ is equivalent to such a subspace of $B(X_2)$, then X_1 is homeomorphic to X_2. II. A Banach space is equivalent to $B(X)$ for some compact X if and only if the following conditions are satisfied: (a) every F_T is linear, (b) B contains an element e such that every $b\in B$ satisfies either $\|b+e\|=\|b\|+1$ or $\|b-e\|=\|b\|+1$, (c) for every $b\in B$ there exists $b'\in B$ such that for all F_T for which $F_T(e)=1$, $F_T(b')=|F_T(b)|$. (Received September 27, 1946.)

405. Everett Pitcher: *Critical point inequalities for a nondegenerate functional.*

A simplified proof is presented of the inequalities of Morse for the case of a positive regular integral of the calculus of variations on curves joining two fixed points on a manifold in the nondegenerate case for which the two points are conjugate on no extremal. Noteworthy features are the use of singular homology, the avoidance of double neighborhood definitions and techniques and of the concept of upper reducibility and the construction of deformations which carry cycles to their inferior limits. If J_c^- denotes the set of curves for which $J<c$ and if g is an extremal on which $J=c$, then the extremal is characterized by the relative homology groups $H_r(J_c^- \cup g \bmod J_c^-)$. The method is related to ideas in the last section of a paper by Morse on functional topology (Ann. of Math. vol. 38 (1937) pp. 386-449). Received September 28, 1946.)

406. Everett Pitcher: *Exact homomorphism sequences for a nest of three spaces.*

The basic construction of an exact homomorphism sequence is set up for relative homology theory of a nest of three chain complexes and for a nest of three sets in the singular and Čech theories. The duality on homology and cohomology sequences is set up in the abstract form and carried over to the two applications. Relative manifold duality is discussed, the duality being between a sequence of relative Čech homology groups for a nest consisting of a manifold and two closed subsets and a sequence of singular homology groups on complementary subsets in complementary dimensions. The work is an extension of a paper with J. L. Kelley titled *Exact homomorphism sequences in homology theory* and announced in Bull. Amer. Math. Soc. Abstracts 52-1-49 and 52-5-209. It is related to the construction of the exact sequence on relative homology groups of three sets as a theorem in the axiomatic homology theory of Eilenberg and Steenrod and to the abstract developments of Mayer. (Received September 28, 1946.)

407. Fred Supnick: *On the problem of Tait.* Preliminary report.

Let an alternating chain of marks, +'s and −'s, be associated with all the vertices of a polygon, except two which are adjacent. Let some two adjacent marks be transformed in a sense preserving manner along the polygon, to the blank spaces. The problem of Tait deals with the number of such transformations which are needed to separate the +'s from the −'s (with different signs adjacent to the blank spaces). This problem has been solved by H. Dellanoy (cf. W. Ahrens, *Mathematische Unterhaltungen und Spiele*, vol. 1, p. 19). In this paper the author studies the above problem subject to the restriction that *the pair of marks which are moved in each transformation must be different.* The number of transformations required to achieve separation becomes a function of the number of blank spaces. Formulas are obtained for the least number of blank spaces for which a separation is possible, and for the least number of steps in which the separation can be effected. A method is given for effecting this separation. It is also noted that these problems of Tait might be interpreted as problems of cooperative phenomena where the marks are considered as spins. (Received September 28, 1946.)

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E. H. Moore, 1901–1902.
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H. S. White, 1907–1908.
Maxime Bôcher, 1909–1910.
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L. E. Dickson, 1917–1918.
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Solomon Lefschetz, 1935–1936.
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ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $71,000, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases*, and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space*, and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's function and the* *problem of Plateau*, *The most general form of the problem of Plateau*, and *Solution of the inverse problem of the calculus of variations.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and Editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebren und ihre Zahlentheorie*, Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem*, which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes*, published in volume 41 of the Annals of Mathematics.

Fifth Award, 1944: To Oscar Zariski, for four papers on algebraic varieties, published in volumes 61 and 62 of the American Journal of Mathematics and volumes 40 and 41 of the Annals of Mathematics.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund is to be used, at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,500.

The Ernest William Brown Fund.

From the estate of Professor Ernest William Brown, a fund of $1,000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

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November 1, 1946

The following list includes the name of every person who is a member of the Society as of November 1, 1946.

So far as information is available the listings are made up as follows: (a) Surname. A dagger before a surname indicates life membership. (b) Initials, or the given name when there is but one, except that the first given name of every woman member is given in full. Parentheses enclosing any part of a name indicate that that part is not generally used in correspondence. (c) Official title or position. (d) Name and address of institution or organization with which item (c) is associated. (e) The address to be used for mail if this is different from (d). The name of the city and postal district number, where necessary, are not repeated if they are the same as in (d). (f) Following the regular listing, information re military service or temporary appointments is given.

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Norton, K. A.
Rhodes, J. S.
Rodabaugh
Scott, M. L.
Snow
Spencer, V. E.
Tepping
Georgetown Univ.
Sohon
George Washington Univ.
Humphreys
Johnston
Mears
Morrow, D. J.
Nelson, D.
Sherwood, J. E.
Taylor, J. H.
Weida
Howard Univ.
Blackwell
Branson
Cox, E. F.
Woodard
Labor Dept.
Jablon
Simmons, W. R.
Miner Teachers Coll.
Haynes, E. L.
Navy Dept.
Brown, A. A.
Bumer
Burington
Campaigne
Cramer, G. F.
Draim
Eachus
Eaton
Fine, N. J.
Fisher, D.
Gillman
Gingerich
Ginsberg
Goldberg, M.
Harkin
Kitchens
Knight, K. V.
Kossack
Lancaster
Lengyel
Marston
May, D. C.
Miller, A.
Morkovin
Newell
Penny
Polachek
Rigby
Shaw, R. H.
Siebielec
Truesdell
Wehausen
Wertheimer
Weyl, F. J.
Whiteman
Whitmore
Wolkow
Wray
Wrench
Wundheiler
Social Security Admin.
Herschfeld
Hoy
State Dept.
Strom
U.S. Coast and Geodetic Survey
Lambert
U. S. Weather Bur.
Leight
Norton, H. W.
War Dept.
Blake, A.
Blanche
Dantzig, G. B.
Dribin
Getchell
Sinkov

FLORIDA

Coral Gables
Univ. of Miami
Del Franco
Meyer, H.
Swingle
Gainesville
Univ. of Florida
Blake, R. G.
Germond
Gillis, M. E.
Lang
Meyer, H. A.
Quade
Simpson, T. M.
Smith, C. B.
Wilson, W. H.
Lakeland
Reinsch
Miami Beach
Ablow
St. Petersburg
Burt
Fraser, F.
Tallahassee
Wade, T. L.

GEORGIA

Athens
Jarnagin
Univ. of Georgia
Bancroft
Barrow
Beckwith, W. S.
Callaway
Fort, T.
Hemmingsen
Huff, G. B.
Levit
Stephens, R. P.
Ward, J. A.
Atlanta
Clark Coll.
Dennis
Georgia School of Tech.
Fulmer
Hefner
Martin, W. A.
Perlin
Smith, D. M.
Collegeboro
Moye
Decatur
Robinson, H. A.
Winter
Emory University
Clark, C. E.
Rohrer
Ft. Valley
Pitts, R. J.
LaGrange
Bailey, E. A.
Milledgeville
Nelson, S. L.
Rome
Hightower

HAWAII

Honolulu
Castle
Lu
Univ. of Hawaii
Watanabe

IDAHO

Caldwell
Rankin
Moscow
Taylor, E.
Nampa
Allison

ILLINOIS

Alton
Kean
Belleville
Segner
Bloomington
Hunt, M.
Carbondale
McDaniel, W. C.
Mayor
Carlinville
Renner
Champaign
Fry, W. J.
Chicago
Andersen
Ballard
Bradfield
Campbell, D. F.
Creagh
Hill, C. G.
Johannes
Jones, G. M.
Korzybski
Mansfield
Murray, F. H.
Nichols, C. A.
Paganelli
Peach, L. C.
Schweitzer, A. R.
Smith, B.
Armour Research Foundation
Barten
Bisshopp
Jaramillo
DePaul Univ.
Corliss
D'Arco
Fischer, F. J.
Svoboda
Illinois Inst. of Tech.
Bibb
Comfort
DeCicco
Dorroh
Esser
Ford, L. R.
Givens
Krathwohl
Matchett
Menger
Oldenburger
Pall
Reingold
Resnikoff
Sadowsky
Wilcox
Loyola Univ.
Gerst
North American Life Ins. Co.
Hornberger
Roosevelt Coll.
Gore
St. Francis Xavier Coll. for Women
Holland
Univ. of Chicago
Albert, A. A.
Barnard
Bartky
Bliss
Charney
Cole, K. S.
Dickson
Dinkines
Dragstedt
Everett, H. S.
Finch
Fouch
Friedman, M.
Graves, L. M.
Halmos
Hartung
Hestenes, M. R.
Hughart
Jacoby

Kaplansky
Karush
Kelley, J. L.
Lane, E. P.
Leavens
Logsdon
Lunn
Marlin
Massey, W. S.
Meyer, H. L.
Moore, J. T.
Newlander
Northrop
Putnam, A. L.
Schilling
Senior, J. K.
Sherman, S.
Stone, M.H.
Tanimoto
Toft
Trombley
Yntema
Young, J. W. A.
Zelinsky
Univ. of Illinois
Bailey, H. W.
Besancon
Davis, James Edgar
Frank
Froseth
Wilson Junior Coll.
Kinney, J. M.
Rasmussen
Sachs
Stoneham
Wright Junior Coll.
Christiansen
Moran
Du Quoin
Carter, G. R.
Elmhurst
Baumgart
Zink
Evanston
Smiley, D. M.
Stange
Northwestern Univ.
Allen, W. R.
Buell
Clark, H. M.
Curtiss, D. R.
Dobbie
Griffiths
Harris, V. C.
Hellinger
Hildebrandt, E. H. C.
von Holdt
Lindstrum
Lonseth
Moulton, E. J.
Olson, F. C. W.
Peck, E. R.
Potts
Reid
Scott, W. T.
Simmons, H. A.
Smiley, M. F.
Stark, R. H.
Wescott
Wolfe, R. S.
Washington National Ins. Co.
Carter, E. R.
Galesburg
Stephens, R.
Joliet
Zeller
Jonesboro
Lence
Lake Forest
Curtis
Mokre
Libertyville
Hart, W. W.
Lisle
Shonka
Monmouth
Beveridge
Mt. Carroll
Baxter
Naperville
Keeler
Seybold
Normal
McCormick, C. T.
Rine
Northbrook
Schweitzer, E. O.
Oak Park
Lane, R. E.
Pekin
Van Orstrand
Peoria
Comstock
Moore, M. G.
Quincy
Velez
River Forest
Steele
Rock Island
Cederberg
Urbana
Cattell
Ketchum, G. S.
Univ. of Illinois
Armstrong
Atkins, D. F.
Baer
Ball
Bedrick
Bourgin
Bower, O. K.
Brahana
Carmichael
Carruth
Chanler
Charnes
Coble
Day, M. M.
Doob
Emch
Ferguson, W. A.
Freundlich
Hanna
Hattan
Hazlett
Hoersch
Hostinsky
Howard, B. E.
Johnson, P. B.
Ketchum, P. W.
Kinney, J. R.
Krzywoblocki
Landin
Langebartel
Levy
Mathews
Mendel
Meserve
Miles, H. J.
Miller, G. A.
Morfoot
Munroe
Nantkes
Pepper, E. D.
Peters, J. W.
Phillips, C. A.
Propes
Schubert
Scott, E. J.
Shanks, E. B.
Sinclair, A.
Springer, M. D.
Stubbe
Townsend
Trjitzinsky
Turquette
Vaughan
Walkley
Wasserman
Welker
Whitman, D. W.
Wilson, W.
Zariski
Wheaton
Brandt

INDIANA

Bloomington
Indiana Univ.
Eilenberg
Gilbarg
Gordon, R. D.
Hamilton, N. T.
Hennel
Peak
Rothrock
Thomas, T. Y.
Williams, K. P.
Wolfe, H. E.
Youngs
Zorn
Collegeville
Maziarz
Zanolar
Crawfordsville
Wabash Coll.
Mielke
Polley
Robinson, V. G.
East Chicago
Burns, H. E.
Ft. Wayne
General Elec. Co.
Satterthwaite
Indiana Tech. Coll.
Olson, H. L.
Lincoln National Life Ins. Co.
Smith, F. C.
Goshen
Hartzler
Greencastle
Edington
Indianapolis
Cox, L. C.
Naval Ordnance Plant
Albert, G. E.
Nielsen
Purdue Univ.
Taylor, L. B.
Lafayette
Purdue Univ.
Alaoglu
Ayres, W. L.
Burr
Byrne, L.
Dyer-Bennet
Golomb
Graves, G. H.
Hazard, C. T.
Holtom
Hughes

Jonah
Lewis, D. J.
Marshall, W.
Miller, G. T.
Niven
Perlis
Robbins, C. K.
Schweigert
Shanks, M. E.
Smith, A. H.
Stone, R. B.
Sturm
Webster
Young, G. S.
North Manchester
Peabody, M. K.
Notre Dame
St. Mary's Coll.
Hackett
Univ. of Notre Dame
Brown, F. L.
DeBaggis
Gassensmith
Isaacs
Kenna
LaSalle
Milgram
Pepper, P. M.
Ross
Paoli
Bentley
St. Mary-of-the-Woods
Hayden
Terre Haute
Hodge
Palmer
West Baden Springs
Hausmann
Lemmer
West Lafayette
Westlund
Whiting
Flodin

IOWA

Ames
McKelvey, M. M.
Iowa State Coll. of Agric. and Mech. Arts
Allen, E. S.
Anderson, E. W.
Atanasoff
Beach
Bortle
Brown, G. W.
Carr
Daniells
Dyer
Gaskell
Gouwens
Herr
Higdon
Hinrichsen
Holl
McKelvey, J. V.
Robinson, P. G.
Rock
Sealander
Smith, E. R.
Smith, H. F.
Thielman
Tintner
Vinograde
Cedar Falls
Kearney
Iowa State Teachers Coll.
Trimble
Van Engen
Cedar Rapids
Coffin
Swanson
Davenport
Hratz
Decorah
Jacobsen
Des Moines
Drake Univ.
Gillam
Harper
Equitable Life Ins. Co. of Iowa
Groth
Hunter, R. G.
Irwin, P. C.
McCankie
MacKinnon
Dubuque
Burns, M. R.
Fairfield
Selz
Grinnell
McClenon
Rusk
Iowa City
State Univ. of Iowa
Carpenter, F. M.
Chittenden
Conkwright
Cosby
Craig, A. T.
Halfar
Knowler
Oberg
Price, H. V.
Woods, R.
Mt. Vernon
Davis, W. M.
Moots
Oskaloosa
Simon, H.
Sioux City
Graber
Rochford

KANSAS

Atchison
Pretz
Sullivan, H.
Emporia
Coll. of Emporia
Linscheid
Kansas State Teachers Coll.
Peterson, O. J.
Tucker, C. B.
Hays
Schaefer, G. R.
Lawrence
Univ. of Kansas
Babcock, W.
Bell, P. O.
Black, F.
Jordan, H. E.
Price, G. B.
Schatten
Smith, G. W.
Stouffer
Wheeler, J. J.
Lindsborg
Marm
Manhattan
Kansas State Coll. of Agric. and Applied Science
Babcock, R. W.
Chako
Greer
Janes
Mossman
Remick
Sanger
Stratton
North Newton
Richert
Pittsburg
Shirk
Smith, R. G.
Salina
Arnoldy
Topeka
Eberhart
WaKeeney
Gibson
Wichita
Beech Aircraft Corp.
Gatewood
Farmers and Bankers Life Ins. Co.
Longenecker
Univ. Of Wichita
Reagan
Winfield
Bernstorf

KENTUCKY

Barbourville
Peters, P. I.
Berea
Hutcherson
Simond
Bowling Green
Yarbrough
Covington
Frisch
Danville
Robinson, W. J.
Lexington
Univ. of Kentucky
Boyd
Church, E. T.

Cohen, L. W.
Downing
Goffman
Grau
Latimer
LeStourgeon
Pence
South
Louisville
Bullitt
Rommel
Nazareth Coll.
Fowler, M. C.
Univ. of Louisville
Moore, W. L.
Parker, S. T.
Stevenson
Ursuline Coll.
Schaeffer, M. R.
Murray
Carman
Richmond
Jenkins, E. D.
Park, R. S.
Smithfield
Atchison
Winchester
Howard, A. S.

LOUISIANA

Baton Rouge
Louisiana State Univ.
Cole, J. P.
Karnes
Parker, W. V.
Rees, P. K.
Rickey
Rutt
Smith, H. L.
Hammond
Davis, M. R.
Lafayette
Loflin
New Orleans
Cordrey
Evans, J. O.
Singer, M.
Tulane Univ. of Louisiana
Beard
Buchanan, H. E.
Duren
Hersh
Humphreys
Many
Miles, E. P.
Riess
Thomson
Weiss
Ruston
Gentry
Shreveport
Griffith

MAINE

Brunswick
Bowdoin Coll.
Christie
Hammond
Holmes
Moody
Kennebunk
Levinson, H. C.
Orono
Univ. of Maine
Bryan
Caton
Comegys
Hohn
Kales
Kimball, S. H.

MARYLAND

Aberdeen Proving Ground
Cox, M. J.
Curran
Dederick
Galbraith
Giese
Hart, B. I.
Landau, H. G.
Levin, J. H.
Reklis
Reno
Smith, T. L.
Sternberg
Annapolis
Torrance, E. M.
U. S. Naval Acad.
Abbott
Bailey, R. P.
Benac
Betz, E. E.
Bleick
Brady, C. P.
Bramble
Church, R.
Clements
Conrad
Craw
Denbow
Downs
Duthie
Gere
Gutzman
Hawkins
Kells
Kiernan
Laden
McLaughlin
Mann, G. J.
Mewborn
Moore, T. W.
Morrow, R. C.
Muhly
Rand
Rawlins
Ritter, E. K.
Root
Scarborough
Seekins
Sohl
Strange
Swafford
Torrance, C. C.
Wilson, L. T.
Baltimore
Bowie, C.
Cook, I. D.
Robinson, L. B.
Goucher Coll.
Bacon, C. L.
Torrey
Johns Hopkins Univ.
Bernstein, A.
Bitterli
Bourne
Carpenter, F. D.
Clifford
Coddington
Cohen, A.
Davids
Doyle, W. L.
Hartman
Haviland
Huck
Isaacson
Jacobson, N.
Larsen, T.
Light
Mannos
Morrill
Murnaghan
Putnam, C. R.
Reed, L. J.
Wallach
Wintner
Loyola Coll.
Dawkins
Mt. St. Agnes Junior Coll.
Thomas, M. P.
College Park
Univ. of Maryland
Bari
Dantzig, T.
Good
Hall, D. W.
Jackson, S. B.
Lewis, D. C.
Martin, M. H.
Ringenberg
Vanderslice
Wagner, T. C. G.
Edgewood Arsenal
Cook, E. A.
Frostburg
Hallett
Hagerstown
Serbin
Patuxent River
Hestenes, A. D.
Silver Spring
Johns Hopkins Univ.
Byers
Cronvich
Edmonson
Munro, G. C.
Sigley
Towson
Jacobson, F.
Westminster
Spicer

MASSACHUSETTS

Amherst
Amherst Coll.
Apostle
Breusch
Brown, B.
Esty
Sprague
Belmont
Burke, B. H.
Franklin, C. W.
Boston
Brainard
Coolidge, T. J.
Dowker, Y. N.
Dysart
Goheen
Gould, A. B.
LeBeau
Miller, A. L.
Riblet
Rich
Rines
Roberts, L. W.
Boston Univ.
Bruce
Johanson
Mode
Franklin Tech. Inst.
Kates
Harvard Univ.
Brown, T. H.
Wilson, E. B.
John Hancock Mutual Life Ins. Co.
Garabedian, H. A.
Hoskins
Northeastern Univ.
Callahan, W. R.
Haskins
Spear
Bridgewater
Wood, F. L.
Cambridge
Agnew, J. L.
Boas, M. L.
Jones, R. C.
Sobczyk
Wilson, E. W.
Zulon
Cambridge Univ.
Widder, V. A.
Harvard Univ.
Ahlfors
Akutowicz
Beatley
Benedict
Bergman
Birkhoff
Bridgman
Buck, R. C.
Burton
Carter, W. C.
Caywood
Cohen, C.
Cohn, H.
Coolidge, J. L.
Crabtree
Davis, P.
Ellis, M.
Epstein, Bernard
Fine, L. A.
Fishback
Gleason, A. M.
Gorenstein
Haimo
Hirschman
Hochschild
Huntington
James, R. C.
Jenkins, J. A.
Kelley, T.
Kemble
Loomis
Lorenz, E. N.
Lyman
Mackey
MacLane, S.
Mises
Mosteller
Mostow
Moyls
Papas
Pierce
Pitcairn
Rulon
Schoenfeld
Seligman
Shapley
Sheffer, H. M.
Shields
Sopka
Springer, G.
Stewart, F. M.
Sutton, F. X.
Van Vleck
Walsh, J. L.
Westergaard
Whitney, H.
Widder, D. V.
Wolfe, J. H.
Zilber
Massachusetts Inst. of Tech.
Andrew
Bartnoff
Bothwell
Clippinger
Crandall
Crout
Franklin, P.
Harvey, G. G.
Hildebrand
Hurewicz
Laning
Levinson, N.
Loud
Martin, A. V.
Martin, W. T.
Mills, B. E.
Mooers
Moon
Painter
Parke
Phillips, H. B.
Pitts, W.
Raisbeck
Reissner
Salem
Secada
Struik
Thomas, G. B.
Thomsen
Wallman
Wiener
Woods, F. S.
Zeldin
Chestnut Hill
Eiardi
O'Donnell
Chicopee
Madden
Danvers
French
Ft. Devens
Basch
Ipswich
Kinsman
Leominster
Bissinger
Lynn
Bauer
Medford
Tufts Coll.
Barnes
Breakwell
Dowker, C. H.
Fulton
Graustein
Spencer, D. E.
Whitman, P. M.
Milton
Pocock
Newton
Dewing
Northampton
Smith Coll.
Busemann
McCoy, N. H.
O'Neill
Rambo
Norton
Wheaton Coll.
Garabedian, C. A.
Geiringer
Nickerson
Pittsfield
Washburne
South Hadley
Jordan, H. A.
Mt. Holyoke Coll.
Bates, G. E.
Johnson, R. E.
Kiokemeister
Litzinger
Southbridge
Boeder
Springfield
Bowie, H. E.
Waltham
Marcou
Smith, C. V. L.
Wellesley
Wellesley Coll.
Ayer
Buck, E. F.
Butcher
Copeland, L. P.
Merrill, H. A.
Russell, H. G.
Stark, M. E.
Young, M. M.
West Somerville
Mancib
Weston
Burke, M. L.
Williamstown
Williams Coll.
Agard
Gillette
Hardy, J. G.
Miser, H. J.
Richmond, D. E.
Wells, V. H.
Woods Hole
Savage, L. J.
Worcester
Wheeler, A. H.
Coll. of the Holy Cross
McBrien
McCormick, J. T.
O'Callahan
Worcester Polytech. Inst.
Gay
Morley
Rice, H

MICHIGAN

Adrian
Alexander, H. W.
Ann Arbor
Grennan
Leisenring
Peterson, J. K.
Roettinger-Kaplan
Univ. of Michigan
Albers
Ambrose
Bartels
Bennett, C. A.
Boothby
Bradshaw
Burks
Churchill, R. V.
Coburn
Coe
Copeland, A. H.
Craig, C. C.
Cronin
Davidson, F. H.
Dickinson
Dolph
Dushnik
Dwyer
Eberlein
Elder, J. D.
Ellis, W.
Field
Fischer, C. H.
Ford, W. B.
Goldstine
Hay
Hildebrandt, T. H.
Hizon
Hopkins, L. A.
Jones, P. S.
Kaplan
Karpinsky
Kincaid
Larguier
Lockhart
Love
Myers
Nesbitt
Northam
Nyswander
Opatowski
Piranian
Poor, V. C.
Rainich
Rainville
Reade
Rothe
Running
Samelson
Spanier
Steenrod
Thrall, R. M.
Tornheim
Wilder, R. L.
Woodbury
Wright, J. B.
Bay City
Ewing, M. M.
Dearborn
Olsztyn
Detroit
Bagby
Dunford, M. J.
Fleiger
Johnson, L. G.
Marshall, J. W.
Savage, I. R.
Weinberg
Univ. of Detroit
Butler, J. F.
McCarthy, E. D.
Mehlenbacher
Smith, W. F.
Wayne Univ.
Borgman
Coral
Folley
Jacobson, A. W.
Morrow, D. C.
Nelson, A. L.
Pixley, E. C.
Pixley, H. H.
Southard
East Lansing
Childs
Plant
Speeker
Michigan State Coll.
Bell, J. H.
Black, H. L.
Blyth
Culbertson
Frame
Grove, V. G.
Herzog
Hill, J. D.
Jones, A. W.
McMillan, O. T.
Nordhaus
Powell
Stewart, B. M.
Wells, C. P.
Zimmer
Fenton
Tryon
Ferndale
Jones, C. D.
Highland Park
Spiro
Holland
Groenewoud
Houghton
Dawson
Kalamazoo
Walton, T. O.
Marquette
Spooner
Marysville
LaRue
Minden City
Denton
Nazareth
Morrison, M. J.
Ypsilanti
Pate
Schneckenburger

MINNESOTA

Duluth
State Teachers Coll.
Cothran, J. C.
Lindquist
Lockwood
Minneapolis
Mickelson
Morgan, W. D.
Northwestern National Life Ins. Co.
Polansky
Univ. of Minnesota
Amundson
Brink
Bussey
Cameron, R. H.
Carlson, E.
Colson
Donsker
Fischer, I. C.
Franck
Gibbens
Hart, W. L.
Hatfield
Hill, E. L.
Jackson, D.
Jensen
Kalisch
Koehler, F.
McEwen, W. R.
Munro, W. D.
Ness
Olmsted
Owchar
Quaid
Shumway
Specht, E. J.
Stoner
Thompson, J. E.
Turrittin
Warschawski
Moorhead
Anderson, M. R.
Northfield
Carleton Coll.
Erickson
May, K.
St. Olaf Coll.
Carlson, C. S.
Norlie
St. Joseph
Muggli
St. Paul
Clamons
Tompkins
Coll. of St. Thomas
Bush, L. E.
Landshoff
Norris, M. J.
Hamline Univ.
Culmer
Macalester Coll.
Camp, E. J.
Winona
Seiler

MISSISSIPPI

Blue Mountain
Trott
Cleveland
Walters
Holly Springs
Caldwell
Jackson
McCoy, D.
State College
Mississippi State Coll.
Ollivier
Pettis, C. R.
Walker, B. M.
Welch, F. P.
University
Hume
Miller, R. A.

MISSOURI

Cape Girardeau
Michel
Clayton
Rosskopf
Columbia
Haynes, N. A.
Stephens Coll.
Whitmer
Univ. of Missouri
Betz, H.
Blumenthal
Ewing, G. M.
Ferguson, W. E.
Kelly
Koken
Wahlin
Westfall
Fayette
Burcham
Helton
Fulton
Sloss
Kansas City
Linton
Melcher
Coll. of St. Teresa
Lackay
Rockhurst Coll.
Doyle, W. C.
Rolla
Erkiletian
Springfield
H'Doubler
Drury Coll.
Finkel
Graves, W. L.
St. Louis
Green, F. W.
Horvay
Central Inst. for the Deaf
Harrison, C. E.
Fontbonne Coll.
Creane
Harris Teachers Coll.
Marth
Maryville Coll.
Kernaghan
St. Louis Coll. of Pharmacy and Allied Sciences
DeFoe
Washington Univ.
Dunkel
Gottlieb
Leighton
Rider
Roberson
Roever
Sholander
Thron
Varney
St. Louis Univ.
Case
Kalinowski
Lorenz, A. J.
Regan
Vezeau

MONTANA

Bozeman
Montana State Coll.
Arnold, B. H.
Hurst
Livers
Schaerf
Butte
Smith, A. J.
Garrison
Canning
Missoula
Montana State Univ.
Carey, E. F. A.
Lennes
Merrill, A. S.
Noble, A. R.

NEBRASKA

Lincoln
Arnold, H. A.
Univ. of Nebraska
Basoco
Brenke
Camp, C. C.
Collins
Doole
Gaba
Hull, R.
Runge
Union Coll.
Ogden
Omaha
Creighton Univ.
Clarkson, H. E.
Duchesne Coll.
Fox, E.
Municipal Univ. of Omaha
Earl
Wayne
Boyce, J. W.

NEVADA

Reno
Univ. of Nevada
Beesley
Lewis, J. V.
Owens, O. G.
Wood, F.

NEW HAMPSHIRE

Bristol
Maker
Concord
Conwell, G. M.
Durham
Univ. of New Hampshire
Giddings
Harvey, A. R.
Slobin
Hanover
Clark School
Morgan, F. M.
Dartmouth Coll.
Brown, B. H.
Doyle, T. C.
Durfee, William H.
Forsyth
Fraser, W. C. G.
King
Mathewson
Perkins
Robinson, R.
Silverman
Wilder, C. E.
Dartmouth Eye Inst.
Luneburg

NEW JERSEY

Bayonne
Thaxton
Belmar
Greenwald
Bridgeton
Miller, W. E.
Caldwell
Fowler, F. H.
Convent
O'Neill, A. C.
Hoboken
Stevens Inst. of Tech.
Davis, H. N.
Deimel
Olmstead
Jersey City
Markou
Lakewood
Wallick
Lawrenceville
Sturley
Long Branch
Rosenthal, J. E.
Madison
Battin
Greenspan
Maplewood
Hazeltine
Montclair
Jacobus
Murray Hill
Bell Telephone Labs.
Bode
Hamming
Ling
McMillan, B.
Mead
Shannon
Shewhart
Newark
Fenichel
Neifeld
Treuhaft
Mutual Benefit Life Ins. Co.
Rhodes, E. E.
Thompson, J. S.
Newark Coll. of Eng.
Gumbel
Hoffmann, P. O.
Jaffe, W. J.
Molina
New Jersey Bell Telephone Co.
Bulloch
Wannemacher
Prudential Ins. Co. of America
Bartleson
Berkeley
Mosesson
Rutgers Univ.
McCarthy, J. J.
New Brunswick
Rutgers Univ.
Barlaz
Bolton
Bunyan
Churchill, E.
Fender
Galuten
Gokhale
Grant, H. S.
Hazard, K. E.
Knobelauch
LeLeiko
Lopata
Meder
Morris, R.
Nelson, C. A.
Ott
Peiser
Phelps
Rauch, L. M.
Robertson, M. S.
Schafer, A. T.
Starke
Walter
Zimmerberg
Pennington
Singleton, R. R.
Princeton
Flood
Inst. for Advanced Study
Alexander, J. W.
Arens
Barankin
Bigelow
Cahn
Erdös
Fan
Gödel
Leibler
MacLane, G. R.
Masani
Mayer
Morse, M.
von Neumann
Robinson, J. B.
Schafer, R. D.
Segal
Slutz
Straus, E. G.
Veblen
Weyl, H.
Princeton Univ.
Adams, E. P.
Artin
Bargmann
Bellman
Bochner
Chevalley
Chung
Church, A.
Crosby, D. R.
Eisenhart, L. P.
Fox, R. H.
Gelbaum
Gillespie
Goldman
Harrold
Henkin
Lefschetz
MacKenzie
Meier
Mills, W. H.
Nering
Rauch, L. L.
Robertson, H. P.
Samuel
Shapiro
Sherman, B.
Treiber
Tucker, A. W.
Tuckerman
Tukey
Votaw
Wedderburn
Whitehead
Wightman
Wigner
Wilks
Red Bank
Newman, P.
Scotch Plains
McMillan, A. W.
South Orange
Herschdorfer
Stanwick
Teaneck
Meyer, H. K.
Trenton
Shuster
Upper Montclair
Boggs
Campbell, G. A.
New Jersey State Teachers Coll.
Davis, D. R.
Mallory
West Orange
Edison

NEW MEXICO

Albuquerque
Univ. of New Mexico
LaPaz
Larsen, H. D.
Rosenthal, A.
Las Vegas
Rodgers
Santa Fe
Gittings
Goldstein
Los Alamos Lab.
Carlson, B. G.
Ulam
Socorro
Cassity

NEW YORK

Albany
Roland
Thompson, W. R.
New York State Coll. for Teachers
Beaver
Birchenough
Lester
Stokes, E. C.
Alfred
Lowenstein

Aurora
Hollcroft, M. P.
Wells Coll.
Carroll-Rusk
Clement, M. D.
Hollcroft, T. R.
Bethpage
Gross
Tripp, R. H.
Brockport
Stopher
Bronxville
Manning, A. W.
Strodt, R. C.
Brooklyn
Annenberg
Berry, W. J.
Hopkins, M. W.
Kalish
Kramer-Lassar
LaSala
Maddaus
Miller, J. L.
Wayne
Wynne
Brooklyn Coll.
Bishop
Boeker
Borofsky
Boyer
Douglas, J.
Fleisher
Griffin, H. M.
Johnson, R. A.
Karlin
Kennison
Kieval
Kormes, J. P.
Landers, A. W.
Luchins
MacNeish
Maria, A. J.
Maria, M. H.
Moore, L. T.
Prenowitz
Richardson, M.
Rosenbaum, I.
Singer, J.
Wolfe, J. M.
Woodbridge
Long Island Univ.
Lieber, H. G.
Lieber, L. R.
Polytech. Inst. of Brooklyn
Fenn
Foster, R. M.
Hutchinson, L. C.
Morduchow
Weber, E.
Whitford
Pratt Inst.
Cowles
Norman
Thompson, J. E.
St. Francis Coll.
Quinn, L.
St. John's Univ.
Cassidy
Tolle
St. Joseph's Coll. for Women
Harvey, V. A.
Larkin
Buffalo
Podmele
American Optical Co.
Kavanagh
Wilkins
Bell Aircraft Corp.
Erickson, W. S.
Mitchell, W. L.
Pell
Curtiss-Wright Corp.
Cramer, R. H.
D' Youville Coll.
Maloney
Univ. of Buffalo
Gehman
Montague
Pound
Canton
Bates, O. K.
Cleveland
Morenus
Clinton
Hamilton Coll.
Ferry
Patterson, B. C.
Williams, L. B.
Cornwall
Greeley
Elmhurst
Eckweiler
Elmira
Suffa
Endicott
Dubisch
Wright, F. M.
Farmingdale
Harrison, M. G.
Far Rockaway
Moore, L.
Flushing
Queens Coll.
Archibald, R. G.
Brown, A. B.
Cope
Critchlow
Feld
Hoffmann, B.
Raudenbush
Sard
Sullivan, M. M.
Williamson, J.
Zippin
Garden City
Adelphi Coll.
Bowden, J.
Sperry Gyroscope Co., Inc.
Harrison, G.
Stewart, S. W.
Geneva
Durfee, Walter H.
Hubbs
Great Neck
Grant, A. M. C.
Hamilton
Munshower
Wardwell
Hempstead
Hofstra Coll.
Hove
Ollmann
Stabler
Houghton
Luckey
Ithaca
Miller, E. A.
Cornell Univ.
Agnew, R. P.
Beinert
Cann
Carver
Ehrlich
Feller
Firestone
Flexner
Goldberg, S.
Goodier
Hurwitz, W. A.
Jones, B. W.
Kac
Karapetoff
Kuo
LeVeque
Malti
Mark
Morton
Pollard, H.
Reiner
Rosser
Sharpe, F. R.
Snyder, V.
Stephan
Ventriglia
Walker, R. J.
Jamestown
Fox, H. H.
Long Beach
McMahon
Loudonville
Lynch, B. A.
Mullen
Mineola
Huntley
Mt. Kisco
Blake, E. M.
Mt. Vernon
Wetzel
New Lebanon
Pflaum
New York
Bal
Baker, E. G.
Becker, M. F.
Bergmann
Boehm
Bowden, M.
Breen
Burgess
Byerly
Carpenter, R. V.
Cloudman
Crane
Croci
Deutsch
Dimsdale
Durding
Dutka
Foyle
Gilbert, N. M.
Grossman, G.
Grunbaum
Hammer, H. K.
Hardin
Hett
Holló
Hopper
Ingersoll
Ingram
Joffe
Kagno
Katz
Keyser, S. Y.
Kimball, B. F.
Kiss
Kormes, M.
Laderman
Landau, D. B.
Lehner
Lewis, J. H.
Luginbuhl
Manring
Meyerson
Miller, K. S.
Mitchell, H. B.
Morrison, N.
Neumann
Nodelman
Parsons
Preston
Rempfer

Riley
Ripandelli
Ritt, R. K.
Roosevelt
Salerno
Senior, J. L.
de Shelly
Shook, R. C.
Simon, L. G.
Snook
Tanzola
Uluçay
Vitale
Weber, C. J.
Wernick
White, M. E.
Widmer
Williams, E. C.
Wolfe, H.
Wurtele
American Math. Society
Hull, E. M.
Kellar
Zucker
American Telephone and Telegraph Co.
Belcher
Halbert
Hodgkinson
Bell Telephone Labs.
Dietzold
Fry, T. C.
Gray
Harvey, W. R.
Hebbert
Hoyt
Jones, P. C.
MacColl
Riordan
Schelkunoff
Sunde
Vaage
Wise
Zobel
Coll. of Mt. St. Vincent
Brady, M. L.
Conlan
O'Brien
Coll. of the City of New York
Barber
Blau
Cortell
Dwork
Fagerstrom
Freilich
Gill, B. P.
Hubert
Hurwitz, S.
Linehan
Lorch, L.
Malin
Nathan
Philip
Post
Robinson, S. L.
Solomon
Updike
Weingarten
Wirth
Wolfenstein
Wright, H. N.
Columbia Univ.
Anderson, T. W.
Arrow
Ascher
Aurora
Bogdanoff
Brown, R. H.
Clark, J. R.
Cohn, R. M.
Ewing, M.
Fialkow
Fite
Goodman, A. W.
Gotz
Gourin
Gutmann
Hoffman, A. J.
Jastrow
Kasner
Keyser, C. J.
Kolchin
Koopman
Ladue
Lazar
Lepson
Levene
Levi, H.
Lorch, E. R.
Mason, B.
Melvin
Mindlin
Mirick
Mittleman
Morin
Mullins
Murray, F. J.
Nagel
Noether
Northcott
Pekeris
Poor, C. L.
Raleigh
Ramsey
Raney
Ritt, J. F.
Secrist
Seeber
Siceloff
Skinner
Smith, P. A.
Steinhardt
Strodt, W.
Supnick
Tannenbaum
Tong
Turner, L. C.
Upton
Wald
Walker, H. M.
Williams, H. B.
Wolfowitz
Cooper Union
Lehmann, C. H.
Levenson
Miller, Frederic H.
Ecole Libre des Hautes Etudes
Kogbetliantz
Econometric Inst.
Boschan
Roos
Equitable Life Assurance Society of the U. S.
Murphy, R. D.
Steinhaus
Federal Telecommunications Labs., Inc.
Assadourian
Fordham Univ.
Kirby, A. R.
Kubis
Lynch, J. J.
Hunter Coll.
Anderson, R. L.
Aroian
Bradley
Bushey, H.
Bushey, J. H.
Cooper
Darkow
Guggenbühl
Hill, L. S.
Kutman
Landers, M. K.
Levin, M.
Norris, N.
Rees, M.
Simons, L. G.
Straus, L. M.
Tuller
Weisner
Whelan
International Telephone and Telegraph Corp.
Arnold, J. W.
Valladares
Manhattan Coll.
Hall, F. C.
Welch, B. A.
Manhattan Life Ins. Co.
Porter, E. A.
Manhattanville Coll. of the Sacred Heart
Kirby, M. D.
Mira
Math. Tables Project
Abramowitz
Blanch
Hillman
Salzer
Metropolitan Life Ins. Co.
Bassford
Brady, G. V.
Campbell, G. C.
Davis, M. E.
Hohaus
Lew
Lotka
Morecroft
Mutual Life Ins. Co. of New York
Monsky
New York Univ.
Adler, C. F.
Bernardi
Bernstein, F.
Blank
Brock
Coler
Cooley, H. R.
Courant
Dressler
Flanders
Friedman, B.
Friedrichs
Graham
John
Kline, M.
Lax
MacDonald
Marx
Payne
Peck, L. G.
Peters, A. S.
Phillips, R. S.
Pohle
Putnam, R. G.
Reddick
Rehberg
Ritter, I. F.
Roth
Schaaf
Shiffman
Stoker
Thorne, P. L.

Tilley
Treuenfels
Wang
Yanosik
Rockefeller Foundation
Weaver
Teachers Ins. and Annuity Asso.
Robbins, R. B.
U. S. Life Ins. Co.
Grossman, E. A.
U. S. Navy Maritime Acad.
Alton
Yeshiva Univ.
Ginsburg
Greitzer
Lowan
Matz
Niagara Falls
Welmers
Niagara University
Banks
North Terrytown
McCauley
Oneonta
Callahan, E. B.
Paul Smiths
Coyne
Perry
Walker, G. W.
Pleasantville
Jones, L. G.
Poughkeepsie
Vassar Coll.
Asprey
Baker, F. E.
Durand, J. C.
McDonald, J.
Newton, A. V.
Smith, G.
Wells, M. E.
Rochester
Betz, W.
MacCheyne
Mersman
Eastman Kodak Co.
Chesna
Herzberger
Univ. of Rochester
Atkins, H. P.
Bagemihl
Bernstein, D. L.
Gale
Ginkel
Glover, I. E.
Gunderson
Komm
Lohwater
Long
Reed, S. G.
Seidel
Watkeys
Wing
Rocky Point
Carter, P. S.
St. Bonaventure
Scheier
Schenectady
General Electric Co.
Alger
Crawford, J. E.
Poritsky
Union Coll.
Burkett
Farrell, O. J.
Fox, A. H.
Garis
Morse, D. S.
Snyder, A. D.
Van Schaack
Staten Island
Young, R. L.
Syracuse
Cowgill
Syracuse Univ.
Aulbach
Bers
Bruns
Cairns, S. S.
Carroll
Decker
Dickerson
Gelbart
Gilbert, P. W.
Goodman, R. O.
Kibbey
Loewner
Ludwig
Morgan, K. A.
Rosenbloom, P. C.
Ryan
Titus
Zatzkis
Troy
Rensselaer Polytech. Inst.
Allen, E. B.
Ames
Bartholomay
Biggerstaff
Campbell, J. D.
Gravalos
Hall, C. F.
Huston
Maly
Russell Sage Coll.
Reschovsky
Utica
Bush, K. A.
West Haverstraw
Nicholl
West Point
Jones, H.
Robinson, V. N.
White Plains
Downey
Willseyville
Seely
Wyoming
Hartnell

NORTH CAROLINA

Black Mountain
Dehn

Chapel Hill
Hotelling, S. P.

Univ. of North Carolina
Bowker, A. H.
Brauer, A. T.
Browne
Cameron, E. A.
Drucker
Hobbs
Hotelling, H.
Hsu
Lasley
Mackie
Madow
Robbins, H. E.
Terry
Wong, Y. K.
Cherry Point
Cole, C. A.
Durham
Duke Univ.
Carlitz
Cohen, E. L.
Dressel
Elliott
Gergen
Hickson
Roberts, J. H.
Thomas, J. M.
Wade, L. I.
Greensboro
Agric. and Tech. Coll. of North Carolina
Jeffries

Greensboro Coll.
Pegram

Univ. of North Carolina
Barton
Lewis, A. L.

High Point
Adams, L.

Raleigh
Efird
Shaw Univ.
Eason
Univ. of North Carolina
Bullock
Cell
Clarkson, J. M.
Fisher, H. A.
Mumford
Nahikian
Park, H. V.
Sutton, P. P.
Williams, C. W.
Winton
Salisbury
Dearborn
Douglas, C. H.

NORTH DAKOTA

Bottineau
Zarling

Fargo
Smith, I. W.

OHIO

Akron
Miller, Frank H.
Univ. of Akron
Lipscombe
Mauch
Athens
Ohio Univ.
Marquis
Reed, F. W.
Starcher
Barberton
Cartwright
Bluffton
Morgan, W. B.
Bowling Green
Bowling Green State Univ.
Krabill
Ogg
Overman
Cincinnati
Our Lady of Cincinnati Coll.
Belal
Lukacs
Rice, M. E.
Univ. of Cincinnati
Barnett, I. A.
Brand
Dunholter
Farnau
Harlow
Justice
Kersten
Lipsich
Lubin
McKissoch
Merriman
Miller, H. L.
Moore, C. N.
Rogers
Smith, E. S.
Soble
Szász
Taylor, W. C.
White, E. F.
Yowell
Western and Southern Life Ins. Co.
Folz
Learson
Xavier Univ.
Marcaccio
Stechschulte
Cleveland
Burwell
Kraft
Livingood
Seielstad
Case School of Applied Science
Brown, O. E.
Focke
Green, L. J.
Guenther
McCuskey
Morris, M.
Rinehart
Thomas, C. F.
Fenn Coll.
McGar
Topp
John Carroll Univ.
Monville
Western Reserve Univ.
Beth
Musselman
Simon, W. G.
Columbus
Bareis
Bell, J. L.
Byrd
Batelle Memorial Inst.
Bell, J. C.
Ohio State Univ.
Bamforth
Blumberg
Fliess
Gardner
Hall, M.
Helsel
Kuhn
Mann, H. B.
Mickle
Miller, L. H.
Moll
Parrish
Rado
Rasor
Scott, W. R.
Swain
Toops
West
Whitney, D. R.
Dayton
Midgley
Army Air Forces Inst. of Tech.
Carson
Wylie
Ohio State Univ.
Chatland
Vickery
Univ. of Dayton
Schraut
Sussman
Wright Field
Block
Eagle
Gorn
Millsaps
Price, I.
Schart
Defiance
Godfrey
MacCullough
Delaware
Rowland
Gambier
Kenyon Coll.
Berg
MacNeille
Transue
Granville
Denison Univ.
Kato
Ladner
Wiley, F. B.
Hiram
Clarke, E. H.
Kent
Kent State Univ.
Brooks
Manchester
Olson, E. J.
Stelson
Marietta
Bennett, T.
Sandt
Marion
Leach
Mt. St. Joseph
Dimond
Molloy
Newark
Wiley, R. B.
New Concord
Knight, L. C.
North Baltimore
Blackall
Oberlin
Yeaton, M. J.
Oberlin Coll.
Cairns, W. D.
Newsom
Randolph
Sinclair, M. E.
Vance
Wagner, R. W.
Yeaton, C. H.
Oxford
Miami Univ.
Anderson, W. E.
Pollard, H. S.
Spenceley
Young, P. M.
Western Coll.
Tappan
Painesville
Peters, R. N.
South Euclid
Garvin
Springfield
Boldyreff
Tripp, M. O.
Toledo
Brandeberry
Wellington
Ruger
Wooster
Coll. of Wooster
Hildner
Williamson, C. O.
Yanney
Wilberforce
Davidson, A.
Yellow Springs
Astrachan
Youngstown
Potor

OKLAHOMA

Bartlesville
Morris, W. L.
Rice, R. B.
Langston
Rouce
Norman
LaFon
Univ. of Oklahoma
Bernhart
Brixey
Court
Hassler
Huff, W. N.
Huskey
McFarland
Meacham
Reaves
Springer, C. E.
Wine

Stillwater
Oklahoma Agric. and Mech. Coll.
Barnett, J.
Hamilton, O. H.
Lewis, P. E.
McKinsey
Oboukhoff
Shepherd
Smith, H. W.
Sugar
Tulsa
Ellis, J. R.
Ricker
Sharpe, J. A.
Woolson
Univ. of Tulsa
Shreve

OREGON

Corvallis
Oregon State Coll.
Clark, C. L.
Eves
Hammer, P. C.
Hostetter
Li
Manning, R.
Milne, W. E.
Poole
Saunders, R. B.
Eugene
Scobert
Univ. of Oregon
Civin
DeCou
Dixon, W. J.
Ghent
Moursund
Peterson, T. S.
Wood, F. E.
Portland
Reed Coll.
Botsford
Griffin, F. L.
Rosenbaum, L. J.
Rosenbaum, R. A.
Univ. of Portland
Vassallo
Salem
Garfin
Willamette Univ.
Luther
Porter, R. E.

PENNSYLVANIA

Allentown
Deck
Koehler, T. L.
Altoona
Herpel
Swope
Annville
Black, A. H.
Bethlehem
Petrie
Lehigh Univ.
Beale
Brown, H. K.
Celauro
Cowling, V. F.
Cutler
Garrison
Hailperin
Lamson
Latshaw
Pitcher
Raynor
Shook, C. A.
Smail
Stoll
Bruceton
Brown, F. W.
Bryn Mawr
Bryn Mawr Coll.
Hewitt
Lehr
Martin, R. M.
Oxtoby, J. C.
Oxtoby, J. S.
Pell-Wheeler
Carlisle
Craver
Dickinson Coll.
Ayers, F.
Smith, Henry E.
Chambersburg
Penn Hall Junior Coll. and Preparatory School
Moser
Wilson Coll.
Davison
Disert
Johnson, R. F.
Priestley
Chester
Bordner
Williams, F. G.
Clarion
Carey, C. E.
Dallas
Garrahan
Easton
Bond
Fuller
Lafayette Coll.
Smith, J. C.
Smith, W. M.
East Pittsburgh
Comenetz
Slepian
Erie
Fasenmyer
Gettysburg
Arms
Glenside
Patterson, J. O.
Greensburg
McNeil
Harrisburg
McKee
Haverford
Haverford Coll.
Allendoerfer
Gummere
Oakley
Wilson, A. H.
Immaculata
Beamish
Jenkintown
Vansant
Kutztown
Knedler
Lancaster
Heyda
Murray, W. R.
LaPlume
Trapani
Latrobe
Yahner
Lewisburg
Gold
Miller, W. I.
Loretto
Martin, N.
McKees Rocks
Arnold, H. C.
Meadville
Allegheny Coll.
Bird
Hurd
Smith, R. E.
Steen
Middletown
Rawhouser
New Wilmington
McGaughey
Philadelphia
Bates, D. M.
Brown, R. D.
Campbell, W. B.
Clark, J. R.
Constable
Fudge
Groat
Jaffe, H.
Kaufman
Kratz
Munster
Porter, J. W.
Ruley
Sherman, J.
Slepin, B.
Smart
Smith, J. P.
Drexel Inst. of Tech.
Davis, James Elmer
McDonough
McNeary
Tartler
Frankford Arsenal
Levenberg
Rosenbloom, J. H.
Franklin Inst.
Carner
Jehle
Temple Univ.
Gleason, R. E.
Moses
Rorer
Stokes, C. N.
Walker, G. L.
Wurster
Univ. of Pennsylvania
Askovitz
Beal
Brainerd
Campbell, R. C.
Carls
Chapline
Clarkson, J. A.
Cohen, I. S.
Evans, H. B.
Goff

Gottschalk
Grove, C. C.
Kirk
Kline, J. R.
Lufkin
Makarov
Newman, M.
Patterson, G. W.
Penico
Rademacher
Saalfrank
Safford
Schoenberg
Schub
Shugert
Wallace, A. D.
Walton, J. B.
Witmer
Wong, Y. C.
Young, J. W.
Zygmund
Pittsburgh
Frankel
Leifer
Carnegie Inst. of Tech.
Díaz
Dines
Duffin
Epstein, Benjamin
Heins, A. E.
Morehead
Moskovitz
Neelley
Olds, E. G.
Peach, M. O.
Rosenbach
Saibel
Synge
Weinstein, A.
Whitman, E. A.
Mellon Inst. of Industrial Research
Briant
Pennsylvania Coll. for Women
Calkins
Univ. of Pittsburgh
Culver
Foraker
Hovey
Knipp
Montroll
Taylor, J. S.
Scranton
Walton, M. B.
Selinsgrove
Robison
State College
Frink, A. H.
Pennsylvania State Coll.
Cohen, T.
Curry
Dotterer
Frink, O.
Gordon, W. O.
Hagen
Harrington, W. J.
Johnson, E.
Krall
Owens, F. W.
Owens, H. B.
Rose, I. H.
Schwartz, A.
Sheffer, I. M.
Story
Thorpe
Umberger
Wagner, C. C.
Swarthmore
Franklin Inst.
Schwartz, H. M.
Swann
Swarthmore Coll.
Brinkmann
Dresden
Marriott
Wasow
Washington
Dorwart
Shaub
Wayne
Feather
Snyder, A. K.

PUERTO RICO

Mayagüez
Univ. of Puerto Rico
García, M.
Morales
Sánchez-Díaz
Rio Piedras
Bobonis
Capó

RHODE ISLAND

Kingston
Pease
Stauffer
Portsmouth
Taliaferro
Providence
Adams, R. B.
Boas, R. P.
Brown Univ.
Adams, C. R.
Archibald, R. C.
Bennett, A. A.
Biot
Carlen
Carrier
Chen
Chernoff
Demers
Eddy
Federer
Fein
Gildea
Gilman
Greenberg
Handelman
Heins, M. H.
Heller
Jónsson
Keck
Lin
Lindsay
Manning, H. P.
Miesse
Neugebauer
Prager
Protter
Richardson, R. G. D.
Saltzer
Slud
Smiley, C. H.
Tolsted
Western, D. W.
Wilansky
Providence Coll.
Gallagher
McKenney
West Warwick
Brant

SOUTH CAROLINA

Anderson
Paschal
Charleston
Shulits
Coll. of Charleston
Bushkovitch
The Citadel
Dye
Hedge
Sutton, C. S.
Clemson
Coker
Columbia
Coleman, J. B.
Univ. of South Carolina
Hedberg
Jackson, J. B.
Lee, T. H.
Lytle
Robinson, L. V.
Williams, W. L.
Mountville
Stokes, R. W.
Rock Hill
Pepper, R. I.
Spartanburg
Pettis, C. S.

SOUTH DAKOTA

Brookings
South Dakota State Coll.
MacDougal
Walder
Wente
Rapid City
March, G. E.
Vermillion
Beaty

TENNESSEE

Alcoa
Harris, L. J.
Jackson
Pulliam
Knoxville
Monsanto Chemical Co.
Dismuke
Garabedian, H. L.
Householder
Univ. of Tennessee
Blackstock
Collier, C. B.
Cooley, J. A.
Eaves
Ficken
Johnson, R. B.
Kiefer
Lee, H. L.
Miller, D. D.
Snyder, W. S.
Watson
Memphis
MacQueen
Nashville
Meharry Medical Coll.
Brown, J. O.
Tennessee Agric. and Industrial State Teachers Coll.
Boswell, A. V.
Vanderbilt Univ.
Boyce, M. G.
Bright, S. K.
Miser, W. L.
Sarratt
Oak Ridge
Johnson, P. C.
Sewanee
Bruton

TEXAS

Abilene
Burnam
Mullings
Arlington
Lassen
Austin
Decherd
Univ. of Texas
Anderson, R. D.
Batchelder
Bing
Coleman, W. B.
Craig, H. V.
Ettlinger
Greenwood
Hurt, J. M.
Jones, F. B.
Lubben
Moise
Moore, R. L.
Porter, G. H.
Porter, M. B.
Titt
Vandiver
Wall
College Station
Agric. and Mech. Coll. of Texas
Basye
Beeman
Brewer
Daum
Hurt, J. T.
Klipple
McCulley
Moore, B. C.
Wells, N. W.
Commerce
Wright, C. B.
Daingerfield
Kennedy, E. C.
Dallas
McDermott
Sorrells
Southern Methodist Univ.
Starr
Southland Life Ins. Co.
Montgomery, P. V.
Denton
North Texas State Teachers Coll.
Brown, M. C.
Texas State Coll. for Women
Miller, H. C.
Mitchell, J. M.
Houston
Jackson, O. B.
Weinstein, A. K.

Houston Coll.
Lattimore

Humble Oil and Refining Co.
Rust
Slotnick

Rice Inst.
Bray
Brunk
Dean
Houston
Lovett
Ulrich
Wilson, H. A.

Univ. of Houston
Aucoin
Hutchinson, J. D.
Newhouse
Huntsville
Querry
Kingsville
Oliphant
Lubbock
Texas Tech. Coll.
Bowman
Michie
Thompson, E. L.
Whyburn, W. M.
Marlin
Falconer
Mercedes
Shaw, W. F.
Prairie View
Stephens, C. F.
San Angelo
Bright, H. F.
San Antonio
Incarnate Word Coll.
Gough
Our Lady Of the Lake Coll.
Hill, M. L.
Trinity Univ.
Newton, G. A.
San Marcos
Cude
Waco
Williams, W. J.

UTAH

Logan
Tingey
Provo
Marshall, M.
Salt Lake City
Univ. of Utah
Biesele
Hayes, J. J.
Henriques
Horsfall
Pehrson
Smith, S. S.
Thorne, C. J.

VERMONT

Bennington
Ellis, G. A.
Kaiser

Burlington
Stibitz
Univ. of Vermont and State Agric. Coll.
Bullard
Larrivee
McClay
Swift, E.

Middlebury
Ballou
Bowker

VIRGINIA

Alexandria
Singleton, H. E.
Arlington
Welsh
Blacksburg
Virginia Polytech. Inst.
DeLury
Jennings, W.
Lawrence
McFadden
Charlottesville
Cooley, N. E.
Patten
Univ. of Virginia
Bernard
Floyd
Fort, M. K.
Gœtchius
Hedlund
Klee
Linfield
McShane
Oglesby
Utz
Whyburn, G. T.
Dahlgren
Barker
Dresch
Ettrick
Hunter, L. S.
McDaniel, R. R.
Falls Church
McBee
Fredericksburg
Carter, H. C.
Hampton
Claytor
Eshleman
Harrisonburg
Partlow
Suter

Hollins College
Brown, A.
Falvey
Hopewell
Fazel
Langley Field
Andrews
Benscoter
Lexington
Paxton
Virginia Military Inst.
Byrne, W. E.
Knox
Washington and Lee Univ.
Meadows
Lynchburg
Sandidge

Randolph-Macon Woman's Coll.
Larew
Wiggin
Richmond
Barr
Staunton
Taylor, M. E.
Sweet Briar
Lee, M. A.
Pollock
University of Richmond
Univ. of Richmond
Gaines
Grable
Wheeler, C. H.
Williamsburg
Phalen
Stetson

WASHINGTON

Lacey
Cebula
Raymond
Pullman
State Coll. of Washington
Butler, L. G.
Hacker
Knebelman

Seattle
Boeing Aircraft Co.
Forsythe
Lanczos
Seattle Coll.
Kinerk
Seattle Pacific Coll.
Beegle
Univ. of Washington
Avann
Ballantine, C. R.
Ballantine, J. P.
Beaumont
Birnbaum
Carpenter, A. F.
Cramlet
Jerbert
Kingston, J. M.
McFarlan
Mullemeister
Smullyan
Stippes
Taub
Winger
Yagi
Zuckerman
Walla Walla
Hutchings

WEST VIRGINIA

Athens
Armbrister
Bethany
Bethany Coll.
Cramblet
Simmons, F. A.
Tye
Charleston
Williams, R.

Morgantown
West Virginia Univ.
Cunningham, A. B.
Davis, H. A.
Eiesland
Murphy, K. H.
Reynolds, C. N.
Turner, B. M.
Vest

WISCONSIN

Appleton
Berry, A. C.
Stewart, J. C.

Beloit
Conwell, H. H.
Huffer

Green Bay
Wagner, R. D.

Kenosha
May, A. E.

LaCrosse
Adkins
Madison
Univ. of Wisconsin
Adler, F. T.
Allen, F. E.
Andree
Arnold, K. J.
Breit
Bruck
Cohen, H. J.
Colvin
Eisenhart, C.
Engel
Evans, H. P.
Everett, C. J.
Fullerton
Goldhaber
Hachmeister
Halteman
Heacock
Ingraham
Kanter
Kleene
Koerting
Langer
Leavitt
MacDuffee
March, H. W.
Noyes
Paige
Rechard, O. W.
Reichelderfer
Rose, G. F.
Ryser
Schecter
Schurrer
Simpson, R. C.
Smith, W. N.
Snowdon
Sokolnikoff, E. S.
Specht, R. D.
Strehler
Trump
Whaples
White, L. R.
Wiegmann
Wilson, R. L.
Wright, R. E.
Wysocki
Milwaukee
Joseph
Krug
Vass
Marquette Univ.
Barron
Fitzpatrick
Moeller
Pettit, H. P.
Wilczewski
Milwaukee-Downer Coll.
Beckwith, E. R.
Milwaukee School of Eng.
Neményi
Mt. Mary Coll.
Vaudreuil
State Teachers Coll.
Norris, R. E.
Univ. of Wisconsin in Milwaukee
Bardell
Brannon
Kenney
Marden
Parkinson
Spitzbart
Wolf, L. A.
Neenah
Nash
Oshkosh
Beenken
River Falls
Eide
Sheboygan
Battig
Superior
Danielson
West Allis
Westbrook
West DePere
DeCleene
Williams Bay
Chandrasekhar

WYOMING

Lamont
Bellamy
Laramie
Univ. of Wyoming
Bristow
Rechard, O. H.
Varineau

PHILIPPINE ISLANDS

Baguio
Melchor

CANADA

ALBERTA
Edmonton
Univ. of Alberta
Campbell, J. W.
Cook, A. J.
Sheldon
Wyman
BRITISH COLUMBIA
Vancouver
Simons, W. H.
Univ. of British Columbia
Buchanan, D.
Chapman
Derry
Duncan, D. G.
Free
Gage
James, R. D.
Jennings, S. A.
Kent
Murdoch
Nowlan
MANITOBA
Brandon
Guinand
Winnipeg
United Coll.
Ritcey
Univ. of Manitoba
Lawson
McEwen, W. H.
NEW BRUNSWICK
Fredericton
Fleming
Sackville
Crawford, W. S. H.
NOVA SCOTIA
Halifax
Adshead
Walmsley
Wolfville
Macphail
Rothberger
ONTARIO
Hamilton
McMaster Univ.
Bankier
Findlay
Johns
Kingston
Queen's Univ.
Halperin
Jeffery
Mendelsohn
Miller, N.
London
Univ. of Western Ontario
Cole, R. H.
Kingston, H. R.
Magee
Toronto
Grant, A. A.
Wolfenden
Univ. of Toronto
Beatty
Brauer, R.
Burk
Coxeter
Gilchrist
Gould, S. H.
Krieger
Pounder
Robinson, G. de B.
Steinberg
Waddell
Webber, W. J.
Windsor
Duffie

QUEBEC
Montreal
Benjamin
Williams, C. S.
Laval Univ.
Putman
Loyola Coll.
O'Connor
McGill Univ.
Gillson
Rosenthall
Shaw, A.N.
Sullivan, C. T.
Wallace, P. R.
Williams, W. L. G.
Univ. of Montreal
Gauthier
Gregoire
SASKATCHEWAN
Regina
Vigder
Saskatoon
Univ. of Saskatchewan
Crosby, W. J. R.
Kozakiewicz
Miller, C. E.
Scherk

AFRICA

SOUTH AFRICA
Capetown
Young, L. C.

ASIA

CHINA
Changsha
Wu
Hangchow
Hsiung
Kunming
Shu
Peiping
Tschen
Tuan
Shanghai
Chern
INDIA
Aligarh
Shah
Annamalainagar
Rao, K. N.
Bangalore
Rao, B. S. M.
Dacca
Vijayaraghavan
Delhi
Seth
Madras
Ananda-Rau
New Delhi
Chand
LEBANON
Beirut
Kennedy, E. S.
PALESTINE
Jerusalem
Hebrew Univ.
Fraenkel
Levitzki
Motzkin
TURKEY
Bolu
Demir

AUSTRALASIA

AUSTRALIA
Adelaide
Schwerdtfeger
Brisbane
Raybould
Canberra
Paranjpye
Hobart
Jaeger, J. C.
Melbourne
Cherry
Perth
Weatherburn
Sydney
Carslaw
Thorne, H. H.
NEW ZEALAND
Auckland
Forder
Christ Church
Gill, P. J.
Dunedin
Martyn

EUROPE

BELGIUM
Brussels
Sips
BULGARIA
Sofia
Lanckton
CZECHOSLOVAKIA
Prague
Myslivec
Schoenbaum
DENMARK
Copenhagen
Denmark Tech. Lib.
Schmidt
Univ. of Copenhagen
Bohr
Jessen
Nørlund
ENGLAND
Alnwick
Collingwood
Bristol
Fraser, P.
Cambridge
Stone, D. M.
Cambridge Univ.
Hall, P.
Hardy, G. H.
Ingham
Mordell
Richmond, H. W.
Smithies
Stone, A. H.
Englefield Green
McCrea
Farnborough
Vermes
Hindhead
Western, A. E.
Leeds
Milne, W. P.
London
Dalal
Univ. of London
Semple
Simpson, H.
Oxford
Univ. of Oxford
Dixon, A. L.
Frecheville
de Wet
Stonehouse
Turner, J. S.
Teddington
Taussky
Wembley Park
Lyndon
FRANCE
Paris
Hademard
Mandelbrojt
IRELAND
Dublin
McConnell, A. J.
NORTHERN IRELAND
Belfast
Gillis, J.
Povey
PORTUGAL
Coimbra
Dias Coimbra
SCOTLAND
Dundee
Copson
Edinburgh
Whittaker

SPAIN
Madrid
Esteban-Carrasco
SWITZERLAND
Basel
Ostrowski
Geneva
Coleman, A. J.
Lausanne
Blanc
Zürich
Plancherel
U.S.S.R.
Moscow
Kryloff
WALES
Bangor
Cowling, T. G.
Cardiff
Livens

CENTRAL AND SOUTH AMERICA

ARGENTINA
Buenos Aires
Mazza

Univ. of Buenos Aires
Baidaff
Barral-Souto
Blaquier
Bula
González-Domínguez

LaPlata
National Univ. of LaPlata
Cesco
DeCesare
Ricabarra
Sagastume Berra
Trejo
Zarantonello
Rosario
Gaspar
Levi, B.

San Martin
Guglialmelli

Tucumán
Terracini

BRAZIL
Paraná
Lepecki

São Paulo
da Silva Rodrigues
Weil

CHILE
Santiago
Parra
COSTA RICA
San Jose
Evans, J. L.
CUBA
Havana
Corral-Alemán
Univ of Havana
González, M. O.
Gutiérrez
ECUADOR
Quito
Thullen
MEXICO
Mexico
National Univ. of Mexico
Barajas
González-Baz
Graef Fernández
Lifshitz
Nápoles-Gándara
National Polytech. Inst.
Vallarta
Monterrey
Treviño García

PERU
Lima
Catholic Univ. of Peru
de Losada y Puga

Univ. of San Marcos
García, G.
Rosenblatt

URUGUAY
Montevideo
Laguardia

VENEZUELA
Caracas
Hodell
Michalup

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, and a Librarian.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article V.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

EDITORIAL COMMITTEES

Section 1. There shall be six Editorial Committees—one of four members for the Bulletin, one of three members for the Transactions, one of three members for the Mathematical Reviews, one of three members for the Colloquium Publications, one of three members for the Mathematical Surveys, and one consisting of three representatives of the Society on the Board of Editors of the American Journal of Mathematics.

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large, and the following *ex officio* members: the officers of the Society specified in Article I, the members of the Editorial Committees, ex-secretaries who have served as secretaries for ten years or more, and ex-presidents for a period of six years after the completion of their respective presidential terms.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. All members of the Council shall be entitled to be present at any meeting of the Council and to take part in the discussion at that meeting.

Section 4. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 5. The voting members at any meeting of the Council shall be the following: the members at large of the Council; the President; the Vice Presidents; the Secretary; the Acting Secretary for the meeting, if one is designated in the absence of the Secretary; the Treasurer; and a member of each Editorial Committee chosen annually by the committee, provided that if such voting member is absent at any meeting of the Council an alternate designated by him from among the membership of the committee shall have the power to vote.

Section 6. The method for settling matters before the Council at any meeting shall be by majority vote of the voting members present. If the result of such a vote is challenged at the time by any member of the Council, it shall be the duty of the Secretary—or of the Acting Secretary for the meeting—or of a member present who shall be designated by the Council for that purpose—to prepare a list of the voting members present, on the basis of which the presiding officer shall determine the true vote by a roll call.

Section 7. A quorum for the transaction of business at any meeting of the Council shall consist of five voting members as defined in Section 5 of this Article.

Section 8. Between meetings of the Council, business may be transacted by a mail vote. Voting members for this purpose shall be the members at large of the Council, the President, the Vice Presidents, the Secretary, the Treasurer, and the voting member of each Editorial Committee as designated in Section 5. An affirmative vote on any proposal by mail shall require at least a majority of the voting members, and at least three-quarters of the total number of votes transmitted to the Secretary before the time announced for the closing of the polls. If two or more voting members request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated.

ARTICLE V

ELECTIONS AND TERMS OF OFFICERS

Section 1. The term of office shall be two years in the case of the trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

Section 2. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an *ex officio* member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 3. Election of officers specified in Article I, trustees, members of Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled; with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

Section 4. If the President of the Society die or resign before the expiration of his term of office, the Council may, with the approval of the trustees, designate one of the Vice Presidents to serve as President until the next Annual Meeting, when the Society shall elect a President for a two-year term. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees may be filled by the Council with the approval of the trustees.

Section 5. If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 6. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VI

ELECTION OF MEMBERS

Section 1. Election of members shall be by vote of the Council.

Section 2. There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

Section 3. A firm, corporation, institution, association or individual interested in the support of mathematics may be elected to sustaining membership. A sustaining member shall receive the *Bulletin*, the *Transactions*, and *Mathematical Reviews* (see Article IX), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 4. A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. An institutional contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 5. Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

Section 6. An ordinary member may become a contributing member by paying the dues for such membership. (See Article VII, Section 4.)

Section 7. An ordinary member or a contributing member shall receive the *Bulletin*, except in the case of members whose dues are remitted under Article VII, Section 10.

ARTICLE VII

DUES

Section 1. Persons elected to ordinary membership in the Society by the Council, under the provisions of Article VI, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article VI, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. Four dollars and fifty cents of the annual dues of those members who receive the *Bulletin* under the provisions of Article VI shall be allocated in payment therefor.

Section 4. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 5. The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

Section 6. The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

Section 7. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article VI, Sections 3–4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

Section 8. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

Section 9. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life member shall have for life the status and privileges of an ordinary member, without further payment of dues. After October 25, 1941, no additional applications for life memberships will be accepted.

Section 10. After retirement from active service on account of age, any member who is not in arrears of dues and with membership extending over at least twenty years may—by giving proper notification to the Secretary—have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the *Bulletin*.

ARTICLE VIII

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

Section 3. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

Section 4. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE IX

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish a journal called the *Transactions of the American Mathematical Society*, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called *Mathematical Reviews* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also coöperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the *Bulletin*, *Transactions*, *Mathematical Reviews*, *Colloquium Publications*, and *Mathematical Surveys*, and the participation of the Society in the management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

ARTICLE X

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

A CHARACTERIZATION OF SEMI-SIMPLE RINGS WITH THE DESCENDING CHAIN CONDITION

OSCAR GOLDMAN

H. Weyl[1] has defined a semi-simple algebra (of finite rank) to be an algebra which admits a faithful semi-simple linear representation. Now, algebras are rings with a field of operators; Artin and others[2] have shown that the theory of semi-simple algebras can be generalized to a theory of semi-simple rings (without the field of operators) provided we replace the condition of finite rank by suitable finiteness conditions. (Both the ascending and descending chain conditions were assumed, but it was later shown that the descending chain condition was sufficient.[3]) The notion of semi-simplicity is defined by the condition that the radical reduces to $\{0\}$, there being several equivalent definitions of the radical. We introduce another one below.

The question now arises whether the Weyl definition could not be extended to the case of rings. To do this, we must extend to an arbitrary ring the notion of a linear representation of an algebra. This can be done by replacing the consideration of the algebra of matrices by the more general notion of the ring of endomorphisms of an abelian group: a *representation* of a ring A will be a homomorphism ρ of A into the ring of endomorphisms of an additive group $\mathfrak{M}$. Let such a representation be given; we can define a law of composition, $(a, m) \to am$, between elements of A and of $\mathfrak{M}$ by writing $am = \{\rho(a)\}(m)$. The composite object formed by $\mathfrak{M}$ and this law of composition is called an *A-module*. A *sub-module* of an *A-module* $\mathfrak{M}$ is a subset $\mathfrak{N}$ of $\mathfrak{M}$ which is a subgroup of the additive group of $\mathfrak{M}$ and is such that $A\mathfrak{N} \subset \mathfrak{N}$. ($A\mathfrak{N}$ is defined to be the set of all finite sums $\sum a_i m_i$, $a_i \in A$, $m_i \in \mathfrak{N}$.) A *homomorphism* of an A-module $\mathfrak{M}$ into an A-module $\mathfrak{M}'$ is a homomorphism h of the additive group of $\mathfrak{M}$ into the additive group of $\mathfrak{M}'$ which is such that $h(am) = ah(m)$ for all $a \in A$, $m \in \mathfrak{M}$.

An A-module $\mathfrak{M}$ is said to be *simple* if its only submodules are $\{0\}$ and itself. Concerning simple modules, we have the well known lemma:

SCHUR'S LEMMA. *A homomorphism h of a simple A-module $\mathfrak{M}$ into*

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[1] H. Weyl, *The classical groups*, Princeton University Press, 1939, p. 85.

[2] E. Artin, Abh. Math. Sem. Hamburgischen Univ. vol. 5 (1927) p. 251; J. Levitzki, Compositio Math. vol. 5 (1937) p. 392.

[3] R. Brauer, Bull. Amer. Math. Soc. vol. 48 (1942) p. 752; C. Hopkins, Duke Math. J. vol. 4 (1938) p. 664; J. Levitzki, Compositio Math. vol. 7 (1939) p. 214.

an A-module $\mathfrak{M}'$ either maps $\mathfrak{M}$ on $\{0\}$ or is an isomorphism of $\mathfrak{M}$ with some sub-module of $\mathfrak{M}'$. If $\mathfrak{M}'$ is also simple, and $h(\mathfrak{M}) \neq \{0\}$, then $h(\mathfrak{M}) = \mathfrak{M}'$.

If $\{\mathfrak{N}_\lambda\}_{\lambda \in L}$ is a family of sub-modules of an A-module $\mathfrak{M}$, we call the *sum* of this family of sub-modules, and we denote by $\sum_{\lambda \in L} \mathfrak{N}_\lambda$, the set composed of all sums $\sum_{\lambda \in L} n_\lambda$, where for each λ, $n_\lambda \in \mathfrak{N}_\lambda$, and $n_\lambda = 0$ except for a finite number of the indices λ. It is clear that $\sum_{\lambda \in L} \mathfrak{N}_\lambda$ is again a sub-module of $\mathfrak{M}$. The sum is said to be *direct* if the representation of an element of the sum in the form $\sum_{\lambda \in L} n_\lambda$ uniquely determines the n_λ. (It is clear that $\mathfrak{N}_1 + \mathfrak{N}_2$ is direct if, and only if, $\mathfrak{N}_1 \cap \mathfrak{N}_2 = \{0\}$.)

A module is said to be *semi-simple* if it can be represented as the direct sum of simple sub-modules. The following facts can easily be shown.

I. If a module is the sum of a family $\mathfrak{F}$ of simple sub-modules, it is also the direct sum of some sub-family of $\mathfrak{F}$.

II. For a module $\mathfrak{M}$ to be semi-simple, it is necessary and sufficient that, given any sub-module $\mathfrak{N}$ of $\mathfrak{M}$, there should exist a sub-module $\mathfrak{N}'$ such that $\mathfrak{M}$ is the direct sum of $\mathfrak{N}$ and $\mathfrak{N}'$.

If $\mathfrak{M}$ is an A-module, define $\mathfrak{M}_T$ to be the set of all $m \in \mathfrak{M}$ for which $Am = \{0\}$. $\mathfrak{M}_T$ is clearly a sub-module of $\mathfrak{M}$; we call $\mathfrak{M}_T$ the *trivial sub-module* of $\mathfrak{M}$. The set $\mathfrak{A}$ of all $a \in A$ such that $a\mathfrak{M} = \{0\}$ is clearly a two-sided ideal in A, called the *annihilator* of $\mathfrak{M}$. The two extreme cases are the one in which $\mathfrak{A} = \{0\}$, in which case we say that $\mathfrak{M}$ is *faithful*, and the case in which $\mathfrak{A} = A$ and therefore $\mathfrak{M}_T = \mathfrak{M}$.

The *radical*[4] is now defined to be the intersection of the annihilators of all simple A-modules. It is clear that the radical is a two-sided ideal in A. The ring A is said to be *semi-simple* if the radical $\mathfrak{R}$ reduces to $\{0\}$. It is readily seen that the factor ring $A/\mathfrak{R}$ is semi-simple.

The following theorem justifies our use of the term "radical":

THEOREM I. *Let A be a ring, and $\mathfrak{R}$ its radical. Then every nilpotent left ideal is contained in $\mathfrak{R}$ and, furthermore, if A satisfies the descending chain condition for left ideals, $\mathfrak{R}$ is itself nilpotent.*

If $\mathfrak{M}$ is any simple A-module, $\mathfrak{A}$ any left ideal in A, $\mathfrak{A}\mathfrak{M}$ is a sub-module of $\mathfrak{M}$, and is therefore $\{0\}$ or $\mathfrak{M}$. If $\mathfrak{A}\mathfrak{M} = \mathfrak{M}$, then $\mathfrak{A}^n\mathfrak{M} = \mathfrak{M}$ for any positive integer n. Thus if $\mathfrak{A}^n = \{0\}$ for some n, we have $\mathfrak{A}\mathfrak{M} = \{0\}$ and $\mathfrak{A} \subset \mathfrak{R}$.

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[4] This definition of the radical is due to C. Chevalley. I am indebted to Professor Chevalley for many interesting discussions on the subject of this paper.

We suppose now that A satisfies the descending chain condition for left ideals and that $\mathfrak{R}$ is not nilpotent. Consider the chain $\mathfrak{R} \supset \mathfrak{R}^2 \supset \cdots$; under our hypotheses, there is some N such that $\mathfrak{R}^N = \mathfrak{R}^{N+1} = \cdots \neq \{0\}$. Call $\mathfrak{R}^N = \mathfrak{S}$; we have $\mathfrak{S}^2 = \mathfrak{S}$ and since $\mathfrak{R}$ is two-sided, $\mathfrak{S}$ is also. Let E be the set of all left ideals $\mathfrak{A} \neq \{0\}$ such that $\mathfrak{S}\mathfrak{A} = \mathfrak{A}$. E is not empty, since $\mathfrak{S}$ is in E. Since the descending chain condition holds, we see that there is a minimal element in E, say $\mathfrak{A}_0$, such that no ideal in E is properly contained in $\mathfrak{A}_0$. Let $\mathfrak{A}_1$ be the set of all $x \in \mathfrak{A}_0$ for which $\mathfrak{S}x = \{0\}$. Since $\mathfrak{S}$ is two-sided, it follows that $\mathfrak{A}_1$ is a left ideal. If $x \in \mathfrak{A}_0$, $x \notin \mathfrak{A}_1$, $\mathfrak{S}x$ is a left ideal, not $\{0\}$, is contained in $\mathfrak{A}_0$, and is such that $\mathfrak{S}(\mathfrak{S}x) = \mathfrak{S}^2x = \mathfrak{S}x$; whence it follows that $\mathfrak{S}x = \mathfrak{A}_0$. $\mathfrak{A}_0$, $\mathfrak{A}_1$ being left ideals, the factor group $\mathfrak{A}_0/\mathfrak{A}_1$ has a natural structure as an A-module. By the preceding remark, we have for any nonzero $\bar{x} \in \mathfrak{A}_0/\mathfrak{A}_1$, $\mathfrak{S}\bar{x} = \mathfrak{A}_0/\mathfrak{A}_1$. But then for any nonzero $\bar{x} \in \mathfrak{A}_0/\mathfrak{A}_1$, we have $A\bar{x} = \mathfrak{A}_0/\mathfrak{A}_1$, from which it is clear that $\mathfrak{A}_0/\mathfrak{A}_1$ is a simple A-module. Because of this we have $\mathfrak{R}\mathfrak{A}_0/\mathfrak{A}_1 = \{0\}$ and therefore $\mathfrak{S}\mathfrak{A}_0/\mathfrak{A}_1 = \{0\}$ which is contrary to our assumptions. From this it follows that $\mathfrak{R}$ is nilpotent.

The natural generalization of Weyl's definition, that a ring is semi-simple if it has a faithful semi-simple module, is, by Theorem II, equivalent to our definition.

THEOREM II. *A necessary and sufficient condition that the radical* $\mathfrak{R}$ *of a ring* A *be* $\{0\}$ *is that there exists a faithful semi-simple* A*-module.*

Let $\mathfrak{M}$ be a semi-simple A-module. If we write $\mathfrak{M}$ as the direct sum, $\sum_{\lambda \in L} \mathfrak{N}_\lambda$, of simple sub-modules, it is clear that the annihilator of $\mathfrak{M}$ is the intersection of the annihilators of the modules $\mathfrak{N}_\lambda$. If $\mathfrak{M}$ is faithful, its annihilator is $\{0\}$, giving us a system of simple modules, the intersection of whose annihilators is $\{0\}$. Thus $\mathfrak{R} = \{0\}$.

If $\mathfrak{R} = \{0\}$, let $\{\mathfrak{M}_\lambda\}_{\lambda \in L}$ be a system of simple A-modules chosen in such a manner that $\bigcap_{\lambda \in L} \mathfrak{A}_\lambda = \{0\}$, where $\mathfrak{A}_\lambda$ is the annihilator of $\mathfrak{M}_\lambda$. Let $\mathfrak{M}'$ be the strong direct product of the additive groups of the $\mathfrak{M}_\lambda$. We make $\mathfrak{M}'$ into an A-module by writing $a(\cdots, m_\lambda, \cdots, m_{\lambda'}, \cdots) = (\cdots, am_\lambda, \cdots, am_{\lambda'}, \cdots)$. Let $\mathfrak{M}_\lambda^*$ be the sub-module of $\mathfrak{M}'$ consisting of all elements of $\mathfrak{M}'$ all of whose coordinates other than the λth are zero. $\mathfrak{M}_\lambda^*$ is clearly isomorphic to $\mathfrak{M}_\lambda$ and is therefore simple. Let $\mathfrak{M} = \sum_{\lambda \in L} \mathfrak{M}_\lambda^*$. Since the $\mathfrak{M}_\lambda^*$ are simple, $\mathfrak{M}$ is semi-simple. The annihilator of $\mathfrak{M}$ is the intersection of the annihilators of the $\mathfrak{M}_\lambda^*$; their intersection was chosen to be $\{0\}$, whence it follows that $\mathfrak{M}$ is faithful. This completes the proof of the theorem.

The class of semi-simple rings is much wider than the class of semi-simple rings with the descending chain condition.[5] Our object is to characterize the latter class of rings by properties of their modules. It is known that if A is a semi-simple ring with the descending chain condition, then every A-module is the direct sum of its trivial sub-module and a semi-simple sub-module. We propose to show that this property is characteristic of the class of rings we are studying:

THEOREM III. *Let A be a ring which has the property that every A-module can be represented as the direct sum of its trivial sub-module and a semi-simple sub-module. Then A is a semi-simple ring in which the descending chain condition holds.*

For convenience we introduce two modules, $\mathfrak{A}^L$ and $\mathfrak{A}^L \times Z$, which can be defined for any ring. The additive group of $\mathfrak{A}^L$ is that of A, while, for $a_1 \in A$, $a_2 \in \mathfrak{A}^L$, $a_1 a_2$ is the element of $\mathfrak{A}^L$ formed by taking the product of a_1 and a_2 in A. Sub-modules of $\mathfrak{A}^L$ correspond to left ideals in A, and conversely. $\mathfrak{A}^L \times Z$ consists of all pairs $\{(a, n)\}$ where $a \in A$ and n is an integer. We define

$$(a, n) + (a', n') = (a + a', n + n'), \qquad a(a', n) = (aa' + na, 0).$$

Since $a(0, 1) = (a, 0)$, the element $(0, 1)$ is annihilated by no nonzero a.

We return to the proof of the theorem. Let us write $\mathfrak{A}^L$ as the direct sum of $\mathfrak{A}^L_T$ and a semi-simple sub-module $\mathfrak{B}$, where $\mathfrak{A}^L_T$ is the trivial sub-module of $\mathfrak{A}^L$, consisting of those $a \in A$ for which $Aa = \{0\}$. We shall show that $\mathfrak{A}^L_T = \{0\}$. If $\mathfrak{M}$ is any A-module, we write $\mathfrak{M} = \mathfrak{N} + \mathfrak{M}_T$, the sum being direct so that $\mathfrak{M}_T \cap \mathfrak{N} = \{0\}$. Since $\mathfrak{N}$ is a sub-module of $\mathfrak{M}$, and $\mathfrak{A}^L_T$ a subset of A, we have $\mathfrak{A}^L_T \mathfrak{N} \subset \mathfrak{N}$. However, $A(\mathfrak{A}^L_T \mathfrak{N}) = (A\mathfrak{A}^L_T)\mathfrak{N} = \{0\}$, since $A\mathfrak{A}^L_T = \{0\}$, so that $\mathfrak{A}^L_T \mathfrak{N} \subset \mathfrak{M}_T$. We have then $\mathfrak{A}^L_T \mathfrak{N} = \{0\}$. Since $\mathfrak{A}^L_T \mathfrak{M} = \mathfrak{A}^L_T \mathfrak{N} + \mathfrak{A}^L_T \mathfrak{M}_T$, $A\mathfrak{M}_T = \{0\}$ and therefore $\mathfrak{A}^L_T \mathfrak{M}_T = \{0\}$, so that $\mathfrak{A}^L_T \mathfrak{M} = \{0\}$ for every $\mathfrak{M}$. But we have already seen that $(0, 1) \in \mathfrak{A}^L \times Z$ is annihilated only by the zero element of A, whence it follows that $\mathfrak{A}^L_T = \{0\}$, which proves our assertion. This result under the condition of the theorem shows that $\mathfrak{A}^L$ is a semi-simple A-module.

We shall need the following two lemmas.

LEMMA I. *If A is a ring which satisfies the condition of the theorem, and $\mathfrak{A}$ is a two-sided ideal in A, then the factor ring $A/\mathfrak{A}$ also satisfies the condition of the theorem.*

Let π be the natural homomorphism of A onto $A/\mathfrak{A}$. If $\mathfrak{M}$ is any $A/\mathfrak{A}$-module, we can consider $\mathfrak{M}$ as an A-module by writing $am=\pi(a)m$ for each $a\in A$, $m\in\mathfrak{M}$, and retaining the additive group structure of $\mathfrak{M}$. It is readily verified that the trivial A-sub-module of $\mathfrak{M}$ is the trivial $A/\mathfrak{A}$-sub-module; and that every A-sub-module of $\mathfrak{M}$ is an $A/\mathfrak{A}$-sub-module, and conversely. Writing $\mathfrak{M}=\mathfrak{M}_T+\mathfrak{N}$, where $\mathfrak{N}$ is a semi-simple A-sub-module, it is clear that $\mathfrak{N}$ is semi-simple as an $A/\mathfrak{A}$-sub-module, which proves the lemma.

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[5] The ring of integers, which does not satisfy the descending chain condition, is seen to be semi-simple by consideration of the cyclic groups of prime order as simple modules for this ring.

LEMMA II. *Let A be a ring which satisfies the condition of the theorem. Furthermore, suppose that there exists an integer $q\neq 0$ such that $qA=\{0\}$. Then the ring A has a unit element.* (qA is the set of all elements of the form qa, $a\in A$.)

Form the module $\mathfrak{A}^L\times Z$. As an A-module, $\mathfrak{A}^L\times Z$ can be expressed as the direct sum $\mathfrak{A}_1+\mathfrak{A}_2$, where $\mathfrak{A}_1$ is the trivial sub-module of $\mathfrak{A}^L\times Z$. Suppose that the element $(0,1)\in\mathfrak{A}^L\times Z$ decomposes into $(-a_0,1-n)$ $+(a_0,n)$, with $(-a_0,1-n)\in\mathfrak{A}_1$, $(a_0,n)\in\mathfrak{A}_2$. We have $(0,qn)$ $=(qa_0,qn)=q(a_0,n)$, which is in $\mathfrak{A}_2$ since (a_0,n) is. However, for each $a\in A$, $a(0,qn)=(qna,0)=(0,0)$, so that $(0,qn)\in\mathfrak{A}_1$. But then $(0,qn)=(0,0)$ or $n=0$. The element $(-a_0,1-n)=(-a_0,1)$ is in $\mathfrak{A}_1$, so that $(-aa_0+a,0)=a(-a_0,1)=(0,0)$ or $aa_0=a$ for all $a\in A$. Since $a(b-a_0b)=0$ for every $a,b\in A$, and since $\mathfrak{A}^L_T=\{0\}$, we have $a_0b=b$. Thus a_0 is a unit element for the ring A, concluding the proof of the lemma.

We shall now show that the ring A has a unit element in every case. We start by showing that the trivial sub-module $\mathfrak{A}_1$ of $\mathfrak{A}^L\times Z$ has at least two elements. If $\mathfrak{A}_1=\{0\}$, then $\mathfrak{A}^L\times Z$ is semi-simple. The set $(A,0)$ is a proper sub-module of $\mathfrak{A}^L\times Z$; we can then write, as a direct sum, $\mathfrak{A}^L\times Z=(A,0)+\mathfrak{A}_3$, with $\mathfrak{A}_3\neq\{0\}$. Since $\mathfrak{A}_3$ is a sub-module of $\mathfrak{A}^L\times Z$, we have $A\mathfrak{A}_3\subset\mathfrak{A}_3$. However every element of $A\mathfrak{A}_3$ has a zero in the second coordinate, so that $A\mathfrak{A}_3\subset(A,0)$. But then $A\mathfrak{A}_3=\{0\}$, and therefore $\mathfrak{A}_3\subset\mathfrak{A}_1$, contradicting the supposition that $\mathfrak{A}_1=\{0\}$, and proving the assertion.

Now let (a',n') be any nonzero element of $\mathfrak{A}_1$, that is, $a(a',n')$ $=(0,0)$ or $aa'+n'a=0$ for all $a\in A$. Clearly n' is not zero, for, if it were, a' would be zero, contrary to the condition that (a',n') is not zero. If $n'a=0$ for all a, Lemma II shows that A has a unit element.

Let $\mathfrak{A}_0$ be the set of all $a\in A$ for which $n'a=0$, and let $\mathfrak{B}=n'A$. Clearly both are two-sided ideals in A. We assert that A is the direct sum of $\mathfrak{A}_0$ and $\mathfrak{B}$. We have already seen that $\mathfrak{A}^L$ is semi-simple; write $\mathfrak{A}^L$ as the direct sum, $\sum_{\lambda\in L}\mathfrak{A}_\lambda$, of simple left ideals. The mapping, $\mathfrak{A}_\lambda\to\mathfrak{A}_\lambda$, which sends $a\in\mathfrak{A}_\lambda$ into $n'a$, is an endomorphism of the simple module $\mathfrak{A}_\lambda$, which, by Schur's Lemma, is either zero or an automorphism. Let $L_0\subset L$ be the set of all λ for which this endomorphism is zero. $\mathfrak{A}_0$ is then $\sum_{\lambda\in L_0}\mathfrak{A}_\lambda$. Furthermore,

$$\sum_{\lambda\in L-L_0}\mathfrak{A}_\lambda = n'\sum_{\lambda\in L-L_0}\mathfrak{A}_\lambda = n'\sum_{\lambda\in L}\mathfrak{A}_\lambda = n'A = \mathfrak{B}.$$

Thus, $A=\mathfrak{A}_0+\mathfrak{B}$, the sum being direct.

Since both $\mathfrak{A}_0$ and $\mathfrak{B}$ are two-sided ideals, $\mathfrak{A}_0$ is isomorphic with the factor ring $A/\mathfrak{B}$. By Lemma I, $\mathfrak{A}_0$ then satisfies the conditions of Lemma II with $q=n'$, and hence $\mathfrak{A}_0$ has a unit element which we shall denote by e_0. Because of the two-sidedness of $\mathfrak{A}_0$ and $\mathfrak{B}$, we have $\mathfrak{A}_0\mathfrak{B}=\mathfrak{B}\mathfrak{A}_0=\{0\}$ (since $\mathfrak{A}_0\cap\mathfrak{B}=\{0\}$), so that it only remains for us to show that $\mathfrak{B}$ has a unit element. In the decomposition $A=\mathfrak{A}_0+\mathfrak{B}$, we write $a'=\alpha+\beta$, $\alpha\in\mathfrak{A}_0$, $\beta\in\mathfrak{B}$. We have already seen that $aa'=-n'a$ for each $a\in A$. For any $b\in\mathfrak{B}$, we have $-n'b=ba'=b(\alpha+\beta)=b\beta$. Since $\beta\in\mathfrak{B}$, $e_1=-(1/n')\beta$ is defined, and is such that $be_1=b$ for every $b\in\mathfrak{B}$. Exactly as in Lemma II, we have $be_1=e_1b$, so that e_1 is the desired unit element for $\mathfrak{B}$. It is clear that e_0+e_1 is the unit element for the entire ring A.

Having established the existence of a unit element in A, we can easily derive that A satisfies the descending chain condition. Again we write the semi-simple module $\mathfrak{A}^L$ as the direct sum, $\sum_{\lambda\in L}\mathfrak{A}_\lambda$, of simple left ideals. The unit element then decomposes in the form $1=\sum_{\lambda\in L}1_\lambda$, with $1_\lambda=0$ for all $\lambda\neq\lambda_i$, $1\leq i\leq N$. But then $a=a\cdot 1=\sum_{\lambda\in L}a\cdot 1_\lambda=\sum_{\lambda\in L}a_\lambda$, $a_\lambda=a\cdot 1_\lambda$, so that $a_\lambda=0$ for all $\lambda\neq\lambda_i$ and all $a\in A$. This shows that $\mathfrak{A}_\lambda=\{o\}$ for $\lambda\neq\lambda_i$ so that $\mathfrak{A}^L$ is the direct sum of a finite number of simple left ideals, giving immediately the descending chain condition. The module $\mathfrak{A}^L$ is semi-simple. Because $1\in\mathfrak{A}^L$, the annihilator of $\mathfrak{A}^L$ reduces to $\{0\}$ or $\mathfrak{A}^L$ is faithful. By Theorem II the ring A is semi-simple. This concludes the proof of Theorem III.

It is well known[6] that every left ideal in a semi-simple ring with the descending chain condition is principally generated by an idempotent. The method of proof of Theorem III enables us to prove the converse:

THEOREM IV. *If every left ideal in a ring A is principally generated by an idempotent, A is a semi-simple ring with the descending chain condition.*

We shall first show that A has a unit element. A being its own left ideal, there is an idempotent e such that $Ae=A$. Thus, given any $a\in A$, there is a $b\in A$ such that $a=be$. But then $ae=be^2=be=a$, since e is idempotent. Thus e is a right unit element in A. Let now $\mathfrak{B}$ be the set of all elements of the form $ea-a$. Since $be=b$, $\mathfrak{B}$ is clearly a left ideal so that there is an $f\in A$, with $f^2=f$ and $\mathfrak{B}=Af$. Since f is idempotent, it is clear that $f\in\mathfrak{B}$. There exists then an element $g\in A$ with $f=eg-g$. But $f=f^2=ff=f(eg-g)=0$ or $\mathfrak{B}=\{0\}$. Thus $ea=a$ or e is a unit element for A.

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[6] N. Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, 1943, p. 65.

From the condition of the theorem, and from the existence of a unit element in A, it follows that $\mathfrak{A}^L$ is semi-simple. For, if $\mathfrak{A}$ is any left ideal, there is an idempotent f with $Af=\mathfrak{A}$. But then $\mathfrak{A}^L$ is the direct sum of Af and $A(e-f)$ so that $\mathfrak{A}^L$ is semi-simple. By the argument in the last paragraph of the proof of Theorem III, it follows that A is a semi-simple ring with the descending chain condition.

PRINCETON UNIVERSITY

SEMI-SIMPLE EXTENSIONS OF RINGS

OSCAR GOLDMAN

In this paper we investigate the conditions under which a given ring is a sub-ring of a semi-simple ring.[1] For convenience, we say that a ring A is an *extension* of a ring B, if B is a sub-ring of A. It is found that the existence of a semi-simple extension is equivalent to the vanishing of the extension radical, a two-sided ideal defined analogously to the ordinary radical. In Theorem II we give an intrinsic characterization of the extension radical, where we find that the latter is determined by the addition in the ring and is independent of the multiplication. This result is summarized in Theorem III.

For the convenience of the reader, we reproduce here some of the definitions given in the paper mentioned in footnote 1. The *radical* of a ring A is obtained as the intersection of the annihilators of all simple A-modules. When the radical consists only of the zero element of A, we say that the ring is *semi-simple*. The radical as defined here contains the ideal classically known as the radical (the sum of all nilpotent left ideals) and is equal to it if one assumes tht A satisfies the descending chain condition on left ideals. A ring which is semi-simple in the present sense has then no nilpotent ideals.

In order to define the extension radical, we must first introduce the auxiliary notion of a quasi-simple module. If $\mathfrak{M}$ is an abelian group, denote by $E(\mathfrak{M})$ the ring of all endomorphisms of $\mathfrak{M}$. We say that $\mathfrak{M}$ is a *quasi-simple group* if $\mathfrak{M}$ is a simple $E(\mathfrak{M})$-module. An A-module $\mathfrak{M}$ is a *quasi-simple module* if the underlying additive group of $\mathfrak{M}$ is quasi-simple. We shall find the following two lemmas useful.

LEMMA I. *If $\mathfrak{M}$ is a simple A-module, then $\mathfrak{M}$ is a quasi-simple module.*

Let $\mathfrak{A}$ denote the annihilator of $\mathfrak{M}$. It is clear that $\mathfrak{M}$ is a simple $A/\mathfrak{A}$-module, and furthermore that $A/\mathfrak{A} \subset E(\mathfrak{M})$. Thus $\mathfrak{M}$ is certainly simple for $E(\mathfrak{M})$.

LEMMA II. *Every quasi-simple group is the underlying additive group of a vector space over a field and conversely.*

Let Z be the center of $E(\mathfrak{M})$. Since $\mathfrak{M}$ is a simple $E(\mathfrak{M})$-module, Z is

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[1] For the definitions and elementary properties of the concepts involved in the study of semi-simple rings, see my previous paper, *A characterization of semi-simple rings*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1021–1027.

a field by Schur's lemma, and $\mathfrak{M}$ can be considered as a vector space over Z. (The precise form of Schur's lemma used here is given in the reference cited in footnote 1.)

If $\mathfrak{M}$ is a vector space over a field K, let E' be the ring of all linear transformations of $\mathfrak{M}$. It can easily be seen[3] that $\mathfrak{M}$ is a simple E'-module; but $E' \subset E(\mathfrak{M})$, so that $\mathfrak{M}$ is a simple $E(\mathfrak{M})$-module, and is therefore a quasi-simple group.

By Lemma II we can associate an integer, either zero or a prime, to each quasi-simple group, $\mathfrak{M}$, namely, the characteristic of those fields over which $\mathfrak{M}$ can be considered as a vector space.

The following fact will be found useful: If every element of an additive group $\mathfrak{M}$ has the same order p, with p a prime, then $\mathfrak{M}$ is a quasi-simple group since it can be considered as a vector space over the primitive field of characteristic p.

We now define the *extension radical* of a ring A to be the intersection of the annihilators of all quasi-simple A-modules. It is clear that the extension radical is a two-sided ideal in A. By Lemma I, the extension radical of A is contained in the radical of the ring.

In order to make use of the notion just introduced, it is necessary to review some elementary facts about simple and semi-simple rings. We say that a ring A is *simple* if there exists a faithful simple A-module. From our definitions it is clear that $E(\mathfrak{M})$ is a simple ring if $\mathfrak{M}$ is a quasi-simple group. We call a ring $E(\mathfrak{M})$, where $\mathfrak{M}$ is quasi-simple, a *full linear ring*. If $\{A_\lambda\}$ is a set of rings, we construct the product of the A_λ by forming the Cartesian product of the underlying sets (the unrestricted product) and defining the ring operations to be coordinate-wise. (The ring formed in this way is sometimes referred to in the literature as the direct sum of the A_λ.) We have the following lemmas.

LEMMA III. *Let $\{A_\lambda\}$ be a set of simple rings, then the product A of the A_λ is semi-simple.*

Let $\mathfrak{M}_\lambda$ be a faithful simple A_λ-module. We can consider $\mathfrak{M}_\lambda$ as an A-module, by writing $(\cdots, a_\lambda, \cdots, a_{\lambda'}, \cdots)x = a_\lambda x$, for all $x \in \mathfrak{M}_\lambda$. As an A-module, $\mathfrak{M}_\lambda$ is still simple; the annihilator of $\mathfrak{M}_\lambda$ is the set of all elements of A whose λ-coordinate is zero. The intersection of the annihilators of all the modules $\mathfrak{M}_\lambda$ considered as A-modules is then $\{0\}$, so that A is semi-simple.

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[3] The statement made is equivalent to the fact that given a subspace $\mathfrak{N}$ of $\mathfrak{M}$, there exists a basis for $\mathfrak{M}$, a subset of which is a basis for $\mathfrak{N}$. See for example, S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, 1942, p. 73.

LEMMA IV. *Every semi-simple ring is contained in the product of full linear rings.*

Let A be a semi-simple ring, $\{\mathfrak{M}_\lambda\}$ a system of simple A-modules, the intersection of whose annihilators is $\{0\}$. Since $\mathfrak{M}_\lambda$ is a simple A-module, $E(\mathfrak{M}_\lambda)$ is a full linear ring, let $E=\prod_\lambda E(\mathfrak{M}_\lambda)$. If $\mathfrak{A}_\lambda$ is the annihilator of $\mathfrak{M}_\lambda$, we have a homomorphism σ_λ of A into $E(\mathfrak{M}_\lambda)$ whose kernel is $\mathfrak{A}_\lambda$. If π_λ is the projection of E onto $E(\mathfrak{M}_\lambda)$, define the homomorphism τ of A into E by $\pi_\lambda\tau=\sigma_\lambda$. It is clear that the kernel of τ is the intersection of the kernels of the σ_λ which we know to be $\{0\}$. Thus A is isomorphically contained in E, the product of full linear rings.

We are now in a position to prove the following theorem.

THEOREM I. *Let A be a ring and $\mathfrak{R}$ its extension radical. Then a necessary and sufficient condition that A has a semi-simple extension is that $\mathfrak{R}$ reduces to $\{0\}$.*

Suppose that B is a semi-simple extension of A; by Lemma IV, B is contained in the product $\prod_\lambda E(\mathfrak{M}_\lambda)$ of full linear rings. The $\mathfrak{M}_\lambda$ are quasi-simple A-modules. By an argument similar to the one used in the proof of Lemma III, the intersection of the annihilators of the $\mathfrak{M}_\lambda$ is $\{0\}$, and therefore the extension radical of A reduces to $\{0\}$.

If on the other hand, the extension radical of A is $\{0\}$, let $\{\mathfrak{M}_\lambda\}$ be a system of quasi-simple A-modules, the intersection of whose annihilators is $\{0\}$. Defining E to be the product of the $E(\mathfrak{M}_\lambda)$, we know that E is semi-simple. Exactly as in the proof of Lemma IV, we can construct an isomorphism of A into E. Thus A has a semi-simple extension.

If n is an integer, define nA to be the set of all elements of A which are of the form na for some $a\in A$. It is clear that nA is a two-sided ideal in A. Denote by T the set of all elements of A which are of finite order in the underlying additive group of A. Again T is a two-sided ideal. We have the following theorem.

THEOREM II. *Let A be a ring, and $\mathfrak{R}$ its extension radical. Then $\mathfrak{R}=T\cap\bigcap_p pA$, taken over all prime numbers p.*

If the characteristic of a quasi-simple A-module $\mathfrak{M}$ is $p\neq 0$, it is clear that pA is contained in the annihilator of $\mathfrak{M}$. Again if $\mathfrak{M}$ is of characteristic zero and $a\in T$, so that $na=0$ for some $n\neq 0$, we have $ax=na\cdot n^{-1}x=0$ for all $x\in\mathfrak{M}$, so that T is contained in the annihilator of $\mathfrak{M}$. Thus we have $\mathfrak{R}\supset T\cap\bigcap_p pA$.

We now consider the special case in which A has a unit element.

We shall first show that it is sufficient to consider only quasi-simple A-modules in which the unit element acts like the identity automorphism. To be precise, if $\mathfrak{M}$ is a quasi-simple A-module, $\mathfrak{A}$ the annihilator of $\mathfrak{M}$, then there is a quasi-simple A-module $\mathfrak{N}$, such that the unit element of A acts like the identity in $\mathfrak{N}$, the annihilator of $\mathfrak{N}$ is $\mathfrak{A}$ and the characteristic of $\mathfrak{N}$ is the same as that of $\mathfrak{M}$. Let $\mathfrak{M}_T$ be the trivial sub-module of $\mathfrak{M}$, that is, the set of all $x \in \mathfrak{M}$ for which $Ax = \{0\}$. If Z is the center of $E(\mathfrak{M})$, Z is a field, all elements of Z commute with the elements of A modulo $\mathfrak{A}$, so that the elements of Z map $\mathfrak{M}_T$ into itself. Then the factor module $\mathfrak{M}/\mathfrak{M}_T$ is a quasi-simple A-module of the same characteristic as $\mathfrak{M}$. Since, for all $x \in \mathfrak{M}$, $1x - x \in \mathfrak{M}_T$, the unit element of A acts like the identity in $\mathfrak{M}/\mathfrak{M}_T$. If $a \in \mathfrak{A}$ it is obvious that $a\mathfrak{M}/\mathfrak{M}_T = \{\bar{0}\}$. If $a\mathfrak{M}/\mathfrak{M}_T = \{\bar{0}\}$, or $a\mathfrak{M} \subset \mathfrak{M}_T$, we have $Aa\mathfrak{M} = \{0\}$ or $a\mathfrak{M} = \{0\}$. Thus the annihilator of $\mathfrak{M}/\mathfrak{M}_T$ is $\mathfrak{A}$.

If p is a prime, the factor group A/pA has a natural structure as an A-module. Furthermore, each element of A/pA has the same prime order so that A/pA is a quasi-simple module. If $aA/pA = \{\bar{0}\}$ or $aA \subset pA$ we have $a \in pA$. Thus the annihilator of A/pA is contained in pA and is therefore equal to pA.

If $T = A$, that is, if $n1 = 0$, for some $n \neq 0$, all nontrivial quasi-simple modules have finite characteristic which divides n. For, if $x \in \mathfrak{M}$, $1x = x$, $nx = n1x = 0$ so that the characteristic of $\mathfrak{M}$ divides n. In that case, by the results of the above paragraph, $\mathfrak{R} \subset T \cap \bigcap_p pA$ and therefore $\mathfrak{R} = T \cap \bigcap_p pA$.

If $T \neq A$, A/T is a nontrivial A-module. It is clear that no element of $\mathfrak{M}_0 = A/T$ has finite order. Consider the tensor product[3] of the field of rationals K with $\mathfrak{M}_0$, $K \times \mathfrak{M}_0$. By the preceding remark, we know that, if $1 \times \bar{a} = 0$, then $\bar{a} = \bar{0}$. $K \times \mathfrak{M}_0$ is a vector space over K, and has a natural structure as an A-module; we define $a'(1 \times \bar{a}) = 1 \times a'\bar{a}$, linearity defines the operations of A on all of $K \times \mathfrak{M}_0$. If now $aK \times \mathfrak{M}_0 = \{0\}$, we have in particular, $a(1 \times \bar{1}) = 1 \times a\bar{1} = 0$, or $a\bar{1} = \bar{0}$ whence $a \in T$. Thus the annihilator of $K \times \mathfrak{M}_0$ is contained in T and is therefore equal to T. We find then again that $\mathfrak{R} \subset T \cap \bigcap_p pA$, so that in the general case, provided A has a unit element, $\mathfrak{R} = T \cap \bigcap_p pA$.

If A is any ring, we introduce the ring A^* in the following way: A^* consists of all pairs $\{(a, n)\}$, $a \in A$, n any integer. We define

$$(a, n) + (a', n') = (a + a', n + n'),$$
$$(a, n)(a', n') = (aa' + na' + n'a, nn').$$

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[3] For the definition of the tensor product, see H. Whitney, *Tensor products of abelian groups*, Duke Math. J. vol. 4 (1938) pp. 495–528.

It is clear that (0, 1) is the unit element of A^*. If $\mathfrak{M}^*$ is an A^*-module, it is immediately an A-module. Conversely, if $\mathfrak{M}$ is an A-module, $\mathfrak{M}$ can be made into an A^*-module by writing $(0, 1)x = x$, for all $x \in \mathfrak{M}$. If $\mathfrak{A}^*$ and $\mathfrak{A}$ are the annihilators in A^* and A, it is clear that $\mathfrak{A}^* \supset (\mathfrak{A}, 0)$. We have then that the extension radical of A^* contains $(\mathfrak{R}, 0)$. But A^* has a unit element, so that its extension radical is $T^* \cap \bigcap_p pA^*$. However, $T^* = (T, 0)$ and $\bigcap_p pA^* = (\bigcap_p pA, 0)$, so that the extension radical of A^* is equal to $(T \cap \bigcap_p pA, 0)$. This together with the preceding remark gives $\mathfrak{R} \subset T \cap \bigcap_p pA$, so that $\mathfrak{R} = T \cap \bigcap_p pA$, concluding the proof of the theorem.

The condition that the extension radical reduces to $\{0\}$ is equivalent to the statement that there are no elements of order p^2 in the additive group of A, for all primes p.[4] For, suppose first that $T \cap \bigcap_p pA = \{0\}$, and $p^2a = 0$ for some p and some a. Then for any prime q, we can write $pa = qra$, where r is any solution of the congruence $p \equiv qr \pmod{p^2}$ (that such an r exists follows from the fact that q and p^2 are relatively prime when $q \neq p$; in the remaining case $r = 1$ will suffice). pa is then in the extension radical, so that $pa = 0$. Conversely, suppose that $a \in T \cap \bigcap_p pA$. Let n be the order of a, and let p be any prime factor of n; we have then $pma = 0$, where pm is the smallest positive integer with this property. Since a is in the extension radical, we have, for some b, $a = pb$. But then $p^2mb = 0$, so that, if there are no elements of order p^2, we have already $pmb = 0$ or $ma = 0$, contradicting the supposition that pm is the order of a. Thus $a = 0$, and hence the extension radical is $\{0\}$.

Combining this result with Theorems I and II, we see that we have already proved the following theorem.

THEOREM III. *Let G be an abelian group, and let Γ be the set of all rings having G as underlying additive group. Then, a necessary and sufficient condition that each element of Γ has a semi-simple extension is that G have no element of order p^2 for any prime number p.*

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[4] I am indebted to the referee for calling my attention to this fact.

ON A PROBLEM OF A. KUROSCH

J. LEVITZKI

The following problem was posed by Kurosch [2]:[1] Suppose that an algebraic algebra[2] A over a field F has a finite number of generators over F, and suppose further that A is of bounded degree over F. Is then A of finite dimensionality over F? Kurosch answered his problem in the affirmative in the case where the elements of the algebra are of degree not greater than 3. As was observed by N. Jacobson [1], Kurosch's problem is equivalent to the following: Is any algebraic algebra of bounded degree locally finite[3] over F? Jacobson succeeded in reducing this problem to the special case of nil-algebras. In the present note we supplement Jacobson's results by proving that *any nil-algebra of bounded index (degree) is locally finite*. Jacobson's results combined with this theorem answer in the affirmative the question raised by Kurosch.

To obtain this result we make use of the notion of semi-nilpotency[4] and of its counterpart, the semi-regularity, which were introduced by the author in a previous paper [3]. As may be easily verified, each finitely generated nilpotent algebra is of finite dimensionality, and conversely, each nil-algebra of finite dimensionality is nilpotent. Hence, for nil-algebras the notion of local finiteness coincides with the notion of semi-nilpotency. Thus the above formulated result concerning nil-algebras is a consequence of the following more general theorem concerning nil-rings which will be proved in this paper: *Each nil-ring of bounded index is semi-nilpotent.* It is of interest to note in this connection that an analogous theorem does not hold for associative multiplicative systems (semi-groups).[5]

It will be convenient to adopt the terminology introduced by the author in [3]. Thus we use here the term radical to denote the sum $N(S)$ of all two-sided semi-nilpotent ideals of a ring S. We shall need the following theorems which were proved in [3]:

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] For the terminology compare Jacobson [1].

[3] An algebra A is called locally finite over F if each finite set of elements of A generates an algebra of finite dimensionality over F.

[4] A ring is called semi-nilpotent if each finite set of elements in this ring generates a nilpotent ring. A ring which is not semi-nilpotent is called semi-regular.

[5] Compare Kurosch [2, proof of Theorem 5]. This was observed independently also by Dr. Th. Motzkin of Jerusalem.

1. *The radical $N(S)$ is a semi-nilpotent two-sided ideal which contains all one-sided semi-nilpotent ideals of the ring.*

2. *The radical contains all nilpotent ideals of the ring.*

3. *The radical of the quotient-ring $S/N(S)$ is zero.*

4. *If the radical of a ring S is zero, and if a is a nonzero element of S, then $a\cdot S$ and $S\cdot a$ are semi-regular rings.*

DEFINITION. An integer $n=n(S)$ is called the upper index of a nil-ring S, if S contains elements a whose index is n (that is, $a^n=0$, $a^{n-1}\neq 0$), but it does not contain elements of higher index. Such an integer n exists of course for each nil-ring of bounded index.

LEMMA. *If S is a semi-regular nil-ring of bounded index, then there exists a semi-regular nil-ring whose upper index is smaller than the upper index of S.*

PROOF. Let $N(S)$ be the radical of S, then by 3 the radical of the ring $T=S/N(S)$ is zero. Since S is semi-regular, it follows that $S\neq N(S)$, and hence that T is a semi-regular nil-ring of bounded index so that $n(T)\leqq n(S)$. In the case where $n(T)<n(S)$ the lemma is proved. Thus it remains to deal with the case $n(T)=n(S)$. Now let a be an element of T so that $a\neq 0$, $a^2=0$, and let us consider the right ideal $U=a\cdot T$ of T. The ring U is by 3 and 4 also semi-regular, and clearly $n(U)\leqq n(T)$. Again in the case where $n(U)<n(T)$ the lemma is proved, and thus it remains to deal with the case where $n(U)=n(T)$. In this case the ring U contains an element b whose index is $n=n(T)$, that is, $b^{n-1}\neq 0$, and since $U=aT$ we may put $b=a\cdot t$, $t\in T$. We consider now the following two cases:

Case 1. $b^{n-1}U\neq 0$. Then $b^{n-1}a\neq 0$, and by putting $c=a+b$, it follows on account of $a^2=0$ that $c^n=(a+b)^n=(a+at)^n=(at)^{n-1}a=b^{n-1}a\neq 0$, which is a contradiction, since n is the upper index of T.

Case 2. $b^{n-1}U=0$ for each element b of U. In this case denote by M the set of all elements x of U satisfying the relation $xU=0$, then

$$b^{n-1}\in M, \qquad b\in U. \tag{1}$$

Since $M^2\subseteq MU=0$, it follows that M is a nilpotent ideal in U, and hence by 1 and 2 we have

$$N(U)\supseteq M \tag{2}$$

where $N(U)$ is the radical of U. Now let us consider the quotient-ring $V=U/N(U)$, which is also a semi-regular nil-ring of bounded index, Since further by (1) all $(n-1)$th powers of the elements of U belong to M, it follows by (2) that for each v of V we have $v^{n-1}=0$, which implies that the upper index of the semi-regular ring V is smaller than $n(S)$. This concludes the proof of the lemma.

THEOREM. *Each nil-ring S of bounded index is semi-nilpotent.*

PROOF. Suppose that S is semi-regular, then it would follow by the foregoing lemma that there exists a semi-regular nil-ring with an upper index 1, which is self-contradictory.

Added in Proof. The problem of Kurosch was also treated by Irving Kaplansky (*On a problem of Kurosch and Jacobson*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 496–500). In this paper Kaplansky succeeds in solving, in conjunction with results of Jacobson, the problem of Kurosch under the restriction that the number of elements in the coefficient field is at least equal to n, where n is a common bound for the degrees of the elements in the algebra.

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NOTE ON MULTIPLY-INFINITE SERIES

I. M. SHEFFER

Let $\sum a_{(i)}$ be a k-tuply infinite series, where the subscript (i) stands for the set of k indices $i_1, \cdots, i_k$ and where for each $s = 1, \cdots, k$ the index i_s ranges from 0 to ∞. If $\sum b_{(i)}$ is a second such series we can determine from them a third series $\sum c_{(i)}$ called the Cauchy-product series, defined by

$$c_{(i)} = \sum a_{(j)} b_{(n)}, \tag{1}$$

where the sum is over all indices for which, simultaneously,

$$j_s + n_s = i_s, \qquad s = 1, \cdots, k. \tag{2}$$

Series $\sum c_{(i)}$ is obtained by formally multiplying the two power series $\sum a_{(i)} t_1^{\alpha} \cdots t_k^{\lambda}$, $\sum b_{(i)} t_1^{\alpha} \cdots t_k^{\lambda}$ ($\alpha = i_1, \cdots, \lambda = i_k$), then setting $t_s = 1$, $s = 1, \cdots, k$.

Given that $\sum a_{(i)}$, $\sum b_{(i)}$ converge to A, B respectively, it is natural to ask if $\sum c_{(i)}$ possesses the *Cauchy-product property*, that is, if it converges and to the sum $C = AB$; and if this is not always true, under what further conditions it will be true. It is the purpose of this note to give an answer to this question. All convergence is to be in the sense of Pringsheim (except in the concluding remark).

For simply-infinite series (hence also for multiple series) mere convergence of the series $\sum a_{(i)}$, $\sum b_{(i)}$ is known to be insufficient to insure that $\sum c_{(i)}$ will converge to the right sum. A theorem of Mertens states however that if both given series converge, one of them *absolutely*, then the Cauchy-product property holds.

Like a number of other properties of simple series, the Mertens theorem does not go over unrestrictedly to multiple series that are Pringsheim convergent. We shall show this by an example, after which we shall find suitable restrictions that will restore the Cauchy-product property.

Example. Let $\sum a_{ij}$, $\sum b_{ij}$ be the following double series: $\sum a_{ij}$ is the absolutely convergent series whose first column has the elements

$$1, \frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}, \cdots,$$

while all other terms are zero; $\sum b_{ij}$ is the convergent series whose

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first two rows are

$$0! + 1! + \cdots + n! + \cdots,$$

$$-0! - 1! - \cdots - n! - \cdots,$$

all other terms being zero. It is readily ascertained that in the series $\sum c_{ij}$,

$$c_{0j} = j!; \quad c_{ij} = -j!/2^i, \qquad i > 0;$$

and that

$$R^C_{pq} = \frac{1}{2^p}(0! + 1! + \cdots + q!), \tag{3}$$

where R^C_{pq} designates the (p, q)th rectangular partial sum in the c-series:

$$R^C_{pq} = \sum c_{ij} \qquad (0 \leqq i \leqq p, 0 \leqq j \leqq q). \tag{4}$$

Relation (3) shows that R^C_{pq} does not have a limit as $p, q \to \infty$; so series $\sum c_{ij}$ does not even converge.

Let $\{R^B_{(i)}\}$ denote the set of rectangular partial sums for the convergent series $\sum b_{(i)}$:

$$R^B_{(i)} = \sum b_{(j)} \qquad (0 \leqq j_s \leqq i_s, s = 1, 2, \cdots, k). \tag{5}$$

The *boundedness* of this set is what we need to validate the Cauchy-product property. This is shown by the theorems that follow.

THEOREM 1. *Let* $A = \sum a_{(i)}$, $B = \sum b_{(i)}$ *be convergent, the former convergence being absolute. If the set* $\{R^B_{(i)}\}$ *is bounded, then* $\sum c_{(i)}$ *converges to the sum* $C = AB$.

For convenience in notation we give the proof for double series ($k = 2$), and it will be seen to carry over to the general case of k-tuple series. Define A', M by

$$A' = \sum |a_{ij}| \qquad (i, j = 0, 1, \cdots), \tag{6}$$

$$M = \text{l.u.b. } |R^B_{ij}| \qquad (i, j = 0, 1, \cdots). \tag{7}$$

Let $\epsilon > 0$ be given. There exist indices a, b such that for all $t \geqq a$, $w \geqq b$,

$$|R^A_{t,w} - A| < \epsilon, \tag{8}$$

and also such that

$$\sum{}^* |a_{ij}| < \epsilon, \tag{9}$$

where $\sum^*$ is the sum of the absolute values of all terms of $\sum a_{rs}$ that are exterior to R^A_{ab}.

Now we have

$$(10)\qquad \begin{aligned} R^C_{tw} &= \sum a_{pq} b_{rs} && (0 \leqq p + r \leqq t,\ 0 \leqq q + s \leqq w) \\ &= \sum a_{pq} R^B_{t-p,w-q} && (0 \leqq p \leqq t,\ 0 \leqq q \leqq w); \end{aligned}$$

and this last sum can be split into two:

$$(11)\qquad R^C_{tw} = \sum{}' + \sum{}'',$$

where $\sum'$ is summed over the set $(0 \leqq p \leqq a,\ 0 \leqq q \leqq b)$, and $\sum''$ over $(0 \leqq p \leqq t,\ 0 \leqq q \leqq w)$ with the additional restriction that in every term either $p > a$ or $q > b$. From (7) and (9), $|\sum''| \leqq M\epsilon$, so

$$(12)\qquad \left| R^C_{tw} - \sum{}' \right| \leqq M\epsilon \qquad (t \geqq a,\ w \geqq b).$$

Let λ_{pq} be defined by

$$(13)\qquad B = R^B_{pq} + \lambda_{pq};$$

then indices u_1, v_1 exist so that

$$(14)\qquad |\lambda_{pq}| < \epsilon \qquad (p \geqq u_1,\ q \geqq v_1).$$

Hence if $t \geqq u = a + u_1$, $w \geqq v = b + v_1$, then

$$R^C_{tw} - \sum{}' = R^C_{tw} - BR^A_{ab} + \sum a_{pq}\lambda_{t-p,w-q} \qquad (0 \leqq p \leqq a,\ 0 \leqq q \leqq b);$$

so from (12),

$$(15)\qquad \left| R^C_{tw} - BR^A_{ab} \right| \leqq (M + A')\epsilon \qquad (t \geqq u,\ w \geqq v).$$

Combining this with (8) we obtain

$$(16)\qquad \left| R^C_{tw} - AB \right| \leqq (M + A' + |B|)\epsilon \qquad (t \geqq u,\ w \geqq v);$$

so series $\sum c_{st}$ converges to $C = AB$, as was to be shown.

It may be asked if the condition of boundedness of the set $\{R^B_{(i)}\}$ is the mildest that will achieve the desired purpose. That this is in a sense so is shown by the following theorem.

THEOREM 2. *Let* $B = \sum b_{(i)}$ *be convergent. If set* $\{R^B_{(i)}\}$ *is unbounded, then there exists an absolutely convergent series* $\sum a_{(i)}$ *for which the Cauchy-product property fails to hold; in fact, for which series* $\sum c_{(i)}$ *does not even converge.*

We first give the proof for $k = 2$, after which we shall outline the corresponding proof in the general k-tuple case. For a given $\epsilon > 0$, indices (p, q) exist so that $|B - R^B_{ij}| < \epsilon$ for all $i > p$, $j > q$. Hence, since $\{R^B_{ij}\}$

is unbounded, either there is a fixed u such that $\{R^B_{uj}\}$ $(j=0, 1, \cdots)$ is unbounded, or there is a fixed v such that $\{R^B_{iv}\}$ $(i=0, 1, \cdots)$ is unbounded. It is no restriction to suppose that $\{R^B_{uj}\}$ is unbounded; and moreover, that u is the *largest* integer having this property. Then for a suitably chosen sequence of increasing integers j_n $(n=1, 2, \cdots)$,

$$\lim_{n\to\infty} \left| R^B_{u,j(n)} \right| = \infty \qquad (j(n) \equiv j_n). \tag{17}$$

We now define the absolutely convergent series $\sum a_{ij}$ as follows:

$$\begin{aligned} a_{ij} &= 0,\ j = 1, 2, \cdots; \quad a_{i0} = 0,\ i \neq r(u+1),\ r = 0, 1, \cdots; \\ a_{r(u+1),0} &= (r+1)^{-2}, \qquad\qquad r = 0, 1, \cdots. \end{aligned} \tag{18}$$

From (10) we readily find that for $s=1, 2, \cdots$,

$$\begin{aligned} R^C_{su+s-1,j(n)} = \frac{1}{1^2} R^B_{su+s-1,j(n)} &+ \frac{1}{2^2} R^B_{(s-1)u+s-2,j(n)} + \cdots \\ &+ \frac{1}{s^2} R^B_{u,j(n)}. \end{aligned} \tag{19}$$

Suppose series $\sum c_{ij}$ converges, say to C. Then indices (p', q') exist so that $|C-R^C_{ij}| < \epsilon$ $(i>p', j>q')$. Let s have a fixed value for which $su+s-1>p'$, and choose N so that $j_n>q'$ for all $n>N$. Then

$$\left| R^C_{su+s-1,j(n)} \right| < |C| + \epsilon \qquad (j(n) \equiv j_n;\ n > N). \tag{20}$$

Also, there exist constants $M_2, \cdots, M_s$ such that

$$\left| R^B_{(s-m)u+s-m-1,j(n)} \right| < M_{s-m} \qquad (n > N;\ m = 0, 1, \cdots, s-2), \tag{21}$$

as is seen from the maximal property of the integer u. Hence from (19),

$$\left| R^B_{u,j(n)} \right| < s^2\{ |C| + \epsilon + M_2 + \cdots + M_s\}, \tag{22}$$

for all $n>N$. As this contradicts (17), it is clear that series $\sum c_{ij}$ does not converge.

Now consider the general (k-tuple) case. Given $\epsilon>0$, there correspond indices $(p \equiv (p_1, \cdots, p_k)$ so that $|B-R^B_{(i)}| < \epsilon$ for all $(i)>(p)$. There is therefore an index, that we may take to be i_1, such that for $i_1=\alpha$ (a certain integer), the set $\{R^B_{\alpha,i(2,\cdots,k)}\}$ is unbounded, where $i(2, \cdots, k) \equiv i_2, \cdots, i_k$. It follows that there is an integer r, in the range $1 \leqq r < k$, with the following properties: (i) When certain r indices, which it is no restriction to take as $i_1, \cdots, i_r$, take on certain fixed values $i_1=\alpha, \cdots, i_r=\lambda$, while the remaining indices $i_{r+1}, \cdots, i_k$ remain arbitrary, the set $\{R^B_{\alpha,\cdots,\lambda,i(r+1,\cdots,k)}\}$ is unbounded. Here

$i(r+1, \cdots, k) = i_{r+1}, \cdots, i_k$. (ii) The integer r is maximal, that is, no larger integer has property (i).

In consequence of (ii), we can assert that it is possible to choose an infinite sequence of sets $i_{r+1}, \cdots, i_k$:

$$(i_{r+1}, \cdots, i_k)_n = i(r+1, \cdots, k)_n = i(r+1, \cdots, k; n), \qquad n = 1, 2, \cdots,$$

with $\lim (i_s)_n = \infty$ as $n \to \infty$ for each $s = r+1, \cdots, k$, for which

$$\lim_{n\to\infty} \left| R^B_{\alpha, \cdots, \lambda, i(r+1, \cdots, k; n)} \right| = \infty.$$

Choose the integer t large enough so that the following r inequalities hold: $t\alpha+t-1>p_1, \cdots, t\lambda+t-1>p_r$; and consider the sets

$$\left\{ R^B_{q\alpha+q-1, \cdots, q\lambda+q-1, i(r+1, \cdots, k; n)} \right\}, \qquad q = 1, 2, \cdots.$$

The set for which $q=1$ is unbounded, we know; and the set for each $q \geqq t$ is *bounded*, since then the sth index exceeds p_s for $s=1, \cdots, k$ (at least for n sufficiently large). Hence there is a largest value of q, say q', for which the corresponding R-set is unbounded.

Define the r integers $a, \cdots, l$ by

$$a = q'\alpha + q' - 1, \cdots, l = q'\lambda + q' - 1.$$

Then the set $\left\{ R^B_{a, \cdots, l, i(r+1, \cdots, k; n)} \right\}$ is unbounded; and since when h is any one of $a, \cdots, l$, and η is the corresponding term in $\alpha, \cdots, \lambda$, we have

$$qh + q - 1 = (qq')\eta + (qq') - 1,$$

we see that for each $q = 2, 3, \cdots$, set $\left\{ R^B_{qa+q-1, \cdots, ql+q-1, i(r+1, \cdots, k; n)} \right\}$ is bounded.

Now we are prepared to define the absolutely convergent series $\sum a_{(j)}$. Let

$$a_{(a+1)t, \cdots, (l+1)t, 0, \cdots, 0} = (t+1)^{-2}, \qquad t = 1, 2, \cdots;$$
$$a_{(i)} = 0 \qquad \text{otherwise.}$$

From the identity

$$R^C_{(i)} = \sum a_{(j)} R^B_{(i-j)} \qquad (0 \leqq j_s \leqq i_s;\ s = 1, \cdots, k)$$

we then obtain the relation

$$R^C_{ta+t-1, \cdots, tl+t-1, i(r+1, \cdots, k; n)} = \sum_{s=1}^{t} \frac{1}{(t-s+1)^2} R^B_{sa+s-1, \cdots, sl+s-1, i(r+1, \cdots, k; n)};$$

and this permits us to conclude, as in the case $k=2$, that series $\sum c_{(\iota)}$ does not converge.

Theorems 1 and 2 combine to give the following theorem.

THEOREM 3. *Let* $B=\sum b_{(\iota)}$ *be convergent. In order that the Cauchy-product series* $\sum c_{(\iota)}$ *converge to the sum* $C=AB$ *for every absolutely convergent series* $A=\sum a_{(\iota)}$, *it is necessary and sufficient that the set* $\{R^B_{(\iota)}\}$ *be bounded.*

We conclude with a remark on *σ-convergent series.*[1] The property of being σ-convergent carries with it the boundedness of the set of all *σ-sums* (which sums correspond to the rectangular sums for Pringsheim convergence), and following the method of Theorem 1 the "Mertens theorem" can be established:

THEOREM 4. *If* $\sum a_{(\iota)}$, $\sum b_{(\iota)}$ *are σ-convergent, one of them being absolutely convergent, then the Cauchy-product property holds: series* $\sum c_{(\iota)}$ is *σ-convergent to the sum* $C=AB$.

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[1] For the definition and some properties of σ-convergence we refer to *Convergence of multiply-infinite series*, Amer. Math. Monthly vol. 52 (1945) pp. 365–376.

NOTE ON ALMOST-ALGEBRAIC NUMBERS

HARVEY COHN[1]

1. **Introduction.** According to a theorem of J. Liouville,[2] if θ is an algebraic number of degree n, then any approximation by rationals, p/q, is of such a nature that

$$|\theta - p/q| \geq kq^{-n} \tag{1}$$

for a positive constant k. Liouville constructed his transcendental numbers as the limit of special sequences of rationals, p/q, which violated condition (1) regardless of the values of k and n, as $q \rightarrow \infty$. Thus Liouville constructed *almost-rational* numbers.

E. Maillet[3] likewise found a lower bound for $\theta-\alpha$ where now θ is approximated by the quadratic numbers, α. He then violated his lower bound by substituting for θ the value of an almost periodic simple continued fraction and for α a quadratic number, namely a periodic simple continued fraction that θ almost represented. Thus he constructed an *almost-quadratic* transcendental.

It is an elementary matter to find a lower bound for $\theta-\alpha$, where we now approximate θ by an algebraic number not necessarily rational or quadratic. We could then try several departures. We could, for example, try to construct almost-cubic or almost-biquadratic transcendentals.[4] On the other hand, we could use a *diagonal* method, that is, we could consider the limit of a rapidly converging sequence of algebraic numbers whose degree becomes indefinite. For example, a root of a power series with rational coefficients is the limit of a sequence of (algebraic) roots of the partial sums, and the speed of convergence is regulated by the remainder. If the remainder is too small we find that the root of our power series can be approximated too closely by algebraic numbers of varying degrees, namely the roots of

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[2] J. Liouville, *Sur des classes très étendues des quantités dont la valeur n'est ni algébrique ni même réductible à des irrationnelles algébriques*, J. Math. Pures Appl. vol. 16 (1851).

[3] E. Maillet, *Théorie des nombres transcendants et des propriétés arithmétiques des fonctions*, Paris, Gauthier-Villars, 1906, chap. 7.

[4] For instance, E. Maillet, op. cit., pp. 22, 100, considers certain "rapidly converging" power series with rational coefficients and algebraic values of the argument. The value of such a series is shown to be almost algebraic when regarded as the limit of the algebraic partial sums (which lie in the field generated by the argument and therefore are of no higher degree than the argument).

the partial sums. Thus the root of our power series must be transcendental. By this method we can obtain some transcendental numbers which seem to have previously escaped notice.

2. Approximation of a fixed algebraic number by an arbitrary algebraic number. We let θ be a *fixed* algebraic number of degree n whose denominator is t_0, that is, θ satisfies the equation with integral coefficients

$$t_0\theta^n + t_1\theta^{n-1} + \cdots + t_n = 0.$$

We consider α the root of an *arbitrary* polynomial with integral coefficients

$$\phi(x) = f_0x^m + f_1x^{m-1} + \cdots + f_m.$$

Denoting $\max|f_i|$ by $A[\phi(x)]$, we shall measure the closeness of α to θ by the smallness of $\phi(\theta)$ in terms of $A[\phi(x)]$.

Now setting $\bar{\phi}(x) = f_0x^m + f_1t_0x^{m-1} + \cdots + f_0t_0^m$, we find $\phi(\theta)$ can be written as $\bar{\phi}(t_0\theta)/t_0^m$, where $t_0\theta$ is an algebraic integer. Thus if we multiply $\phi(\theta)$ by its conjugates, we obtain

$$\prod \phi(\theta) = \left\{ \prod \bar{\phi}(t_0\theta) \right\} / t_0^{nm} = M[\phi(\theta)]/t_0^{nm}, \tag{2}$$

where $M[\phi(\theta)]$ is an integer. If $\phi(\theta) \neq 0$, then the same inequality holds for the conjugates of θ and thus

$$| M[\phi(\theta)] | \geqq 1. \tag{3}$$

For any conjugate of θ, we find easily

$$| \phi(\theta') | \leqq A[\phi(x)]T_1^m, \tag{4}$$

where T_1 is a constant determined by θ.

Hence if $\phi(\theta) \neq 0$, we obtain from (2), (3), and (4)

$$| \phi(\theta) | \geqq 1/\left\{ (A[\phi(x)])^{n-1} T_1^{m(n-1)} t_0^{mn} \right\}.$$

In short, *if θ is algebraic of degree n, then for an arbitrary polynomial $\phi(x)$ of degree m with integral coefficients no greater than $A[\phi(x)]$ in absolute value,*

$$| \phi(\theta) | \begin{cases} = 0 \text{ or} \\ \geqq 1/\left\{ (A[\phi(x)])^{n-1} T_2^m \right\}, \end{cases} \tag{5}$$

where T_2 depends only on θ. We should note in passing that if $\phi(x) = -p + qx$, then (5) is Liouville's theorem.

3. Construction of transcendentals. We shall construct a power series $\sigma(x) = a_0x^{e_0} + a_1x^{e_1} + a_2x^{e_2} + \cdots$ $(e_i \geqq 0)$, where a_i are rational fractions r_i/s_i $(\neq 0)$, and e_i are increasing integral exponents. For simplicity set $a_0 = 1$. Let $\sigma_k(x)$ be the kth partial sum (up to and including the exponent e_k). Then of the coefficients a_i in $\sigma_k(x)$ let g_k be the maximum $|a_i|$ $(g_k \geqq 1)$ and let d_k be the least common multiple of the denominators s_i in $\sigma_k(x)$. Then if as $k \to \infty$, e_{k+1} is of a higher order of infinity[5] than e_k, $\log g_k$, and $\log d_k$, we find that $\sigma(x)$ *is transcendental for all algebraic x within its circle of convergence*, except $x = 0$.

To see this, let x take the algebraic value θ of degree n within a circle of convergence of radius ρ. We suppose $E(\sigma)$ to be an arbitrary polynomial of degree k with integral coefficients of which the greatest has the absolute value $A[E(\sigma)]$. Then we wish to show $E(\sigma(\theta)) \neq 0$.

First we write $E(\sigma(x))$ as partial sum and remainder

$$E(\sigma(x)) = E(\sigma_k(x)) + R_k(x).$$

Now $\phi_k^k(x) = d_k E(\sigma_k(x))$ is a polynomial of degree ke_k with integral coefficients. Its maximum coefficient $A[\phi_k(x)]$ easily satisfies the condition

$$\begin{aligned} A[\phi_k(x)] &\leqq d_k^k \sum_{\kappa=0}^{k} A[E(\sigma)](k+1)^{\kappa} g_k^{\kappa} \\ &\leqq d_k^k (k+1) A[E(\sigma)](k+1)^k g_k^k \end{aligned}$$

since any coefficient in the polynomial $\phi_k(x)$ is less than the value this polynomial assumes when we replace each a_i by g_k and set $x = 1$.

Furthermore, since $E(\sigma)$ has only a limited number of roots, $E(\sigma_k(\theta))$ can not vanish for all k. For, when k changes to $k+1$, $\sigma_k(\theta)$ changes by $a_{k+1}\theta^{e_{k+1}}$, which approaches zero as k approaches infinity, although this term never vanishes, $\theta = 0$ having been excluded. Thus by (5), for an infinity of values of k,

$$\left| d_k^k E(\sigma_k(\theta)) \right| \geqq \left\{ d_k^k (k+1) A[E(\sigma)](k+1)^k g_k^k \right\}^{-n+1} T_2^{-ke_k}.$$

But since k can not exceed 2^{e_k} (or even e_k), we can simplify the last inequality and obtain

$$(6) \qquad \left| E(\sigma_k(\theta)) \right| \geqq d_k^{-nk} g_k^{-(n-1)k} T_3^{-e_k},$$

where T_3 depends only on θ and on $E(\sigma)$ as a polynomial in σ.

On the other hand $R_k(\theta) = E(\sigma(\theta)) - E(\sigma_k(\theta))$ and, by elementary considerations,

$$(7) \qquad \left| R_k(\theta) \right| \leqq T_4 \left| \sigma(\theta) - \sigma_k(\theta) \right| \leqq T_5 \left| (\theta/\rho)^{e_{k+1}}/(1 - \theta/\rho) \right|$$

where T_5 is determined by $\sigma(x)$ and $E(\sigma)$, independently of k.

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[5] If we have two quantities u and v each depending on k we say u is of a higher order of infinity than v, if $u/v \to \infty$ with k.

By our condition on the order of magnitude of e_{k+1}, it follows from (6) and (7) that for some k, $|E(\sigma(\theta))| \geqq |E(\sigma_k(\theta))| - |R_k(\theta)| > 0$; and $\sigma(\theta)$ can not satisfy the algebraic equation $E(\sigma) = 0$. Q.E.D.

A simple method of satisfying the conditions on $\sigma(x)$ is to take $\sigma_1(x) = 1 + x^{1!} + x^{2!} + \cdots + x^{n!} + \cdots$, $|x| < 1$. Then $\sigma_1(x)$ and x are not both algebraic unless $x = 0$, and indeed, the values of x $(\neq 0)$ for which $\sigma_1(x)$ is algebraic are some instances of the type of transcendentals under discussion.

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ON THE LOWER ORDER OF INTEGRAL FUNCTIONS

S. M. SHAH

Let $f(z)=\sum_0^\infty a_n z^n$ be an integral function of order ρ. It is known that[1]

$$(1)\quad \limsup_{n\to\infty}\frac{n\log n}{\log\{1/|a_n|\}}=\rho=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}\quad (0\leqq\rho\leqq\infty).$$

A similar result for the lower[2] order λ, namely

$$\liminf_{n\to\infty}\frac{n\log n}{\log\{1/|a_n|\}}=\lambda=\liminf_{r\to\infty}\frac{\log\log M(r)}{\log r},$$

does not always hold. In fact for

$$\exp(z^2)+\exp(z)=2+z+z^2\left(\frac{1}{1!}+\frac{1}{2!}\right)+\cdots,$$

$$\liminf_{n\to\infty}\frac{n\log n}{\log\{1/|a_n|\}}=1$$

whereas $\lambda=\rho=2$.

We prove here the following theorem.

THEOREM 1. *If* $f(z)=\sum_0^\infty a_n z^n$ *is an integral function of order* ρ *and lower order* λ $(0\leqq\lambda\leqq\infty)$ *then*

$$(2)\qquad \lambda\geqq\liminf_{n\to\infty}\frac{n\log n}{\log\{1/|a_n|\}}\geqq\liminf_{n\to\infty}\frac{\log n}{\log|a_n/a_{n+1}|}.$$

COROLLARY 1.[3]

$$(3)\quad \begin{aligned}&\liminf_{n\to\infty}\frac{\log|a_n/a_{n+1}|}{\log n}\leqq\liminf_{n\to\infty}\frac{\log\{1/|a_n|\}}{n\log n}=\frac{1}{\rho}\leqq\frac{1}{\lambda}\\&\leqq\limsup_{n\to\infty}\frac{\log\{1/|a_n|\}}{n\log n};\leqq\limsup_{n\to\infty}\frac{\log|a_n/a_{n+1}|}{\log n}.\end{aligned}$$

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[1] E. C. Titchmarsh, *Theory of functions*, pp. 253–254; E. T. Copson, *Theory of functions of a complex variable*, pp. 175–178.

[2] For the definition, and so on, see (i) J. M. Whittaker, *The lower order of integral functions*, J. London Math. Soc. vol. 8 (1933) pp. 20–27; (ii) S. M. Shah, *The lower order of the zeros of an integral function* (II), Proceedings of the Indian Academy of Sciences (A) vol. 21 (1945) pp. 162–164.

[3] Cf. a similar result (1) in S. M. Shah, *The maximum term of an entire series*, Mathematics Student vol. 10 (1942) pp. 80–82.

COROLLARY 2. *If* $\lim_{n\to\infty} n \log n/\log\{1/|a_n|\} = L$ *where* $0 \leqq L < \infty$ *then* $f(z) = \sum_0^\infty a_n z^n$ *is an integral function of regular growth*[4] *and of order* L.

THEOREM 2. *If* (i) $f(z) = \sum_0^\infty a_n z^n$ *is an integral function of order* ρ *and lower order* λ $(0 \leqq \lambda \leqq \infty)$ *such that* (ii) $|a_n/a_{n+1}|$ *is a nondecreasing function of* n *for* $n > n_0$, *then*

$$\lambda = \liminf_{n\to\infty} \frac{n \log n}{\log\{1/|a_n|\}} = \liminf_{n\to\infty} \frac{\log n}{\log|a_n/a_{n+1}|}, \tag{4}$$

$$\rho = \limsup_{n\to\infty} \frac{\log n}{\log|a_n/a_{n+1}|}. \tag{5}$$

We note that the hypothesis (ii) of Theorem 2 does not imply that $f(z)$ is of regular growth. In fact we have the following theorem.

THEOREM 3. *There exists an integral function* $f(z) = \sum_0^\infty a_n z^n$ *for which* (i) $a_n > 0$, (ii) a_n/a_{n+1} *is a steadily increasing function of* n, *and* (iii) $\rho > \lambda$.

An interesting application of these results can be made to the series $F(z) = \sum_0^\infty a_n \epsilon_n z^n$ where $\{\epsilon_n\}$ are a set of numbers such that $|\epsilon_n| = 1$ or 0 and such that $\sum_0^\infty a_n \epsilon_n z^n$ consists of an infinite number of terms. $F(z)$ is an integral function. Let its order be $\rho(F)$ and lower order be $\lambda(F)$. Since

$$M(r, f) \geqq |a_n| r^n \geqq |a_n \epsilon_n| r^n$$

for every n and r, and so if $\mu(r)$ denotes the maximum term, $M(r, f) \geqq \mu(r, F)$. Hence

$$\lambda(f) \geqq \lambda(F); \qquad \rho(f) \geqq \rho(F). \tag{6}$$

If $|a_n/a_{n+1}| = \psi(n)$ (say) is a nondecreasing function of n then

$$\lambda(f) = \liminf_{n\to\infty} \frac{n \log n}{\log\{1/|a_n|\}} \leqq \limsup_{n\to\infty} \frac{n \log n}{\log\{1/|a_n \epsilon_n|\}} = \rho(F) \tag{7}$$

and so we have the following theorem.

THEOREM 4. *If* $f(z) = \sum_0^\infty a_n z^n$ *is an integral function of order* ρ *and of lower order* λ *and is such that* $|a_n/a_{n+1}|$ *is a nondecreasing function of* n *for* $n > n_0$, *then* $F(z) = \sum_0^\infty a_n \epsilon_n z^n$ *is of order* $\rho(F) \geqq \lambda$.

For instance every function $F = \sum_0^\infty \epsilon_n z^n/n!$ is of order 1.

An example, to illustrate the point that by an appropriate choice of ϵ_n the order $\rho(F)$ of $F(z)=\sum a_n\epsilon_n z^n$ can be made equal to any number x where $\lambda(f)\leqq x\leqq\rho(f)$, is given in the proof of Theorem 3.

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[4] Cf. G. Valiron, *Lectures on the general theory of integral functions*, pp. 41–44.

The function $\exp z=\sum_0^\infty z^n/n!$ for which $\psi(n)$ is an increasing function of n is bounded on the real negative axis and the series

$$F(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

is bounded on the real axis. *If $\psi(n)$ is increasing sufficiently rapidly then we prove that $f(z)$ and $F(z)$ are not bounded on any line* $\arg z=\alpha$ $(0\leqq\alpha\leqq 2\pi)$. In fact we have the following theorem.

THEOREM 5. *If $f(z)=\sum_0^\infty a_n z^n$ is an integral function of lower order λ such that $|a_n/a_{n+1}|\geqq\vartheta^2|a_{n-1}/a_n|$ for $n>n_0$ then*

$$(8)\qquad \limsup_{r\to\infty}\frac{\log\log m(r,f)}{\log r}\geqq\lambda;\qquad \limsup_{r\to\infty}\frac{\log\log m(r,F)}{\log r}\geqq\lambda$$

where $m(r,f)=\min_{|z|=r}|f(z)|$ and $\vartheta=2\cdot 2$.

LEMMA. *a_n is any sequence of real or complex numbers such that*[6]

$$\text{(i)}\qquad |a_n|<1 \qquad \textit{for } n>n_0.$$

Let

$$\theta(n)=\frac{\log\{1/|a_n|\}}{n\log n};\qquad \phi(n)=\frac{\log|a_n/a_{n+1}|}{\log n};$$

$$\alpha=\liminf_{n\to\infty}\phi(n);\qquad \gamma=\liminf_{n\to\infty}\{1/\phi(n)\};$$

$$\beta=\limsup_{n\to\infty}\phi(n);\qquad \delta=\limsup_{n\to\infty}\{1/\phi(n)\};$$

$$A=\liminf_{n\to\infty}\theta(n);\qquad C=\liminf_{n\to\infty}\{1/\theta(n)\};$$

$$B=\limsup_{n\to\infty}\theta(n);\qquad D=\limsup_{n\to\infty}\{1/\theta(n)\};$$

then

$$(9)\qquad \alpha\leqq A=1/D;\qquad 1/C=B\leqq\beta;\qquad C\geqq\gamma.$$

(ii) *If*[6] *further $\psi(n)$ is a nondecreasing function of n for $n\geqq N$ and $\psi(N)\geqq 1$ then*

$$(10)\qquad C=\gamma=1/\beta;\qquad D=\delta=1/\alpha.$$

The proof of (9) is straightforward and omitted.

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[6] Some of the relations in (9) and (10) hold under less restrictive conditions.

PROOF OF (10). By hypothesis (ii), α, β, γ and δ are non-negative and $\beta = 1/\gamma$, $\alpha = 1/\delta$. We prove $B \geqq \beta$. Suppose first $0 < \beta < \infty$. Then

$$\psi(n) > n^{\beta-\epsilon} \qquad \text{for } n = N_1, N_2, \cdots, N_p, \cdots .$$

Let $N_1 > \max\{n_0, N\}$. Then

$$\left|\frac{1}{a_n}\right| = k(N_1)\psi(N_1+1)\cdots\psi(n-1),$$

$$\theta(n) = o(1) + \frac{\log\psi(N_1+1) + \cdots + \log\psi(n-1)}{n\log n}.$$

Let $n = [N_p \log^2 N_p] + 1$. Then

$$\theta(n) \geqq o(1) + \frac{(n-N_p)\log N_p^{\beta-\epsilon}}{n\log n}.$$

Hence $B \geqq \beta$ which holds also when $\beta = 0$. If β be infinite the above argument with an arbitrary large number instead of $\beta-\epsilon$ gives that $B = \infty$. Hence from (9) we get that $B = \beta$ and so $C = \gamma = 1/\beta$. The second relation in (10) follows similarly.

PROOF OF THEOREM 1. Since $\sum a_n$ is convergent, $|a_n| < 1$ for $n > n_0$. As $C \geqq \gamma$ we need prove $\lambda \geqq C$ only. Suppose first $0 < C < \infty$. Then

$$\frac{n\log n}{\log\{1/|a_n|\}} > C - \epsilon, \qquad \text{for all } n \geqq N(\epsilon).$$

$$|a_n| > n^{-n/(C-\epsilon)},$$

Let $r_n = 2n^{1/(C-\epsilon)}$. If $r_n \leqq r \leqq r_{n+1}$ $(n > N)$ then

$$M(r) \geqq |a_n| r^n \geqq |a_n| r_n^n > n^{-n/(C-\epsilon)} \exp(n\log r_n) = \exp(n\log 2).$$

Hence $\log M(r) \geqq \log 2\{(r/2)^{C-\epsilon} - 1\}$ for all large r and so $\lambda \geqq C$, which holds when $C = 0$. If $C = \infty$, the above argument shows that $\lambda = \infty$.

Corollary 1 follows from (1), (2) and (9), and Corollary 2 from (1) and (2). The example given at the beginning of the paper shows that $f(z)$ may be of regular growth and $\lim_{n\to\infty}\{n\log n/\log\{1/|a_n|\}\}$ may not exist.

PROOF OF THEOREM 2. Let $\mu(r)$ denote the maximum term, $\nu(r)$ its rank. By hypothesis (ii), $\psi(n) > \psi(n-1)$ for an infinity of n; for if otherwise $\psi(n) = \psi(n+1) = \cdots$ ad inf for $n > p$, say, and hence the radius of convergence of the series $\sum a_n z^n$ would be finite. $\psi(n)$ tends to infinity with n.

When $\psi(n) > \psi(n-1)$ the term $a_n z^n$ becomes a maximum term and we have $\mu(r)=|a_n|r^n$, $\nu(r)=n$ for $\psi(n-1)\leqq r<\psi(n)$. Now $\lambda=\liminf_{r\to\infty}\log \nu(r)/\log r$. Suppose first that $0<\lambda<\infty$. Then $\nu(r)>r^{\lambda-\epsilon}$ for $r>R=R(\epsilon)$. Let $|z|=r>R$ and let $a_{m_1}z^{m_1}$ and $a_{m_2}z^{m_2}$ $(m_1>n_0;\psi(m_1-1)>R)$ be two consecutive terms so that $m_1\leqq m_2-1$ and let $m_1<n\leqq m_2$. Since $a_{m_1}z^{m_1}$ is maximum term we have $\nu(r)=m_1$ for $\psi(m_1-1)\leqq r<\psi(m_1)$. Hence for every r in this interval $m_1=\nu(r)>r^{\lambda-\epsilon}$. In particular $m_1>\{\psi(m_1)-C\}^{\lambda-\epsilon}$ where $C=\min\{1,((\psi(m_1)-\psi(m_1-1))/2\}$. Further we have

$$\psi(m_1)=\psi(1+m_1)=\cdots=\psi(n-1).$$

Hence

$$\begin{aligned}\psi(n_0+1)\cdots\psi(n-1)=\left|\frac{a_{n_0+1}}{a_n}\right| &\leqq \{\psi(n-1)\}^{n-n_0-1}\\ &< \{C+m_1^{1/(\lambda-\epsilon)}\}^{n-n_0-1}\\ &< K(n_0)2^n n^{(n-n_0-1)/(\lambda-\epsilon)}.\end{aligned}$$

Hence for all large n

$$\left|\frac{1}{a_n}\right|<K_1(n_0)2^n\cdot n^{(n-n_0-1)/(\lambda-\epsilon)}$$

and so

$$C\geqq\lambda \tag{11}$$

which holds when $\lambda=0$. If $\lambda=\infty$ the above argument gives $C=\infty$. Hence from (2), $\lambda=C$ and so from (10) we get (4); and from (1) and (10) we have (5).

PROOF OF THEOREM 3. Let $n_1=2$, $n_{s+1}=n_s^4$ $(s=1,2,3,\cdots)$,

$$r_1=1,\qquad r_m=m \qquad\qquad \text{for } n_s\leqq m<n_s^2,$$

$$r_m=n_{s+1}-\frac{n_{s+1}-m}{\{(n_{s+1})!\}^{(n_{s+1})!}}\qquad \text{for } n_s^2\leqq m<n_{s+1},$$

$s=1,2,3,\cdots$, and let

$$f(z)=1+\sum_1^\infty\frac{z^n}{r_1r_2\cdots r_n}\cdot$$

Then $a_n>0$ and $a_n/a_{n+1}=r_{n+1}$ which is a steadily increasing function of n. Also

$$\theta(n)=\frac{\log r_1+\cdots+\log r_n}{n\log n}\cdot$$

Hence

$$\theta(n_{s+1}) \sim \frac{(n_s^4 - n_s^2)\log(n_s^4)}{4n_s^4 \log n_s} \sim 1,$$

$$\theta([n_s^2 \log n_s]) \sim \frac{(n_s^2 \log n_s - n_s^2)\log(n_s^4) + O(n_s^2 \log n_s)}{n_s^2 \log n_s^4 \log\{n_s^2 \log n_s\}} \sim 2.$$

It is easily seen that $\limsup_{n\to\infty}\theta(n) = 2$; $\liminf_{n\to\infty}\theta(n) = 1$. Hence $f(z)$ is an integral function of order 1 and lower order 1/2. Let now

$$\epsilon_m = \begin{cases} 1 & \text{when } m = [n_s^2 \log n_s] \qquad (s = 1, 2, 3, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$F(z) = \sum_1^\infty a_n\epsilon_n z^n = \sum_1^\infty \frac{\epsilon_n z^n}{r_1 r_2 \cdots r_n}$$

is an integral function of order 1/2. If

$$\epsilon_m = \begin{cases} 1 & \text{when } m = n_s \qquad (s = 1, 2, 3, \cdots) \\ 0 & \text{otherwise} \end{cases}$$

then $F(z)$ is of order 1. Let $1/2 < x < 1$ and $\epsilon_m = 1$ when $m = [\exp(4x \log n_s)]$ $(s = 1, 2, 3, \cdots)$ and zero otherwise; then $F(z)$ is of order x.

PROOF OF THEOREM 5. Let $|\epsilon_n| = 1$ for $n = N_1, N_2, \cdots N_p, \cdots$ $(N_1 > n_0)$. We write $N_p = N$. Let $R_N = \vartheta\psi(N-1)$ and $|z| = R_N = R$.

$$\mu(r, f) = |a_N| r^N = \mu(r, F) \qquad \text{for } \psi(N-1) \leqq r < \psi(N)$$

and R lies inside this interval.

$$\begin{aligned} |f(z)| &= \left|\sum_0^{N-1} a_n z^n + a_N z^N + \sum_{N+1}^\infty a_n z^n\right| \\ &\geqq \mu(R, f) - \left|\sum_0^{N-1} a_n z^n\right| - \left|\sum_{N+1}^\infty a_n z^n\right|. \end{aligned}$$

Now

$$\begin{aligned} \left|\sum_0^{N-1} a_n z^n\right| &\leqq |a_{N-1}| R^{N-1} + \cdots \\ &\leqq \mu(R)\left\{\frac{1}{\vartheta} + \frac{1}{\vartheta^4} + \frac{1}{\vartheta^9} + \cdots + \frac{1}{\vartheta^{(N-n_0-2)^2}} + o(1)\right\} \\ &\leqq \mu(R)\left\{\frac{1}{\vartheta} + \frac{1}{\vartheta^4} + \frac{1}{\vartheta^9} + \cdots \text{ ad inf}\right\} + \frac{\mu(R)}{10^{10}} \end{aligned}$$

for all large N.

$$\left|\sum_{N+1}^{\infty} a_n z^n\right| \leqq |a_{N+1}| R^{N+1} + \cdots$$

$$\leqq \mu(R)\left\{\frac{1}{\theta} + \frac{1}{\theta^4} + \frac{1}{\theta^9} + \cdots\right\}.$$

Hence for all large R

$$|f(z)| > \frac{\mu(R, f)}{10000}.$$

Similarly

$$|F(z)| > \frac{\mu(R, f)}{10000}.$$

Hence f and F are not bounded on any line $\arg z = \alpha$.

Since

$$\liminf_{r\to\infty} \frac{\log\log \mu(r, f)}{\log r} = \lambda$$

the theorem follows.

Added in proof. A short note containing a part of each of the Theorems 1, 2, and 3 appeared in J. Indian Math. Soc. vol. 9 (1945) pp. 50–54.

MUSLIM UNIVERSITY

ON THE SINGULARITIES OF A CLASS OF FUNCTIONS ON THE UNIT CIRCLE

S. M. SHAH

Pólya has suggested and Szegö and others have proved the following theorem.[1]

THEOREM. *Let $f(z)$ be a function regular in the whole plane including $z=\infty$ except at $z=1$. Let*

$$f(z) = \begin{cases} \sum a_n z^n, & |z| < 1, \\ \sum b_n / z^n, & |z| > 1. \end{cases}$$

If $a_n = O(n^k)$ and $b_n = O(n^k)$ then $f(z)$ is a rational function.

The above theorem is generalized in this paper as follows.

THEOREM 1. *Let $f(z)$ be regular in the whole plane including $z=\infty$, except possibly at a certain set S of points on $|z|=1$ (the set S being not everywhere dense on the complete circumference of the unit circle). Let*

$$f(z) = \begin{cases} \sum a_n z^n, & |z| < 1, \\ \sum b_n / z^n, & |z| > 1, \end{cases}$$

and let $a_n = O(n^k)$, $b_n = O(n^k)$; then the following results hold.

(i) *Every isolated singularity on $|z|=1$ will be a pole of order not exceeding $k+1$.*

(ii) *If there are only a finite number of singularities on $|z|=1$, then $f(z)$ is a rational function.*

THEOREM 2. *There exists a function satisfying the hypothesis of Theorem 1 and having an infinite number of singularities on the unit circle; also there exists a function satisfying the same hypothesis and having no isolated singularities.*

LEMMA 1. *Let $f(z)$ be an integral function and let*

$$I_p(r) = \int_0^{2\pi} |f(re^{i\phi})|^p d\phi \qquad \text{where } p > 0$$

be bounded on a sequence of circles $r = r_n$ tending to infinity, for some $p > 0$. Then $f(z)$ reduces to a constant.

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[1] J. Deutschen Math. Verein vol. 40 (1931) Aufgaben und Lösungen p. 81 (Polyá); ibid. vol. 43 (1934) Aufgaben und Lösungen pp. 13–16 (Szegö and others).

PROOF. $|f(z)|^p$ is subharmonic in any region of the z-plane. By Poisson's integral formula

$$|f(z)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,|f(Re^{i\phi})|^p d\phi}{R^2 + r^2 - 2rR\cos(\phi - \theta)}$$

where $|z| = r < R = r_n$. Hence

$$|f(z)|^p \leq \frac{1}{2\pi}\,\frac{R^2 - r^2}{(R - r)^2}\int_0^{2\pi} |f(Re^{i\phi})|^p d\phi \leq K\,\frac{R+r}{R-r}.$$

Putting $R = 2r$ we get

$$|f(z)|^p \leq 3K/2 \quad \text{on} \quad |z| = R/2.$$

Hence $f(z)$ is bounded on $|z| = r_n/2$ and so it reduces to a constant.

LEMMA 2. *Let $f(z)$ be regular for $|z| \geq H$ except probably at infinity; and let*

$$\int_0^{2\pi} |f(z)|^p d\phi \qquad (p > 0)$$

be bounded on a sequence of circles $|z|r = r_n$ tending to infinity. Then $f(z)$ is regular at infinity.

PROOF. We can write $f(z) = g(z) + h(z)$ where $g(z)$ is an integral function and $h(z)$ regular at infinity. Since $h(z)$ is bounded at infinity, it follows from Minkowski's inequality when $p>1$ (and still simpler when $p \leq 1$) that $\int_0^{2\pi}|g(z)|^p d\phi$ $(z = re^{i\phi})$ is bounded on a sequence of circles $r = r_n$ tending to infinity. Hence $g(z)$ is constant by Lemma 1 and so $f(z)$ is regular at infinity.

PROOF OF THEOREM 1. To prove that every isolated singularity will be a pole, it is enough to prove that if $z = 1$ is an isolated singularity, it is a pole since every other singularity can be brought to $z = 1$ by a rotation. We suppose that k is a positive integer. We have already supposed that $z = 1$ is an isolated singularity of $f(z)$. Let $x = (1+z)/(1-z)$. This transforms the unit circle in the z-plane into the imaginary axis in the x-plane and $z = 1$ corresponds to $x = \infty$. The function $\phi(x) = f(z)$ given by the above relation is therefore regular for $|x| \geq R_0$ (where R_0 is some number), except at $x = \infty$.

From our assumptions about the coefficients we obtain

$$|f(z)| \leq \frac{c}{|1 - |z||^{k+1}}$$

in the neighbourhood of the circle $|z| = 1$. Hence

$$|(1-z)^{k+1}f(z)| \leqq c\left|\frac{1-z}{1-|z|}\right|^{k+1}.$$

Let $\psi(x)=(1-z)^{k+1}f(z)$ where $z=(x-1)/(x+1)$. Then

$$|\psi(x)| \leqq \frac{2^{k+1}}{|x+1|^{k+1}}\,\frac{c}{|1-|(x-1)/(x+1)||^{k+1}}$$

$$\leqq \frac{c_1}{||x+1|-|x-1||^{k+1}}.$$

Let $x=\rho e^{i\gamma}$. Then $|x+1|^2-|x-1|^2=4\rho\cos\gamma$. Hence

$$|\psi(x)| \leqq \frac{c_1\{|x+1|+|x-1|\}^{k+1}}{\{||x+1|^2-|x-1|^2|\}^{k+1}}$$

$$\leqq c_1\left\{\frac{|x+1|+|x-1|}{4\rho|\cos\gamma|}\right\}^{k+1}$$

$$\leqq \frac{c_2}{|\cos\gamma|^{k+1}}$$

if $\rho \geqq \rho_0$ is sufficiently large. Hence

$$|\psi(x)|^{1/2(k+1)} \leqq \frac{c_3}{|\cos\gamma|^{1/2}}$$

except when $\gamma=\pm\pi/2$. Hence

$$\int_0^{2\pi}|\psi(x)|^{1/2(k+1)}d\gamma \leqq \int_0^{2\pi}\frac{c_3 d\gamma}{(|\cos\gamma|)^{1/2}}$$

which is a convergent integral. Hence

$$\int_0^{2\pi}|\psi(x)|^{1/2(k+1)}d\gamma$$

is bounded and so, by Lemma 2, $\psi(x)$ is regular at infinity. Hence

$$(1-z)^{k+1}f(z)$$

is regular at $z=1$, which shows that $f(z)$ has a pole of order not exceeding $k+1$ at $z=1$.

This proves (i). To prove (ii) it follows by part (i) that each of the finite number of singularities on $|z|=1$ is a pole. Hence $f(z)$ is a regular function throughout the z-plane including infinity, except for a finite number of poles. Hence $f(z)$ is a rational function.

PROOF OF THEOREM 2. Consider the function

$$f(z) = \sum_{1}^{\infty} \frac{1}{2^n} \frac{1}{(z - \alpha_n)}$$

where (α_n) is any sequence of points on $|z| = 1$. *If* the sequence (α_n) has only one limit point the above function $f(z)$ has a pole at each of these points α_n and an essential singularity at the limit point of the sequence (α_n). It is regular elsewhere. If

$$f(z) = \sum_{0}^{\infty} a_p z^p \qquad \text{for } |z| < 1$$

then

$$a_p = -\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\alpha_n^{p+1}}$$

and therefore $|a_p| \leq 1$. Similarly if $f(z) = \sum b_p/z^p$ $(|z| > 1)$ then $|b_p| \leq 1$. Hence a_p and b_p are certainly $O(n^k)$ for any $k \geqq 0$. To prove the second part, it is enough to take (α_n) in the above example to be everywhere dense on some arc of the unit circle, the arc not being the whole of the circumference. The function $f(z)$ will have a non-isolated essential singularity at every point of this arc and the coefficients a_n and b_n are bounded.

This example shows that the part (ii) of Theorem 1 is in a sense the best possible result, for the function constructed satisfies the conditions on the coefficients while it is not a rational function since it has an infinite number of singularities on the unit circle.

MUSLIM UNIVERSITY

ON A GENERALIZATION OF THE STIELTJES MOMENT PROBLEM

W. H. J. FUCHS

The "generalised moment problem"

$$\int_0^\infty t^{\lambda_n} d\alpha(t) = \mu_n \qquad (0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_n \to \infty) \tag{1}$$

is said to be determined if there is at most one increasing function $\alpha(t)$ satisfying (1) and normalized by $\alpha(0)=0$. R. P. Boas, Jr., who first considered this problem [1][1] gave conditions under which (1) is determined. These do not include the best known result in the classical case $\lambda_n = n$, namely Carleman's criterion: If $\lambda_n = n$ and $\sum \mu_n^{-1/2n} = \infty$, then (1) is determined. I shall now prove a theorem including Carleman's test as a special case. On the other hand this theorem will not include the results of Boas, as I shall from now on assume

$$\lambda_{n+1} - \lambda_n > c \qquad (n = 1, 2, \cdots) \tag{2}$$

for some $c>0$.

Let

$$\psi(r) = \exp\left\{ \sum_{0<\lambda_\nu \leq r} \lambda_\nu^{-1} \right\}.$$

THEOREM. *If there are a non-increasing function $\phi(r)$ and positive constants A and a such that*

$$\psi(r) > A(r/\phi(r))^a$$

and if

$$\sum_1^\infty \frac{\lambda_n - \lambda_{n-1}}{\mu_n^{1/a\lambda_n}\phi(\lambda_{n-1})} = \infty, \tag{3}$$

then (1) *is determined.*

The proof is based on the following lemma.

LEMMA. *If* (2) *is the case, then*

$$G(z) = \prod_{\nu=1}^\infty \frac{\lambda_\nu + z}{\lambda_\nu - z} e^{-2z/\lambda_\nu}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

is regular apart from poles at the λ_ν *and for some constant* B

$$|G(z)| < B^{x}(\psi(r))^{-x} \qquad (z = x + iy = re^{i\theta})$$

in $x \geqq 0$ *except in circles of radius* $c/3$ *round the* λ_ν.

This lemma is proved in [2].

Proof of the Theorem. We must prove that two increasing solutions, $\alpha_1(t)$ and $\alpha_2(t)$, of (1) can differ by a constant only. Consider

$$F(z) = \frac{1}{2}\int_0^\infty t^z d(\alpha_1 - \alpha_2).$$

$F(z)$ is regular in $\Re z = x > 0$ and

$$|F(z)| < \frac{1}{2}\int_0^\infty t^x d(\alpha_1 + \alpha_2) \leqq (v(x))^{ax},$$

say. Since $(\int_0^\infty t^x d(\alpha_1+\alpha_2)/\int_0^\infty d(\alpha_1+\alpha_2))^{1/x}$ is an increasing function of x, by Hölder's inequality, we may choose

$$v(x) = K\mu_n^{1/a\lambda_n} \qquad (\lambda_{n-1} < x \leqq \lambda_n). \tag{4}$$

Also $F(\lambda_n) = 0$ $(n = 1, 2, \cdots)$, but unless $\alpha_1(t) - \alpha_2(t) =$ const., $F(z)$ does not vanish identically. It is therefore sufficient to prove that $F(z)$ is identically zero.

If $G(z)$ is the function defined in the lemma, let

$$H(z)s^{-z} = F(z/a)G(z/a)z^z e^{-C(1+z)}(1 + z)^{-2}s^{-z}.$$

This function is regular in $\Re z > 0$. Also, if $z = x + iy = re^{i\theta}$

$$\begin{aligned} |F(z/a)G(z/a)| &\leqq (v(x/a)\phi(r/a)BA^{-1}ar^{-1})^x \\ &\leqq (v(x/a)\phi(x/a)BA^{-1}ar^{-1})^x, \end{aligned}$$

$$|z^z| = r^x e^{-r\theta\sin\theta} \leqq r^x e^{-\pi|y|/2+x},$$

since $\theta \sin \theta \geqq \pi|\sin\theta|/2 - \cos\theta$ for $|\theta| \leqq \pi/2$;

$$|s^{-z}| = |s|^{-x}e^{y\arg s}.$$

Therefore

$$|H(z)s^{-z}| < (v(x/a)\phi(x/a)|s|^{-1})^x e^{-(\pi/2-|\arg s|)|y|}(1 + r)^{-2}, \tag{5}$$

provided that C is taken sufficiently large. Consider now

$$g(s) = \int_{1-i\infty}^{1+i\infty} H(z)s^{-z}dz. \tag{6}$$

Because of (5) the integral is uniformly convergent for $|s| \geqq 1$, $|\arg s| \leqq \pi/2$. In particular $g(s)$ is a regular function of s in $|s| > 1$, $|\arg s| < \pi/2$. It also follows from (5) that the line of integration in (6) may be shifted to any other line $x = b > 0$. Taking $b = \xi$ and using (5) gives

$$|g(s)| < 2(v(\xi/a)\phi(\xi/a))^{\xi}|s|^{-\xi} \qquad (|\arg s| \leqq \pi/2) \tag{7}$$

for every $\xi > 0$.

By a theorem due to Carleman and Ostrowski (7) implies that $g(s)$ vanishes identically, if

$$\int_1^{\infty} (v(\xi/a)\phi(\xi/a))^{-1}d\xi = \infty \tag{8}$$

(see [3], in particular Satz IV and §14).

By (4)

$$\int_{a\lambda_{n-1}}^{a\lambda_n} (v(\xi/a)\phi(\xi/a))^{-1}d\xi \geqq a\frac{\lambda_n - \lambda_{n-1}}{\mu_n^{1/a\lambda_n}\phi(\lambda_{n-1})},$$

so that (3) implies (8). Therefore $g(s)$ vanishes identically. By a well known uniqueness theorem for the Mellin transform this implies that $H(z)$ is zero and so $F(z)$ must be equal to zero everywhere. Q.e.d.

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UNIVERSITY COLLEGE, SWANSEA

ON FRACTIONAL DERIVATIVES OF UNIVALENT FUNCTIONS[1]

H. P. ATKINS

It has been shown by F. Marty [6][2] that if $f(z)$ is analytic and univalent in the unit circle, $f(0)=0$, and $f'(0)=1$, and if the Bieberbach conjecture [2] that $|f^{(n)}(0)| \leq n \cdot n!$ is assumed when $n>3$, then

$$|f^{(n)}(z)| \leq n!(n+r)(1-r)^{-n-2}, \qquad n = 0, 1, 2, 3, \cdots, \tag{1}$$

where $|z|=r$, and that equality is attained for real positive z by the function $f(z)=z(1-z)^{-2}$. For $n=0$ and $n=1$ the inequality reduces to the well known relations obtained by Pick in evaluating the constant in the Verzerrungssatz of Koebe (see [2]).

The purpose of this paper is to generalize this relation to include fractional derivatives and integrals. The bound obtained will be expressed in terms of the ratio of the incomplete to the complete beta function, defined for p and q real and one or the other positive by the equations

$$I_r(p, q) = \frac{1}{B(p, q)} \int_0^r x^{p-1}(1-x)^{q-1}dx, \qquad p > 0,$$

$$I_r(p, q) = 1 - \frac{1}{B(p, q)} \int_r^1 x^{p-1}(1-x)^{q-1}dx, \qquad q > 0,$$

which are equivalent if both p and q are positive. Two separate definitions of the fractional derivative will be found useful; these may be shown equivalent for the values for which both are defined. For $a<0$ the Abel-Riemann definition [1, 9] is more easily applied:

$${}_0D_z^a f(z) = \frac{1}{\Gamma(-a)} \int_0^z f(w)(z-w)^{-a-1}dw, \qquad a < 0,$$

$${}_0D_z^a f(z) = \frac{d^m}{dz^m}\,{}_0D_z^{a-m} f(z), \qquad m - 1 \leq a < m,$$

where m is a positive integer. For $a \geq 0$ the Laurent definition [4] is more satisfactory:

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[1] The material of this paper forms part of a thesis, prepared under Professor W. Seidel, to be presented to the Graduate School of The University of Rochester for the degree Doctor of Philosophy.

[2] Numbers in brackets refer to the bibliography.

$$ {}_0D_z^a f(z) = \frac{\Gamma(a+1)}{2\pi i}\int_P f(w)(w-z)^{-a-1}\,dw, \qquad a \geqq 0, $$

where P is a path from 0 once around z in the positive sense and returning to 0. To avoid ambiguity in the Laurent definition we shall use that determination of the multiple valued quantity $(w-z)^{-a-1}$ obtained by letting

$$ (w-z)^{-a-1} = r^{-a-1}e^{-(t-\pi)(a+1)i}, $$

where $-\pi < t \leqq \pi$, at the start of the path P. The corresponding ambiguity in the Abel-Riemann definition will cause no difficulty. It should be noted that the fractional derivative may be undefined for $z=0$.

THEOREM. *If $f(z)$ is analytic and univalent in the unit circle with $f(0)=0$ and $f'(0)=1$, and if a is any real number and $n+1$ the smallest non-negative integer greater than a, then (without restriction if $a\leqq 3$, but assuming the Bieberbach conjecture if $a>3$)*[3]

$$ \left| {}_0D_z^a f(z) - \sum_{j=1}^{n} {}_0D_z^{a-j} f^{(j)}(0) \right| $$
$$ \leqq (1-r)^{-a-2}\Gamma(a+1)\left\{aI_r(n+1-a,\, a+2) + rI_r(n-a,\, a+2)\right\} $$

for $z\neq 0$. Equality is attained for real positive z by the function $f(z)=z(1-z)^{-2}$.

(The summation from 1 to n is to be interpreted as 0 if $n<1$, and ${}_0D_z^{a-j}f^{(j)}(0)$ means the derivative of order $a-j$ of the constant $f^{(j)}(0)$.)

PROOF. If $a<0$, the desired relation follows from the Abel-Riemann definition, for making the substitutions

$$ w = \frac{x-1+r}{x}e^{it}, \qquad z = re^{it} $$

and noting that, from equation (1) with $n=0$,

$$ \left| f\left(\frac{x-1+r}{x}e^{it}\right)\right| \leqq x^2(1-r)^{-2} - x(1-r)^{-1}, $$

we have

---

[3] Using the Littlewood result [5] that $|f^{(n)}(0)| < en\cdot n!$, the theorem may be changed so as to be valid without assuming the Bieberbach conjecture by inserting a factor e on the right side of the relation, making the relation a definite inequality, and deleting the last sentence.

$$|{}_0D_z^a f(z)| \leqq \frac{1}{\Gamma(-a)} \int_{1-r}^{1} (1-r)^{-a-2} x^a (1-x)^{-a-1} (x-1+r)\,dx$$
$$= (1-r)^{-a-2}\Gamma(a+1)\{(a+1)I_r(-a, a+2) - (1-r)I_r(-a, a+1)\}.$$

Using the reduction formula [8]

$$I_r(p, q) = rI_r(p-1, q) + (1-r)I_r(p, q-1),$$

which may easily be verified for all cases, we obtain the desired result for $n=-1$,

$$|{}_0D_z^a f(z)|$$
$$\leqq (1-r)^{-a-2}\Gamma(a+1)\{aI_r(-a, a+2) + rI_r(-1-a, a+2)\}.$$

If $a \geqq 0$ we use the Laurent definition of the fractional derivative. Let $b = a - n$, where $n = [a]$, and temporarily restrict b to be different from 0. Let $g(z) = f^{(n)}(z) - f^{(n)}(0)$, so that $g(z)$ is analytic and $g(0) = 0$.

Select c between 0 and r, and take the path of integration P to consist of a straight line from 0 to $(r-c+cr)e^{it}$, a circle with z as center traversed once in the positive sense, and a straight line back to 0.

On the circular portion of this path, putting $w = z + c(1-r)e^{i\theta}$ and integrating by parts with $e^{-ib\theta}d\theta$ as the differential,

$$\frac{\Gamma(b+1)}{2\pi i} \int g(w)(w-z)^{-b-1}dw$$
$$= \frac{1}{\Gamma(1-b)}(1-r)^{-b}c^{-b}e^{-bit}g([r-c+cr]e^{it}) + O(c^{1-b}),$$

where $c^{b-1}O(c^{1-b})$ is bounded, for fixed z, as c tends to 0.

On the two straight portions of the path P, taken together,

$$\frac{\Gamma(b+1)}{2\pi i} \int g(w)(w-z)^{-b-1}dw$$
$$= \frac{\Gamma(b+1)}{2\pi i} \int_0^{r-c+cr} g(|w|e^{it})(r-|w|)^{-b-1}e^{-(b+1)(t-\pi)i}e^{it}d|w|$$
$$+ \frac{\Gamma(b+1)}{2\pi i} \int_{r-c+cr}^{0} g(|w|e^{it})(r-|w|)^{-b-1}e^{-(b+1)(t+\pi)i}e^{it}d|w|$$
$$= \frac{1}{\Gamma(-b)} e^{-bit} \int_0^{r-c+cr} g(|w|e^{it})(r-|w|)^{-b-1}d|w|.$$

The substitution $|w| = (x-1+r)/x$ and integration by parts with $(1-x)^{-b-1}dx$ as the differential give

$$
\begin{aligned}
&-\frac{1}{\Gamma(1-b)}e^{-bit}(1-r)^{-b}(1+c)c^{-b}g([r-c+cr]e^{it})\\
&+\frac{1}{\Gamma(1-b)}e^{-bit}(1-r)^{-b}(b-1)\int_{1-r}^{1/(1+c)}x^{b-2}(1-x)^{-b}g\left(\frac{x-1+r}{x}e^{it}\right)dx\\
&+\frac{1}{\Gamma(1-b)}e^{(1-b)it}(1-r)^{1-b}\int_{1-r}^{1/(1+c)}x^{b-2}(1-x)^{-b}g'\left(\frac{x-1+r}{x}e^{it}\right)dx.
\end{aligned}
$$

Combining this with the value obtained on the curved portion of P, and allowing c to approach 0, we have

$$
\begin{aligned}
&{}_0D_z^b g(z)\\
&\quad=\frac{1}{\Gamma(1-b)}e^{-bit}(1-r)^{-b}(b-1)\int_{1-r}^{1}x^{b-2}(1-x)^{-b}g\left(\frac{x-1+r}{x}e^{it}\right)dx\\
&\qquad+\frac{1}{\Gamma(1-b)}e^{(1-b)it}(1-r)^{1-b}\int_{1-r}^{1}x^{b-2}(1-x)^{-b}g'\left(\frac{x-1+r}{x}e^{it}\right)dx.
\end{aligned}
$$

Integrating the first integral by parts, with $x^{b-2}(1-x)^{-b}dx$ as the differential, gives finally

$$
\begin{aligned}
&{}_0D_z^b g(z)\\
&\quad=\frac{1}{\Gamma(1-b)}e^{(1-b)it}(1-r)^{1-b}\int_{1-r}^{1}x^{b-2}(1-x)^{-b}g'\left(\frac{x-1+r}{x}e^{it}\right)dx.
\end{aligned}
$$

This formula may be verified directly for $b=0$ so the restriction on b may be removed. If we note that, by univalence of $f(z)$, formula (1) gives (see also [7])

$$
\begin{aligned}
&\left|g'\left(\frac{x-1+r}{x}e^{it}\right)\right|=\left|f^{(n+1)}\left(\frac{x-1+r}{x}e^{it}\right)\right|\\
&\qquad\qquad\leqq (n+1)!\{(n+2)x^{n+3}-(1-r)x^{n+2}\}(1-r)^{-n-3},
\end{aligned}
$$

the right side of the equation for ${}_0D_z^b g(z)$ may be reduced to the bound given in the theorem by a procedure exactly analogous to that for $a<0$. Thus it remains necessary to show only that

$$
\left|{}_0D_z^b g(z)\right|=\left|{}_0D_z^a f(z)-\sum_{j=1}^{n}{}_0D_z^{a-j}f^{(j)}(0)\right|.
$$

From

$$
{}_0D_z^b f^{(n)}(z)=\frac{\Gamma(b+1)}{2\pi i}\int_P f^{(n)}(w)(w-z)^{-b-1}dw
$$

we can obtain, using the proper determination of $(w-z)^{-b-1}$ and integration by parts with $f^{(n)}(w)dw$ as the differential,

$$ {}_0D_z^b f^{(n)}(z) = -\frac{1}{\Gamma(-b)} f^{(n-1)}(0) r^{-b-1} e^{-(b+1)it} + {}_0D_z^{b+1} f^{(n-1)}(z). $$

The process may be repeated to get

$$ {}_0D_z^b f^{(n)}(z) = {}_0D_z^a f(z) - \sum_{j=0}^{n-1} \frac{1}{\Gamma(1-a+j)} f^{(j)}(0) r^{j-a} e^{i(j-a)t}. $$

But

$$ {}_0D_z^{a-j} f^{(j)}(0) = \frac{1}{\Gamma(1-a+j)} f^{(j)}(0) r^{j-a} e^{i(j-a)t} $$

by direct evaluation, so

$$ {}_0D_z^b f^{(n)}(z) = {}_0D_z^a f(z) - \sum_{j=0}^{n-1} {}_0D_z^{a-j} f^{(j)}(0) $$

and

$$ \begin{aligned} {}_0D_z^b g(z) = {}_0D_z^b f^{(n)}(z) - {}_0D_z^b f^{(n)}(0) &= {}_0D_z^a f(z) - \sum_{j=0}^{n} {}_0D_z^{a-j} f^{(j)}(0) \\ &= {}_0D_z^a f(z) - \sum_{j=1}^{n} {}_0D_z^{a-j} f^{(j)}(0), \end{aligned} $$

since $f(0) = 0$.

Since every relation in the entire proof, except those limiting the range of a and b, becomes an equality for real positive z when $f(z) = z(1-z)^{-2}$, this completes the proof.

The result reduces to that of Marty for non-negative integral values of a.

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A GENERALIZATION OF A THEOREM OF LEROY AND LINDELÖF

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1. **Introduction.** Consider a Taylor series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ with radius of convergence unity. Let the coefficients a_n be the values taken on by a regular function $a(z)$ for $z=0, 1, \cdots$.

The object of this paper is to study the Taylor series under the assumption that $a(z)$ is regular in certain domains.[1] The results obtained are of the nature of domains in which the function defined by $\sum_{n=0}^{\infty}a_nz^n$ is regular and of domains which contain the singularities of the function defined by the series. In terms of $a(z)$ fairly general sufficient conditions are given such that the circle of convergence is not a cut for the function.

The results may be regarded as a generalization of a theorem due to LeRoy and Lindelöf.[2]

THEOREM (LEROY AND LINDELÖF). *Suppose* (a) $a(x+iy)$ *is regular in the semiplane* $x\geqq\alpha$, (b) *there is a* $\theta<\pi$ *such that for every arbitrary small positive* ϵ *and for sufficiently large* ρ

$$\left| a(\alpha+\rho\exp(i\psi))\right| < \exp\left[\rho(\theta+\epsilon)\right], \qquad -\pi/2\leqq\psi\leqq\pi/2.$$

Then

$$f(z)=\sum_{n=0}^{\infty}a(n)z^n, \qquad z=r\exp(i\phi)$$

is regular in the angle

$$\theta<\phi<2\pi-\theta.$$

The generalization of this theorem that we prove consists, under suitable restrictions, in replacing the semiplane $x\geqq\alpha$ by an angular opening including the axis of positive reals in its interior.

The singularities of the function $f(z)$ studied in this paper are those of a "principal branch" obtained by immediate continuation of the series.

Consider an angular opening with vertex on the positive real axis which includes the axis of reals in its interior. Suppose $a(z)$ has no singularities in this angular opening with the possible exception of

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[1] By the term domain we mean an open connected set.

[2] See Dienes [1]. Numbers in brackets refer to the bibliography at the end of the paper.

the point at infinity. Let the sides of the angular opening make angles ψ_1 and ψ_2 with the axis of reals.

Our problem is to characterize the behavior of the function $f(z)$ in terms of the magnitudes of the angles ψ_1 and ψ_2 and the type of singularity $a(z)$ has at infinity.

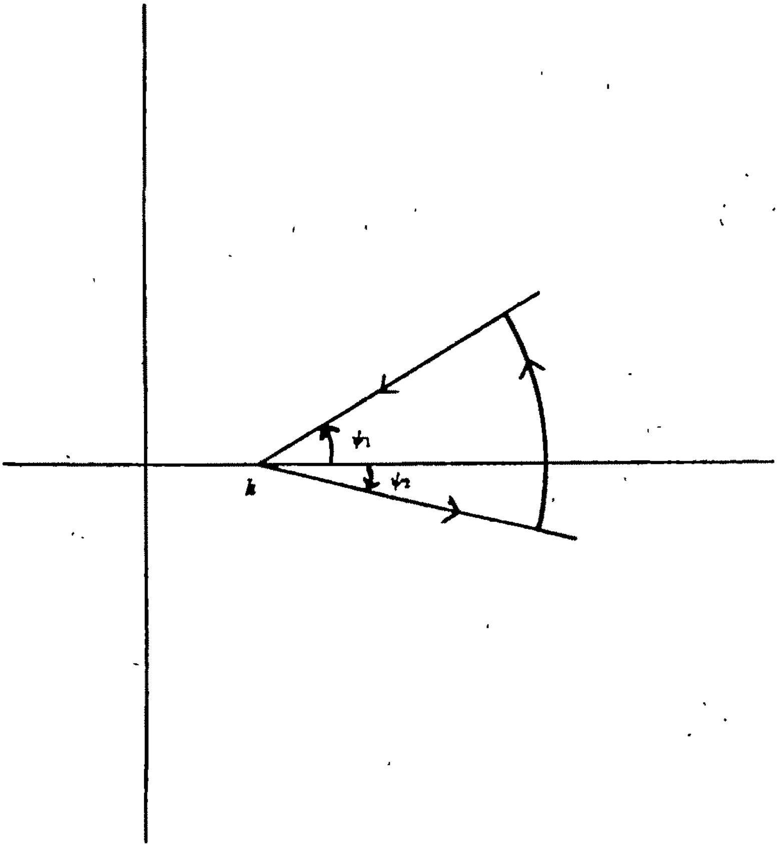

FIG. 1

We consider first the case where $a(z)$ may have a pole of order K at infinity and then the case in which infinity may be an essential singularity for $a(z)$.

2. **$a(z)$ may have a pole at infinity.** Suppose now that $a(z)$ is regular interior to and on the sides of the angular opening shown in Figure 1, except for at most a pole of order K at infinity. In Figure 1 let $l-1<h<l$ where l is a positive integer.

By the calculus of residues if $F(\omega)$ and $G(\omega)$ are uniform functions in a domain and if $G(\omega)$ has only simple zeros α_i in this domain, then (integration being understood in the positive sense)

$$\int_C \frac{F(\omega)}{G(\omega)}\,d\omega = \frac{F(\alpha_i)}{G'(\alpha_i)} \tag{1}$$

where C is a path enclosing α_i but no other zero of $G(\omega)$ and $G'(\alpha_i)$ is the derivative of $G(\omega)$ evaluated at $\omega=\alpha_i$.

Let $F(\omega)=a(\omega)z^\omega$ and $G(\omega)=\exp(2\pi i\omega)-1$.

For a given value of $z=r\exp(i\theta)$ we shall understand by z^ω

$$z^\omega = \exp[\omega(\log r + i\theta)], \qquad 0 \leqq \theta < 2\pi.$$

Here $\exp z$ is the principal value of e^z. This convention will be adhered to throughout the paper.

Then by (1)

$$\frac{1}{2\pi i}\int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega = a(n)z^n \tag{2}$$

where C is a path enclosing n and no other real integer. $a(\omega)$ is a function regular in the angular opening. This choice of $F(\omega)$ and $G(\omega)$ led to the well known theorem of LeRoy and Lindelöf [1] and is of course a well known method for the summation of certain series. The analysis of this paper follows lines similar to those in the analysis of Lindelöf.

Consider

$$\int_{C_{k,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega$$

where $C_{k,R}$ is the path formed by the two sides of the angular opening, with vertex $k>0$, and the arc of a circle of radius R, where R is a positive integer. This is indicated in Figure 1.

By application of (2) it follows that

$$\frac{1}{2\pi i}\int_{C_{k,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega = \sum_{n=l}^{R+l-1} a(n)z^n. \tag{3}$$

Denote the sides of the angular opening corresponding to ψ_1 and ψ_2 by I_1 and I_2 and the arc of a circle by C. Then

$$\begin{aligned}\int_{C_{k,R}} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega &= \int_{I_2} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega \\ &\quad + \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega + \int_{I_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega.\end{aligned} \tag{4}$$

It will now be shown that, if we place certain restrictions on ψ_1, ψ_2 and z,

$$\int_C \frac{a(\omega)z^{\omega}}{\exp(2\pi i\omega) - 1}\,d\omega$$

converges uniformly to zero as R becomes infinite.

Let $z = r\exp(i\pi)$ and $\omega = k + R\exp(i\psi)$ with $-\pi < v \leq \psi \leq u < \pi$. By hypothesis there exists an R_0 such that in the angular opening with vertex $k > 0$

$$(5) \qquad |a(k + R\exp(i\psi))| < AR^K, \qquad R > R_0,$$

where A is a constant and K is a positive integer. On the arc C

$$(6) \qquad \left|\frac{1}{\exp(2\pi i\omega) - 1}\right| < B,$$

where B is a constant. This follows from the fact that ω is bounded away from an integer by our choice of k, l and R.

Clearly

$$\left|\frac{1}{\exp(2\pi i\omega) - 1}\right| = \left|\frac{1}{1 - \exp(-2\pi i\omega)}\right| \left|\exp(-2\pi i\omega)\right|.$$

Hence on C,

$$(7) \qquad \left|\frac{1}{\exp(2\pi i\omega) - 1}\right| < D\exp(2\pi R\sin\psi),$$

where D is a constant. This inequality proves useful for $-\pi < v \leq \psi < 0$. Now

$$(8) \qquad \left|\int_C \frac{a(\omega)z^{\omega}}{\exp(2\pi i\omega) - 1}\,d\omega\right| \leq \int_C \frac{|a(\omega)|\,|z^{\omega}|}{|\exp(2\pi i\omega) - 1|}\,|d\omega|,$$

and

$$\begin{aligned} |z^{\omega}| &= |\exp[(k + R\cos\psi + iR\sin\psi)(\log r + i\pi)]| \\ &= r^k\exp[-R(\log r^{-1}\cos\psi + \pi\sin\psi)]. \end{aligned}$$

If $r < 1$, then

$$(9) \qquad b(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi > 0, \qquad 0 \leq \psi \leq \pi/2.$$

If $r > \exp(\pi\tan\psi)$, then

$$(10) \qquad b_1(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi > 0, \qquad \pi/2 \leq \psi \leq u < \pi.$$

Clearly $r > \exp(\pi\tan u)$ is sufficient for (10) to hold. We note here and also in (11) below, that the case $|\psi| = \pi/2$ is included. If $r<1$ then

$$(11) \qquad b_2(r,\psi) = \log r^{-1}\cos\psi - \pi\sin\psi > 0, \qquad -\pi/2 \leqq \psi \leqq 0.$$

If $r > \exp(-\pi\tan\psi)$, then

$$(12) \quad b_3(r,\psi) = \log r^{-1}\cos\psi - \pi\sin\psi > 0, \quad -\pi < v \leqq \psi \leqq -\pi/2.$$

Clearly $r > \exp(-\pi\tan v)$ is sufficient for (12) to hold.

We have seen that if $-\pi/2 \leqq \psi \leqq \pi/2$, $r<1$ implies $b(r,\psi)>0$ and $b_1(r,\psi)>0$. Denote, for $\alpha>0$ but otherwise arbitrarily small, by r_1 the larger of $\exp(\pi\tan u)+\alpha$ and $\exp(-\pi\tan v)+\alpha$. Set $r_2 = 1-\alpha'$, $\alpha'>0$ but otherwise arbitrarily small. Then $r_1 \leqq r \leqq r_2$ implies $\pi/2 < u < \pi$ and $-\pi < v < -\pi/2$. We choose α and α' sufficiently small, so that $r_1 \leqq r \leqq r_2$ is an interval consisting of more than one point. Then, for a given α and α' and $r_1 \leqq r \leqq r_2$, (9), (10), (11), and (12) take on their minimum values on their respective intervals of definition. Let these be $b'>0$, $b_1'>0$, $b_2'>0$ and $b_3'>0$. Denote by $\beta>0$ the smallest of these four values. Then by application of (5), (6) and (7) for $R>R_0$, (8) becomes

$$(13) \quad \begin{aligned} \left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega \right| &\leqq T_0(A,B,D)R\int_v^u \exp(-\beta R)\,d\psi \\ &= T_0(A,B,D)R\exp(-\beta R)(u-v). \end{aligned}$$

Given an arbitrary $\epsilon>0$, there exists an R_1 such that, for all $R>R_0$, R_1 and $r_1 \leqq r \leqq r_2$, the right-hand member of (13) is less than ϵ. That is

$$(14) \qquad \left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega \right| < \epsilon, \qquad R > R_0,\ R_1;\ r_1 \leqq r \leqq r_2.$$

By (14) it follows that the integral along the arc converges uniformly to zero as R becomes infinite. Hence by (3) and (4) we may write

$$(15) \quad \sum_{n=l}^{\infty} a(n)z^n = \int_{l_2} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega + \int_{l_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega.$$

Let

$$J_{\psi_1}(z) = \int_{l_1} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega$$

and

$$J_{\psi_2}(z) = \int_{l_2} \frac{a(\omega)z^\omega}{\exp(2\pi i\omega)-1}\,d\omega.$$

It will be shown that $J_{\psi_1}(z)+J_{\psi_2}(z)$ is regular for z in a domain which contains the interval $-r_2 \leqq z \leqq -r_1$ in its interior.

In this discussion four cases with regard to the angles ψ_1 and ψ_2 present themselves naturally.

A: $-\pi/2 < \psi_2 < 0, \quad 0 < \psi_1 < \pi/2$

B: $-\pi/2 < \psi_2 < 0, \quad \psi_1 = \pi/2$

C: $\psi_2 = -\pi/2, \quad \psi_1 = \pi/2$

D: $-\pi < v \leqq \psi_2 \leqq -\pi/2, \quad \pi/2 < \psi_1 \leqq v < \pi.$

Consider case A. Here

$$J_{\psi_1}(z) = \int_{I_1} \frac{a(\omega) z^{\omega}}{\exp(2\pi i \omega) - 1} d\omega$$
$$= -\int_0^{\infty} \frac{a(\omega)\exp[(k + R\exp(i\psi_1))(\log r + i\theta)]}{\exp(2\pi i\omega) - 1} \exp(i\psi_1) dR.$$

From (5) and (6) it follows that

$$(16) \qquad |J_{\psi_1}(z)| < ABr^k \int_0^{\infty} R^K \exp[-R(\log r^{-1}\cos\psi_1 + \theta\sin\psi_1)]dR.$$

If

$$(17) \qquad 0 < r < \exp(\theta\tan\psi_1), \qquad 0 \leqq \theta < 2\pi,$$

then $b(r, \theta, \psi_1) = \log r^{-1}\cos\psi_1 + \theta\sin\psi_1 > 0$.

Suppose $z = r\exp(i\theta)$ is in some closed domain[3] contained in the domain defined by (17). Denote by $b'(\psi_1) > 0$ the minimum assumed by $b(r, \theta, \psi_1)$ in this closed domain. Then (16) becomes

$$|J_{\psi_1}(z)| < ABr^k \int_0^{\infty} R^K \exp[-b'(\psi_1)R]dR < M,$$

where M is a constant.

Therefore $J_{\psi_1}(z)$ converges uniformly for z in any closed domain contained in (17). For ω on I_1 and z contained in the domain defined by (17) the integrand of $J_{\psi_1}(z)$ is continuous in ω and z. It follows from our definition of z^{ω} that for a fixed ω on I_1 the integrand is a regular function of z for z in any closed domain contained in (17). Then by well known theorems [2], $J_{\psi_1}(z)$ is regular for z in any closed domain contained in (17).

Consider

---

[3] D is said to be a closed domain if there exists a domain E with the property that $D = \bar{E}$. Here if E is a given set and E' its derived set then $\bar{E} = E + E'$.

$$
\begin{aligned}
J_{\psi_2}(z) &= \int_{l_2} \frac{a(\omega)z^{\omega}}{\exp(2\pi i\omega) - 1}\,d\omega \\
&= \int_0^{\infty} \frac{a(\omega)\exp\left[(k + R\exp(i\psi_2))(\log r + i\theta)\right]}{\exp(2\pi i\omega) - 1}\exp(i\psi_2)\,dR.
\end{aligned}
$$

By (5) and (7) we may write

$$
\begin{aligned}
&|J_{\psi_2}(z)| \\
&\quad < ADr^k \int_0^{\infty} R^K \exp\left[-R\log r^{-1}\cos\psi_2 + (\theta - 2\pi)\sin\psi_2\right]dR.
\end{aligned} \tag{18}
$$

If

$$
0 < r < \exp\left[(\theta - 2\pi)\tan\psi_2\right] \tag{19}
$$

then $b(r, \theta, \psi_2) = \log r^{-1}\cos\psi_2 + (\theta - 2\pi)\sin\psi_2 > 0$.

Let $z = r\exp(i\theta)$ be in any closed domain contained in the domain defined by (19). Denote by $b'(\psi_2) > 0$ the minimum taken on by $b(r, \theta, \psi_2)$ in this closed domain; then (18) becomes

$$
|J_{\psi_2}(z)| < ADr^k \int_0^{\infty} R^K \exp\left[-b'(\psi_2)R\right]dR < N,
$$

where N is a constant.

It then follows by the same analysis employed in the case of $J_{\psi_1}(z)$ that $J_{\psi_2}(z)$ is regular in any closed domain contained in the domain defined by (19). The function $J_{\psi_1}(z) + J_{\psi_2}(z)$ will therefore be regular for $z = r\exp(i\theta)$ in any closed domain contained in the domain common to (17) and (19).

It has been shown (15) that

$$
\sum_{n=l}^{\infty} a(n)z^n = J_{\psi_1}(z) + J_{\psi_2}(z)
$$

where $z = r\exp(i\pi)$ and $r_1 \leqq r \leqq r_2$. But $J_{\psi_1}(z) + J_{\psi_2}(z)$ has been shown to be regular in a domain which includes the interval $-r_2 \leqq z \leqq -r_1$ in its interior. Hence $J_{\psi_1}(z) + J_{\psi_2}(z)$ provides the analytic continuation of the function defined by $\sum_{n=l}^{\infty} a(n)z^n$ to any closed domain contained in the domain common to (17) and (19). Since

$$
f(z) = \sum_{n=0}^{l-1} a(n)z^n + \sum_{n=l}^{\infty} a(n)z^n,
$$

it is evident that $J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n)z^n$ provides the analytic continuation of $f(z)$ to the same domain. Hence in this case $f(z)$ will in general be regular in a domain bounded by two spirals as indicated in Figure 2.

Consider case B. By an analysis similar to that given in case A it is easily shown that $J_{\psi_1}(z)+J_{\psi_2}(z)$ is regular in any closed domain contained in the domain common to

$$0 < r < \exp\,[(\theta - 2\pi)\tan\psi_2], \qquad 0 \leq \theta < 2\pi, \tag{20}$$

and the whole complex plane excluding the segment 1 to $+\infty$. By an argument similar to that employed in case A it is easily seen that $f(z)$ is regular in any closed domain contained in the domain common to (20) and the whole complex plane excluding the segment 1 to $+\infty$.

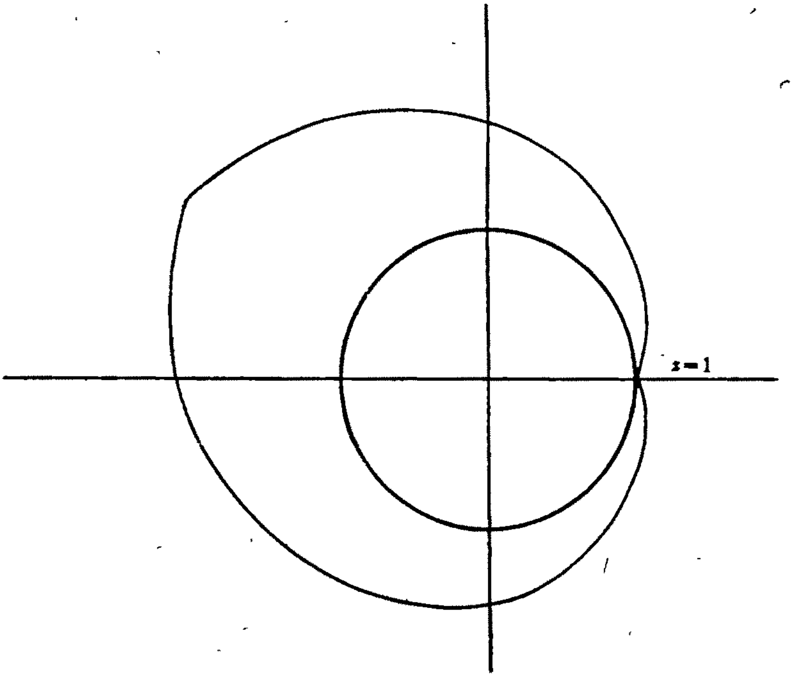

Fig. 2

Case C. Here we have a very special case of the theorem of LeRoy and Lindelöf [1]. It could be shown by an analysis similar to the preceding that $f(z)$ is regular in any domain of the complex plane excluding the segment 1 to $+\infty$.

Finally we consider case D. By a method similar to that employed in case A it is simple to show that $J_{\psi_1}(z)$ converges uniformly for z in any closed bounded domain contained in the domain defined by

$$r > \exp(\theta \tan \psi_1), \qquad 0 \leqq \theta < 2\pi, \tag{21}$$

and that $J_{\psi_2}(z)$ converges uniformly for z in any closed bounded domain contained in the domain defined by

$$r > \exp[(\theta - 2\pi) \tan \psi_2], \qquad 0 \leqq \theta < 2\pi. \tag{22}$$

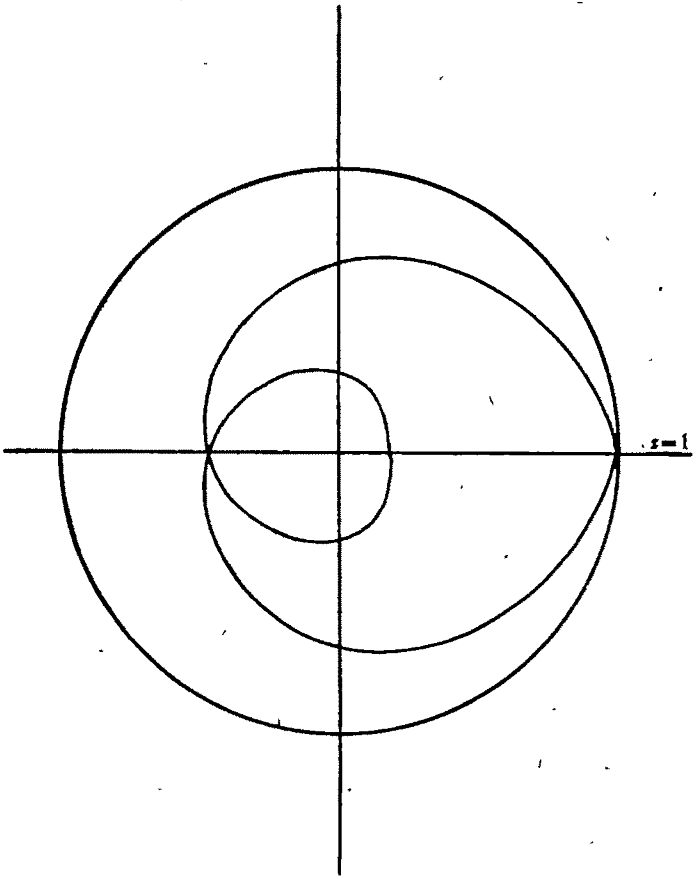

FIG. 3

Now the domain common to (21) and (22) contains the segment $-r_2 \leqq z \leqq -r_1$ in its interior. This is easily seen by setting $\theta = \pi$ in (21) and (22) and noting that r_1 is the larger of $\exp(\pi \tan u) + \alpha$ and $\exp(\pi \tan v) + \alpha$, $\alpha > 0$. Hence by arguments similar to those used in case A

$$J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{k-1} a(n) z^n$$

provides the analytic continuation of $f(z)$ to any closed bounded domain contained in the domain common to (21) and (22). For the case $\psi_1=\psi_2$ the spirals (21) and (22) are indicated in Figure 3. In calculating $J_{\psi_1}(z)$ and $J_{\psi_2}(z)$ for a given $z=r\exp(i\theta)$ we recall that by our convention

$$z^{\omega}=\exp\left[\omega(\log r+i\theta)\right], \qquad 0\leqq\theta<2\pi.$$

Hence we have the following theorem.

THEOREM 1. *Let* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *with radius of convergence unity. Let the coefficients* a_n *be the values taken on by an analytic function* $a(z)$ *at* $z=0, 1, 2, \cdots$. *Suppose* $a(z)$ *is regular with the possible exception of a pole of order* K *at infinity, in an angle with vertex* $h>0$ *(non-integral) on the axis of reals and including the axis of positive reals in its interior. Let the sides of this angle make angles* ψ_1 *and* ψ_2 *with the axis of reals. Then, if* $\gamma>0$ *but otherwise arbitrarily small, and*

A: $$0<\psi_1<\pi/2, \qquad -\pi/2<\psi_2<0,$$

$f(z)$ *is regular in the domain common to*

$$r\leqq\exp\left[\theta\tan\psi_1\right]-\gamma, \qquad 0\leqq\theta<2\pi,$$

and

$$r\leqq\exp\left[(\theta-2\pi)\tan\psi_2\right]-\gamma, \qquad 0\leqq\theta<2\pi.$$

If

B: $$\psi_1=\pi/2, \qquad -\pi/2<\psi_2<0,$$

$f(z)$ *is regular in any closed domain common to*

$$r\leqq\exp\left[(\theta-2\pi)\tan\psi_2\right]-\gamma, \qquad 0\leqq\theta<2\pi,$$

and the whole complex plane excluding the segment 1 *to* $+\infty$. *If*

C: $$\psi_1=\pi/2, \qquad \psi_2=-\pi/2,$$

$f(z)$ *is regular in any closed domain of the complex plane excluding the segment* 1 *to* $+\infty$. *If*

D: $$\pi/2<\psi_1\leqq u<\pi, \qquad -\pi<v\leqq\psi_2<-\pi/2,$$

$f(z)$ *is regular in any bounded domain common to*

$$r\geqq\exp\left(\theta\tan\psi_1\right)+\gamma, \qquad 0\leqq\theta<2\pi,$$

and

$$r \geqq \exp\left[(\theta - 2\pi)\tan\psi_2\right] + \gamma, \qquad 0 \leqq \theta < 2\pi.$$

It is clear[4] that the theorem above is still true even though $a(z)$ does not have a pole at infinity. All that is necessary is that $a(z)$ be single-valued in the angle and that $|a(h+R\exp(i\psi))| < AR^K$, $R > R_0$.

3. **$a(z)$ may have an essential singularity at infinity.** Suppose $a(z)$ is regular interior to and on the sides of the angular opening of Figure 1. Suppose there exists an R_0 such that for $R > R_0$ and $z = h + R\exp(i\psi)$ in this angle

$$|a(h + R\exp(i\psi))| < \exp(\delta R). \tag{23}$$

In order to simplify the work to follow suppose $\delta \leqq \pi - d$, $d > 0$.

It will be shown that, if we place certain restrictions on ψ_1, ψ_2 and z,

$$\int_C \frac{a(\omega)z^{\omega}}{\exp(2\pi i\omega) - 1}\, d\omega \tag{24}$$

converges uniformly to zero as R becomes infinite. Let $z = r\exp(i\pi)$ and $\omega = h + R\exp(i\psi)$ with $-\pi/2 \leqq \psi \leqq \pi/2$.

If $r < \exp[\pi\tan\psi - \delta\sec\psi]$, then

$$b_1(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi - \delta > 0. \tag{25}$$

If $r < \exp[-\pi\tan\psi - \delta\sec\psi]$, then

$$b_2(r, \psi) = \log r^{-1}\cos\psi - \pi\sin\psi - \delta > 0. \tag{26}$$

Set $g(\psi) = \pi\tan\psi - \delta\sec\psi$ where $0 \leqq \psi \leqq \pi/2$ and $\delta \leqq \pi - d$, $d > 0$. Now $g'(\psi) = \sec\psi(\pi\sec\psi - \delta\tan\psi)$. Hence for $\delta \leqq \pi - d$ and $0 \leqq \psi < \pi/2$, $g'(\psi)$ is positive.

As ψ approaches $\pi/2$, $g(\psi)$ approaches $+\infty$. Hence the minimum value of $g(\psi)$ on the interval $0 \leqq \psi \leqq \pi/2$ is $g(0)$, that is $-\delta$. In a similar manner we see that the minimum of $-\pi\tan\psi - \delta\sec\psi$ on the interval $-\pi/2 \leqq \psi \leqq 0$ is $-\delta$.

Hence if $r < \exp[-\delta - p]$ where $p > 0$ but otherwise arbitrarily small, (25) holds for $0 \leqq \psi \leqq \pi/2$ and (26) holds for $-\pi/2 \leqq \psi \leqq 0$.

Denote by $b_1' > 0$ the minimum assumed by $b_1(r, \psi)$ on the interval $0 \leqq \psi \leqq \pi/2$ and by $b_2' > 0$ the minimum assumed by $b_2(r, \psi)$ on the interval $-\pi/2 \leqq \psi \leqq 0$ where $0 \leqq r \leqq \exp[-\delta - p]$, $p > 0$. Let $b_0 > 0$ be the smaller of $b_1' > 0$ and $b_2' > 0$.

Then from (6), (7) and (23) we have

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[4] The author is indebted to the referee for this observation.

$$(27)\qquad \left|\int_C \frac{a(\omega)z^\omega d\omega}{\exp(2\pi i\omega)-1}\right| \leqq Rr^kT_1(B, D)\int_{-\pi/2}^{\pi/2}\exp\left[-b_0R\right]d\psi$$
$$= Rr^kT_1(B, D)\exp\left[-b_0R\right]\pi.$$

Given an $\epsilon>0$ we can choose an R_1 such that for $R>R_0$, R_1 and $0\leqq r\leqq\exp\left[-\delta-p\right]$, $p>0$, the quantity on the right of (27) is less than ϵ. That is, (24) converges uniformly to zero as R becomes infinite.

Let $\pi/2\leqq\psi\leqq u<\pi$. If $r>\exp\left[\pi\tan\psi-\delta\sec\psi\right]$ then

$$(28)\qquad b_1(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi - \delta > 0.$$

Let $-\pi<v\leqq\psi\leqq-\pi/2$. If $r>\exp\left[-\pi\tan\psi-\delta\sec\psi\right]$ then

$$(29)\qquad b_2(r, \psi) = \log r^{-1}\cos\psi + \pi\sin\psi - \delta > 0.$$

Denote, for a given $q>0$ but otherwise arbitrarily small, by F the smaller of the numbers $\pi\sin u-q$ and $\pi\sin v-q$. Let $\delta\leqq F$. The maximum for $\pi/2\leqq\psi\leqq u<\pi$ of $\pi\tan\psi-\delta\sec\psi$ is $\pi\tan\psi-F\sec\psi$. The maximum for $-\pi<v\leqq\psi\leqq-\pi/2$ of $-\pi\tan\psi-\delta\sec\psi$ is $\pi\tan v-F\sec v$. Denote by E the larger of $\pi\tan u-F\sec u+t$ and $\pi\tan v-F\sec v+t$ where $t>0$ but sufficiently small that $E<0$. If $r>\exp(E)$ then (28) and (29) hold. Suppose now in addition that u and v have the property that

$$(30)\qquad \cos u > -\frac{\pi\sin u - F}{\delta+p+t}$$

and

$$(31)\qquad \cos v > -\frac{\pi\sin v - F}{\delta+p+t}\cdot$$

Here p and t are positive but otherwise may be chosen arbitrarily small. Let $z=r\exp(i\pi)$ with

$$(32)\qquad \exp E\leqq r\leqq\exp(-\delta-p).$$

That there is an interval consisting of more than one point satisfying (32) follows from the restrictions (30) and (31) placed on u and v. For if u and v satisfy (30) and (31) then $E<-\delta-p$. Denote by $b_1''>0$ the minimum assumed by $b_1(r, \psi)$ on $\pi/2\leqq\psi\leqq u<\pi$ and by $b_2''>0$ the minimum assumed by $b_2(r, \psi)$ on $-\pi<v\leqq\psi\leqq-\pi/2$ where r satisfies (32). Denote by $b_0''>0$ the smaller of b_1'' and b_2''. Let $b^*>0$ be the smaller of b_0 and b_0''. Then by (6), (7) and (23) if $-\pi<v\leqq\psi\leqq u<\pi$, $\delta\leqq F$ and $\exp E\leqq r\leqq\exp(-\delta-p)$, where u and v satisfy (30) and (31) we have

$$(33)\qquad \left| \int_C \frac{a(\omega)z^\omega}{\exp(2\pi i\omega) - 1}\, d\omega \right| \leqq Rr^k T_2(B, D) \int_v^u \exp(-b^* R)\, d\psi \leqq Rr^k T_2(B, D) \exp(-b^* R)(u - v).$$

Given an $\epsilon > 0$ there exists an R_1 such that for $R > R_0$, R_1 and $\exp E \leqq r \leqq \exp(-\delta - p)$ the quantity on the right of (33) is less than ϵ. Hence as R becomes infinite we again obtain (15) since (24) converges uniformly to zero.

It is now possible to consider again the four cases of §2; however, for brevity we shall consider only those corresponding to A and D. Let us denote these by A′ and D′.

A′: $\quad 0 < \psi_1 < \pi/2, \qquad -\pi/2 < \psi_2 < 0,$

D′: $\quad \pi/2 < \psi_1 \leqq u, \qquad v \leqq \psi_2 < -\pi/2.$

Case A′. Suppose that $\delta \leqq \pi - d$, $d > 0$. It will be shown that $J_{\psi_1}(z) + J_{\psi_2}(z)$ is regular for z in a domain which includes all or part of the segment $-\exp[-\delta - p] \leqq z < 0$ in its interior. By a method similar to that employed in §2 we could show that for $J_{\psi_1}(z)$ to converge for a fixed $z = r \exp(i\theta)$ it is sufficient that

$$(34)\qquad 0 < r < \exp[\theta \tan \psi_1 - \delta \sec \psi_1], \qquad 0 \leqq \theta < 2\pi.$$

It follows then that $J_{\psi_1}(z)$ will converge uniformly for z in any closed domain contained in the domain defined by (34). Hence by reasoning similar to that employed in §2, it is easily seen that $J_{\psi_1}(z)$ is regular for z in any closed domain contained in (34).

Consider $J_{\psi_2}(z)$. It is easily shown that it converges for a fixed $z = r \exp(i\theta)$ contained in the domain defined by

$$(35)\qquad 0 < r < \exp[(\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2], \quad 0 \leqq \theta < 2\pi.$$

If then $z = r \exp(i\theta)$ is in any closed domain contained in the domain defined by (35), $J_{\psi_2}(z)$ converges uniformly and hence represents a regular function. The function $J_{\psi_1}(z) + J_{\psi_2}(z)$ will therefore be regular in any closed domain contained in the domain common to (34) and (35). We have seen that if $z = r \exp(i\pi)$ with $\exp E \leqq r \leqq \exp[-\delta - p]$, $p > 0$, that

$$\sum_{n=l}^{\infty} a(n) z^n = J_{\psi_1}(z) + J_{\psi_2}(z).$$

But $J_{\psi_1}(z) + J_{\psi_2}(z)$ has been shown to be regular in a domain which includes all or part of the interval $\exp E \leqq r \leqq \exp[-\delta - p]$ in its interior. Therefore $J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n) z^n$ provides the analytic continuation of $f(z)$ to any closed domain contained in the domain common to (34) and (35). Hence in this case $f(z)$ will be regular in a domain bounded by two spirals as indicated in Figure 4.

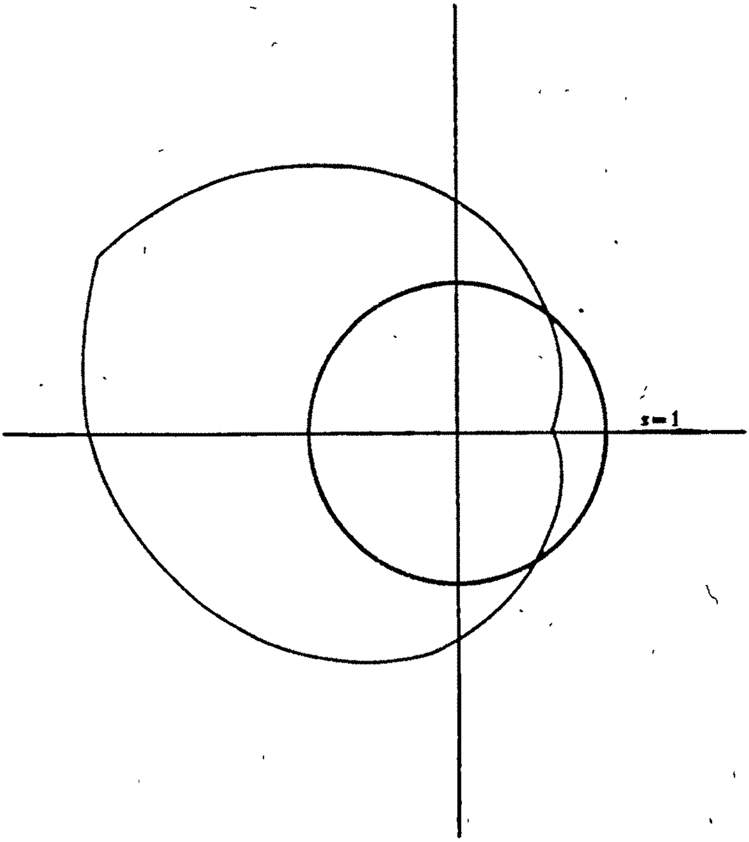

FIG. 4

Consider case D′. Suppose $\delta \leqq F$ and that v and u satisfy (30) and (31). Then for $\exp E \leqq r \leqq \exp(-\delta - p)$ it follows from (33) that (15) holds. It is easily seen that $J_{\psi_1}(z)$ will converge for a fixed $z = r \exp(i\theta)$ if $\log r^{-1} \cos \psi_1 + \theta \sin \psi_1 - \delta > 0$. This will be the case if

$$(36) \qquad r > \exp\left[\theta \tan \psi_1 - \delta \sec \psi_1\right], \qquad 0 \leqq \theta < 2\pi.$$

It is evident that $J_{\psi_1}(z)$ will converge uniformly for $z = r \exp(i\theta)$ in any closed bounded domain contained in the domain defined by (36). In order that $J_{\psi_2}(z)$ converge for a fixed $z = r \exp(i\theta)$ it is sufficient that $\log r^{-1} \cos \psi_2 + (\theta - 2\pi) \sin \psi_2 - \delta > 0$. That is,

$$(37) \qquad r > \exp\left[(\theta - 2\pi) \tan \psi_2 - \delta \sec \psi_2\right], \qquad 0 \leqq \theta < 2\pi.$$

Then $J_{\psi_2}(z)$ will converge uniformly for $z = re^{i\theta}$ in any closed bounded domain contained in the domain defined by (37). Hence

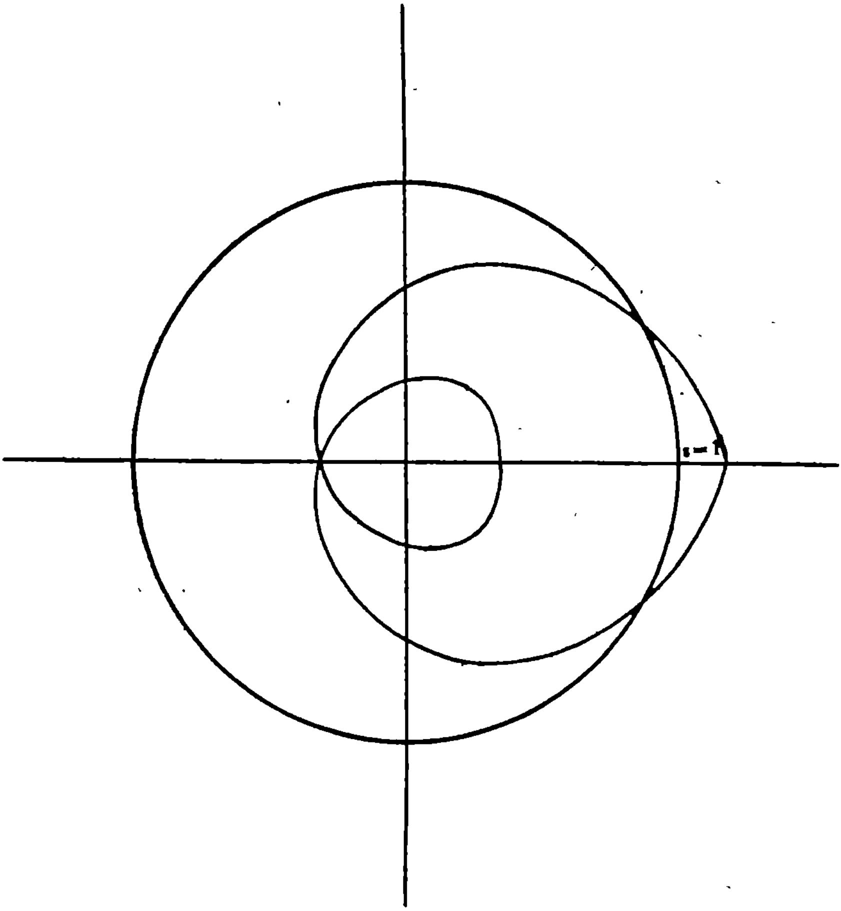

FIG. 5

$J_{\psi_1}(z) + J_{\psi_2}(z)$ will converge uniformly for $z = re^{i\theta}$ in any closed bounded domain contained in the domain common to (36) and (37). It is evident that the region common to (36) and (37) contains the interval $-\exp(-\delta - p) \leqq z \leqq -\exp E$ in its interior. For E is by definition the larger of $\pi \tan u - F \sec u + t$ and $\pi \tan v - F \sec v + t$, $t > 0$, and if we set $\theta = \pi$ in (36) and (37) it is clear that both exponents are smaller than E. Therefore $J_{\psi_1}(z) + J_{\psi_2}(z) + \sum_{n=0}^{l-1} a(n) z^n$ provides the analytic continuation of $f(z)$ to any closed bounded domain contained in the domain common to (36) and (37). For the case $\psi_1 = \psi_2$ this will be such a domain as indicated in Figure 5.

Hence we have the following theorem.

THEOREM 2. *Let* $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *with radius of convergence unity. Let the coefficients* a_n *be the values taken on by an analytic function* $a(z)$ *at* $z=0, 1, 2, \cdots$. *Suppose* $a(z)$ *is regular with the possible exception of an essential singularity at infinity in an angle with vertex* $h>0$ *(non-integral) on the real axis, including the axis of positive reals in its interior. Let the sides of this angle make angles* ψ_1 *and* ψ_2 *with the axis of positive reals. Then if, for* $z=h+R\exp(i\psi)$ *in this angular opening,* $a(z)$ *satisfies the inequality*

$$|a(h+R\exp(i\psi))|<\exp(\delta R), \qquad R>R_0,$$

where $\delta\leqq\pi-d$, $d>0$, *and*

$$\text{A}': \qquad 0<\psi_1<\pi/2, \qquad -\pi/2<\psi_2<0,$$

$f(z)$ *is regular in the domain common to*

$$r\leqq\exp[\theta\tan\psi_1-\delta\sec\psi_1]-\gamma, \qquad \gamma>0$$

and

$$r\leqq\exp[(\theta-2\pi)\tan\psi_2-\delta\sec\psi_2]-\gamma, \qquad \gamma>0,$$

for $0\leqq\theta<2\pi$.

For a given $q>0$ *but otherwise arbitrarily small let* F *be the smaller of the numbers* $\pi\sin u-q$ *and* $\pi\sin v-q$. *Suppose* u *and* v *may be chosen such that for a given* p *and* t *positive but otherwise arbitrarily small*

$$\cos u>-\frac{\pi\sin u-F}{\delta+p+t}$$

and

$$\cos v>-\frac{\pi\sin v-F}{\delta+p+t}.$$

Then if $\delta\leqq F$ *and*

$$\text{D}': \qquad \pi/2<\psi_1\leqq u<\pi, \qquad -\pi<v\leqq\psi_2<-\pi/2,$$

$f(z)$ *is regular in any bounded domain common to*

$$r\geqq\exp[\theta\tan\psi_1-\delta\sec\psi_1]+\gamma, \qquad \gamma>0,$$

and

$$r\geqq\exp[(\theta-2\pi)\tan\psi_2-\delta\sec\psi_2]+\gamma, \qquad \gamma>0,$$

where $0\leqq\theta<2\pi$.

4. **Conclusions.** We note first that if, in Theorem 1, $\psi_1>0$ and $\psi_2<0$ but otherwise arbitrarily small, that $f(z)$ has $z=1$ as its only singularity on the circle of convergence.

In part D of Theorem 1 if both ψ_1 and ψ_2 are greater than 90° in magnitude, that is, the sector of regularity is greater than 180°, we have the rather remarkable result that $z=1$ is the only singularity of $f(z)$ in the finite plane. Thus for example the function defined by the series

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^\beta},$$

where β is an integer and α is not equal to zero or a negative integer but otherwise arbitrary, has the point $z=1$ as its only singularity in the finite plane.

It is of course clear that we may use the results obtained in a different manner, that is, if $f(z)$ has a singular point on the circle of convergence other than $z=1$ then $a(z)$ cannot be analytic in an angular opening including the axis of positive reals in its interior with at most a pole of finite order at infinity.

If the inequality for $a(z)$ in Theorem 2 is satisfied for every $\delta>0$ however small, then under the condition of case D′ of Theorem 2, $z=1$ is the only singularity in the finite plane. This result[5] is analogous to the following theorem due to Faber [1].

THEOREM (FABER). *If* $g(z)$ *is an integral function such that* $|g(re^{i\theta})|<e^{\epsilon r}$ *for an arbitrary positive* ϵ *and* $r>r'$, *the function* $f(z)$ *defined by* $\sum_{n=0}^{\infty} g(n)z^n$ *and its analytic continuation has the point* 1 *as its only singular point.*

We observe now that the bounding curves

$$r < \exp\left[\theta \tan\psi_1 - \delta \sec\psi_1\right], \qquad 0 \leqq \theta < 2\pi,$$

and

$$r < \exp\left[(\theta - 2\pi)\tan\psi_2 - \delta\sec\psi_2\right], \quad 0 \leqq \theta < 2\pi,$$

of (34) and (35) cut the unit circle at the points $\exp\left[i\delta\csc\psi_1\right]$ and $\exp\left[i(2\pi+\delta\csc\psi_2)\right]$. If now, in addition to the requirements of part A of Theorem 2, ψ_1, ψ_2 and δ satisfy the inequality

$$2\pi > \delta(\csc\psi_1 - \csc\psi_2)$$

it is easily seen that the region common to (34) and (35) will extend beyond the unit circle. We then have the following theorem.

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[5] The author is indebted to the referee for pointing out the analogy.

THEOREM 3. *If the conditions of Theorem 2 part* A *are satisfied and if in addition the quantities ψ_1, ψ_2 and δ satisfy the inequality*

$$2\pi > \delta(\csc \psi_1 - \csc \psi_2)$$

then the circle of convergence is not a cut for the function.

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A NOTE ON THE HILBERT TRANSFORM

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The Hilbert transform of $f(t)$, $-\infty < t < \infty$, is $1/\pi$ times the Cauchy principal value

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \lim_{\delta \to 0+} \int_{\delta}^{\infty} \frac{f(x+t) - f(x-t)}{t}\,dt.$$

If $f(t) \in L_p$, $p > 1$, then $\bar{f}(x) \in L_p$, and a considerable literature is devoted to studying the relationship of such pairs of "conjugate" functions to the theory of functions analytic in a half-plane. More to the point of the present note is a series of papers studying the Hilbert transform along strictly real variable lines ([2, 3]; further bibliography in [2]).[1]

Much less is known about $\bar{f}(x)$ when $f(t) \in L_1$. Plessner found by applying complex variable methods to the theory of Fourier series that if $f(t) \in L_1$ then $\bar{f}(x)$ exists almost everywhere (see [1, p. 145]). Besicovitch [4] proved Plessner's result using only the theory of sets, starting from his own previous real variable investigation of the L_2 transform case. S. Pollard [5] showed how Besicovitch's proof could be extended to prove the existence a.e. of the principal value of the Stieltjes integral

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{dF(t)}{t-x},$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $F(t)$ is continuous and of bounded variation over $(-\infty, \infty)$. In general $\bar{f}(x)$ is not summable, but Kolmogoroff [6] found, using a contradiction argument, that there exists a constant A such that the set where $\bar{f}(x) > M > 0$ has measure at most $A\|f\|/M$, where $\|f\| = \int_{-\infty}^{\infty} |f(t)|\,dt$. Titchmarsh [7] was able to refine Besicovitch's existence proof so that it implied this bound, with a numerical value for A.

The present note contains a new direct real-variable proof of the Plessner existence theorem and the Kolmogoroff bound. In fact, this bound in a sense is the central tool for the existence proof, a device which allows for the first time the L_1 results to be obtained without recourse to the L_2 transform theory.

LEMMA 1. *If* $c_i > 0$ *and*

$$g(x) = \sum_{i=1}^{n} \frac{c_i}{x - a_i},$$

then the set of points where $g(x) > M$ $(M > 0)$ *consists of* n *intervals whose total length is precisely* $(\sum c_i)/M$. *The set where* $g(x) < -M$ *has the same length.*

Since $g(a_i-) = -\infty$, $g(a_i+) = \infty$ and $g'(x) < 0$ for all x, there are precisely n points m_i such that $g(m_i) = M$, and $a_i < m_i < a_{i+1}$, $i = 1, \cdots, n-1$, $a_n < m_n$. The set where $g(x) > M$ thus consists of the intervals (a_i, m_i) and has the total length

$$\sum_{i=1}^{n} (m_i - a_i) = \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} a_i. \tag{1}$$

But the numbers m_i are the roots of the equation

$$\sum_{i=1}^{n} \frac{c_i}{x - a_i} = M,$$

whose cross-multiplied form is

$$\sum_{i=1}^{n} c_i \left[\prod_{j \neq i} (x - a_j) \right] = M \prod_{i=1}^{n} (x - a_i),$$

or

$$Mx^n - \left[M \sum a_i + \sum c_i\right] x^{n-1} + \cdots = 0,$$

so that

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} a_i + \frac{1}{M} \sum_{i=1}^{n} c_i. \tag{2}$$

The first part of the lemma follows from (1) and (2); the proof for $g(x) < -M$ is almost identical.

LEMMA 2. *Let $F(t)$ be increasing over $(-\infty, \infty)$ with finite total variation $V(F)$. If $(x_j-\delta_j, x_j+\delta_j)$, $j=1, \cdots, n$, are disjoint intervals such that*

$$(3) \qquad \int_{-\infty}^{x_j-\delta_j} + \int_{x_j+\delta_j}^{\infty} \frac{dF(t)}{t-x_j} > M > 0,$$

then $\sum \delta_j \leqq 4V(F)/M$. The same inequality is implied if the integral is less than $-M$, $j=1, \cdots, n$.

Let t_i, $i=1, \cdots, N$ be a finite subdivision including the points $x_j-\delta_j$, x_j, $x_j+\delta_j$ for $j=1, \cdots, n$, and such that the approximating Riemann sums for (3), with the integrand evaluated at the left-hand end points, remain greater than M. Thus, if $\Delta_i = F(t_{i+1}) - F(t_i)$,

$$(4) \qquad \sum_{i \notin I_j} \frac{\Delta_i}{t_i - y} > M$$

for $y = x_j$, where the set I_j of omitted indices is defined by

$$\bigcup_{i \in I_j} (t_i, t_{i+1}) = (x_j - \delta_j, x_j + \delta_j).$$

Since the left member of (4) is an increasing function of y for $x_j - \delta_j < y < x_j + \delta_j$, the inequality (4) holds for $x_j \leqq y < x_j + \delta_j$. For every such y one of the following inequalities is therefore satisfied:

$$\sum_{i=1}^{N-1} \frac{\Delta_i}{t_i - y} > \frac{M}{2}, \qquad \sum_{i \in I_j} \frac{\Delta_i}{t_i - y} < -\frac{M}{2}.$$

Applying Lemma 1 and summing over j, we have

$$\sum \delta_j \leqq \sum_{i=1}^{N-1} \frac{2\Delta_i}{M} + \sum_{j=1}^{n} \sum_{i \in I_j} \frac{2\Delta_i}{M} \leqq \frac{4}{M} \sum_{i=1}^{N-1} \Delta_i \leqq \frac{4}{M} V(F).$$

To prove the second part of the lemma we only need to observe that the integral in (3) is less than $-M$ if and only if after replacing $F(t)$ by $-F(-t)$ and x_j by $-x_j$ it is greater than M.

COROLLARY. *If $F(t)$ is of bounded variation in Lemma 2 then $\sum \delta_j \leqq 8V(F)/M$.*

This follows at once upon applying the lemma to the increasing and decreasing parts, F_1 and F_2, of F, using $V(F) = V(F_1) + V(F_2)$.

Preliminary to the theorem we remark that if $f(t)$ has the value 1 in

(a, b) and 0 elsewhere, then its Hilbert transform exists except at the two points a and b, and has the value $\log |(x-b)/(x-a)|$. In particular the Hilbert transform of any step function exists except at a finite number of points.

THEOREM. *Let $F(t)$ be of bounded variation over $(-\infty, \infty)$. Then its Hilbert-Stieltjes transform*

$$\bar{f}(x) = P\int_{-\infty}^{\infty} \frac{dF(t)}{t-x}$$

exists almost everywhere, and, for every positive M, the set where $\bar{f}(x) > M$ has measure at most $16V(F)/M$, as does the set where $\bar{f}(x) < M$.

We first prove the existence of $\bar{f}(x)$. It is sufficient to show that, given ϵ, for every x except in a set of measure less than ϵ

$$\left| \int_{x-\delta}^{x-\delta'} + \int_{x+\delta'}^{x+\delta} \frac{dF(t)}{t-x} \right| \leqq \epsilon \tag{5}$$

for all sufficiently small δ and δ'. Now the absolutely continuous part of F can be approximated to within ϵ' by the integral F_1 of a step function h, $F_1(t) = \int_{-\infty}^{t} h(t)dt$, and the singular part of F can be approximated to within ϵ' by a singular function F_2 whose variation is confined to a closed set of measure 0, that is, which is constant on the intervals of an open set M whose complement has measure zero. Thus, taking $\epsilon' = \epsilon^2/192$, we have $F = F_1 + F_2 + F_3$, where $V(F_3) < \epsilon^2/96$. Let E_ϵ be the set of x for which the inequality

$$\left| \int_{x-\delta}^{x-\delta'} + \int_{x+\delta'}^{x+\delta} \frac{dF_3(t)}{t-x} \right| \leqq \frac{\epsilon}{3} \tag{6}$$

fails to hold for arbitrarily small δ and δ' $(\delta' < \delta)$. Then for every x in E_ϵ

$$\left| \int_{-\infty}^{x-\Delta} + \int_{x+\Delta}^{\infty} \frac{dF_3(t)}{t-x} \right| > \frac{\epsilon}{6} \tag{7}$$

for arbitrarily small Δ. By Vitali's theorem a disjoint sequence of intervals $(x_i - \Delta_i, x_i + \Delta_i)$ satisfying (7) can be chosen so as to cover E_ϵ except for a set of measure 0. Then by Lemma 2, corollary, $m(E_\epsilon) \leqq 2\sum \Delta_i \leqq 2 \cdot 8V(F_3) \cdot 6/\epsilon < \epsilon$.

Since $F_1(t)$ is the integral of a step function, its Hilbert-Stieltjes transform (the Hilbert transform of $h(t)$) exists except at a finite number of points, which we add to E_ϵ. Since $F_2(t)$ is constant on the intervals of M, its Hilbert-Stieltjes transform obviously exists except on the complement of M, which we add to E_ϵ. Thus if x is not in the enlarged E_ϵ there is a Δ such that, for all δ and δ' less than Δ, (6) holds for F_1, F_2 and F_3, and hence (5) holds, as was to be proved.

The second part of the theorem follows immediately from Lemma 2, corollary, where the intervals $(x_j-\delta_j, x_j+\delta_j)$ are chosen by Vitali's theorem to cover almost entirely the set where $\bar{f}(x)>M$ $(<-M)$, so that the measure of this set is not greater than $2\sum\delta_j \leqq 16V(F)/M$.

COROLLARY. *If* $0<p<1$ *and* $p+q>1$, *then* $|\bar{f}(x)|^p/(1+|x|)^q \in L_1$.

This follows immediately from the fact that the decreasing function on $(0, \infty)$ which is equimeasurable with $|\bar{f}(x)|$ is dominated by K/x.

In case $F(t)$ is singular and increasing, it can be shown with little difficulty that the constant 16 can be replaced by 1, and this is best possible since $1/x$ itself is the Hilbert-Stieltjes transform of the function $F(t)$ which is 1 when $t<0$ and 0 when $t \geqq 0$. This is probably the correct value of the constant in the general case.

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HARVARD UNIVERSITY

ON THE STRUCTURE OF LINEAR GRAPHS

P. ERDŐS AND A. H. STONE

Introduction. If the numbers of vertices and edges of a (linear) graph are suitably restricted, it is to be expected that something can be said about the configurations which the graph contains. As far as we know the first result in this direction is due to Turán.[1] He proved that a graph with kn vertices and $C_{k,2}n^2+1$ edges always contains a complete graph of order $k+1$. We shall here prove one such theorem (which arose originally out of a topological problem),[2] and then list (without proofs) several other theorems and conjectures of this nature.

Notations. For the present purposes, a *graph* is simply a finite set of "*vertices*," together with an assignment of certain pairs of vertices (possibly none) as being "*edges*." Two vertices in an edge are said to be *joined*; the *order* of a vertex is the number of vertices to which it is joined. The *complementary* graph G^* to a graph G has the same vertices as G, but two vertices are joined in G^* if and only if they are *not* joined in G. A *complete graph of order* k is a graph having k vertices, every two of which are joined. When $k=3$, this configuration is called simply a triangle. If E is any set, $|E|$ denotes the cardinal number of E. For any real number x, $[x]$ denotes the greatest integer not greater than x, and $[x]^*$ the least integer not less than x. We write $l_1(x)=\ln(x)$, $l_2(x)=\ln(\ln(x))$, and generally $l_r(x)=\ln(l_{r-1}(x))$. Letters like m, n, p, k, N, r, and so on, usually denote positive integers, and ϵ always denotes a positive number less than 1.

THEOREM. *Given ϵ and an integer $r\geqq 2$, there exists $n_0(\epsilon, r)$ such that, for every $n>n_0$, every linear graph having n vertices and fewer than $(1/2(r-1)-\epsilon)n^2$ edges contains r mutually exclusive groups of k vertices each, for some $k\geqq (l_{r-1}(n))^{1/2}$, such that no two vertices in different groups are joined.*

The proof will go by induction over r. First we need a combinatorial lemma.

LEMMA. *Given N subsets Q_1, Q_2, $\cdots$, Q_N (not necessarily all distinct).*

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[1] In fact Turán determined for every k and n the maximum number of edges a graph of n vertices can have without containing a complete graph of k vertices (Matematikai és physikai lapok (1941)) (in Hungarian).

[2] See A. H. Stone, *Connectedness and coherence*, Annals of Mathematics Studies.

of a set E of n elements, such that each $|Q_i| \geqq p \geqq k$, *there exist at least* $NC_{p,k}/C_{n,k}$ *of the sets* Q_i *whose intersection has at least* k *elements.*

Let M be the greatest number of sets Q_i having k or more elements in common (or, more exactly, the greatest number of distinct suffixes for which the corresponding sets have this property). In each Q_i choose a subset R_i with $|R_i| = p$, and let M' be the greatest number of sets R_i having not less than k elements in common (with the same convention as before). Consider now the subsets S_j of E with $|S_j| = k$; there are $C_{n,k}$ of them. Each R_i contains exactly $C_{p,k}$ sets S_j, and each S_j is contained in at most M' sets R_i. Hence $N \leqq M'C_{n,k}/C_{p,k}$; and since $M' \leqq M$, the result follows.

COROLLARY 1. $M/N \geqq ((p-k+1)/n)^k$.

COROLLARY 2. *If further* $p \geqq \epsilon n$ *and* $k \leqq \epsilon \ln n$ (*where* $0<\epsilon<1$), *then* $M/N \geqq 1/n^{3/4}$.

Corollary 2 is readily deduced from Corollary 1. (In fact, this estimate can be considerably improved, but will suffice for our purpose.)

Restating the theorem in terms of the complementary graph, we have to prove:

If G is a graph having n vertices and not less than $(1/2-1/2(r-1)+\epsilon)n^2$ edges, and if n is large enough (depending on r and ϵ), then G contains r mutually exclusive groups of k vertices each, where $k = [(l_{r-1}(n))^{1/2}]^*$, such that two vertices in different groups are always joined.

Suppose first that $r=2$. Let G have N vertices $P_1, \cdots, P_N$ of orders not less than $\epsilon n/2$. Since the total number of edges of G is less than $Nn+\epsilon n(n-N)/2$, and yet is not less than ϵn^2, by hypothesis, it follows that $N>\epsilon n/2$. Let Q_i be the set of vertices to which P_i is joined. An application of Corollary 2 shows that at least $N/n^{3/4}$ of the sets Q_i intersect in at least $[(\epsilon \ln n)/2]$ points. If n is large enough, and $k=[(\ln n)^{1/2}]^*$, we have $N/n^{3/4}>\epsilon n^{1/4}/2>k$ and $[(\epsilon \ln n)/2]>k$; thus we may suppose that $Q_1 \cap Q_2 \cap \cdots \cap Q_k$ contains the k distinct vertices $R_1, \cdots, R_k$. The vertices $P_1, \cdots, P_k$ and $R_1, \cdots, R_k$ now form the two groups required, since every P_i and R_j $(i, j \leqq k)$ are joined.

Now suppose that $r \geqq 3$ is given, and that the theorem is true for $r-1$. We say that a number $\epsilon>0$ is "admissible" if the theorem is true for it (for the given value of r); thus evidently $1/2(r-1)$ is admissible. Let c be the greatest lower bound of the admissible numbers ϵ; thus $0 \leqq c \leqq 1/2(r-1)$, and it will clearly suffice to prove that $c=0$. Suppose, then, $c>0$; we shall derive a contradiction.

By the theorem for $r-1$, with $\epsilon=1/2(r-1)(r-2)$, there exists n_0 such that every graph with $n \geqq n_0$ vertices and not less than $(1/2-1/2(r-1))n^2$ edges contains $r-1$ mutually exclusive groups of $2q$ vertices each, where $q=[\{l_{r-2}(n)\}^{1/2}/2]$, such that every two vertices in different groups are joined. Choose $\delta>0$ with $\delta<c/2r$. Since $c-\delta$ is *not* admissible, there exists for arbitrarily large n a graph G having n vertices and not less than $(1/2-1/2(r-1)+c-\delta)n^2$ edges, but *not* containing the r groups of k vertices each (where $k=[(l_{r-1}(n))^{1/2}]^*$) demanded by the theorem. Choose such a graph G for which n is large compared with n_0 and $1/\delta$.

G must contain $r-1$ groups of q (and in fact of $2q$) vertices each, say P_{ij} $(i \leqq r-1, j \leqq q)$, with P_{ij} and P_{gh} joined whenever $i \neq g$. Of the $n-q(r-1)$ other vertices, let N be the number which are joined to $(r-2)q+kq/\ln q$ or more of the vertices P_{ij}. Each of these N vertices is thus joined to at least $kq/\ln q$ vertices P_{ij} from each group (i fixed). An application of Corollary 2 of the lemma (with $\epsilon=k/\ln q$) shows that at least $N/q^{3/4}$ of the N vertices are all joined to the *same* k vertices P_{1j}—say $P_{11}, \cdots, P_{1k}$. Similarly, at least $N/(q^{3/4})^2$ of these $N/q^{3/4}$ vertices are also all joined to (say) $P_{21}, \cdots, P_{2k}$; and so on. We finally obtain $h<N/(q^{3/4})^{r-1}$ vertices (say) $P_{r1}, \cdots, P_{rh}$, each of which is joined to each P_{ij} $(i \leqq r-1, j \leqq k)$. Thus $h<k$, since otherwise G contains the r groups of k vertices required by the theorem. It readily follows that (n being large) $N<kq^{3(r-1)/4}<n^{1/2}<nk/\ln q$.

Now consider the subgraph G_1 of G obtained by discarding the $q(r-1)$ vertices P_{ij} and all edges involving them. The number of discarded edges is thus at most

$$Nq\,(r-1)+(n-q\,(r-1)-N)((r-2)\,q+kq/\ln q)+(r-1)^2q^2/2$$

(from the way in which N was defined); and this is easily seen to be less than $nq\{(r-2)+kq/\ln q+N/n\}<nq\{(r-2)+2k/\ln q\}$ $<nq(r-2)(1+\delta)$ if n is large enough (since $2k/\ln q \to 0$).

We now repeat the whole argument on G_1. If G_1 has n_1 vertices, it is easy to see that if n was large enough n_1 will also be large enough and the number of edges of G_1 will be large enough for the argument to apply to G_1 (with the same values of q and k as before; note that $2q_1>q$, where $q_1=[\{l_{r-2}(n_1)\}^{1/2}]$). Thus in the same way we omit $q(r-1)$ vertices and the edges through them from G_1, leaving a subgraph G_2, and so on, repeating the argument as long as it continues to apply.

We assert: the argument can be applied at least $s=[cn/(r-1)q]$ times. For the total number of vertices this removes is not greater than $cn \leqq n/4$, so each of the graphs $G_1, \cdots, G_{s-1}$ has not less than

$3n/4$ vertices, which is large enough if n was chosen large enough to begin with. Further, the total number of edges removed is less than $cn^2(1+\delta)(r-2)/(r-1)$, so that each G_t $(t<s)$ has more than $\{1/2-1/2(r-1)\}n^2$ edges (as readily follows from the choice of δ), and so certainly has more than $\{1/2-1/2(r-1)\}n_t^2$ edges (where n_t is the number of its vertices), which are all that are required.

Write $d=(r-1)qs/n$; thus $0<d\leqq c$, and $d\to c$ as $n\to\infty$. Then G_s has $n(1-d)$ vertices; and since $c+\delta$ is an admissible ϵ, it follows that on taking n large enough G_s can have at most $(1/2-1/2(r-1)+c+\delta)(1-d)^2n^2$ edges. Thus, on counting the number of edges of G, we obtain

$$\begin{aligned}&n^2(1/2-1/2(r-1)+c-\delta)\\&\quad< n^2(1/2-1/2(r-1)+c+\delta)(1-d)^2+n^2c(1+\delta)(r-2)/(r-1).\end{aligned}$$

Divide through by n^2, and observe that (by making $n\to\infty$) we can make $d\to c$, and also can now make $\delta\to 0$. Thus $(r-2)/2(r-1)+c \leqq ((r-2)/2(r-1)+c)(1-c)^2+(r-2)c/(r-1)$, which leads, on simplification, to $c\geqq 2-(r-2)/2(r-1)>3/2$. But this is absurd, since $c\leqq 1/2(r-1)\leqq 1/4$; and the theorem is proved.

Remark. The estimate for k given by this theorem can be improved; in fact, the same proof will show that we may take $k\geqq (l_{r-1}(n))^{1-\delta}$ for each fixed $\delta>0$. It is plausible though unproved that $l_{r-1}(n)$ would be about the "best" value.[3]

On the other hand the number of edges required by the theorem is substantially "best possible." To see this, suppose for convenience that $r-1$ is a divisor of n, and consider a graph G which is the sum of $r-1$ pairwise disjoint complete graphs of order $n/(r-1)$. G has n vertices and fewer than $n^2/2(r-1)$ edges; yet, of any r vertices, at least two are joined, so that the conclusion of the theorem is false, even with $k=1$.

Further theorems and conjectures. Rademacher[4] proved that every graph of $2n$ vertices and n^2+1 edges contains at least n triangles. It has been conjectured that every graph with $2n$ vertices and n^2+k edges must contain at least kn triangles if $k<n$, and this has been proved for $k\leqq 3$ (Erdős;[5] it is false for $k=n$).

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[3] By using the method of the paper *Some remarks on the theory of graphs* (P. Erdős, to be published in Bull. Amer. Math. Soc.), we can show that for $r=2$, $c\log n$ is best possible. The method fails for $r>2$.

[4] Oral communication.

[5] Unpublished.

Similar problems arise for complete subgraphs of higher order. By analogy with Rademacher's theorem we can perhaps conjecture that every graph of kn vertices and $C_{k,2}n^2+1$ edges must contain n^{k-1} complete subgraphs of order $k+1$.

STANFORD UNIVERSITY AND
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L. K. Hua, *A remark on a result due to Blichfeldt.*

p. 539, last display. Insert $n^{n/e-n/en}$ after $\geq$.

T. Motzkin, *A 5 curve theorem generalising the theorem of Carnot.*

p. 974, lines 4 and 6 from the bottom. Insert "$\prod$" after the word ratio.

T. Motzkin, *The hypersurface cross ratio.*

p. 976, lines 13 and 14. These two lines should read "surfaces, in §3. This section is parallel to §§1 and 2, much of the contents of which is known, on the ordinary resultant and intersection."

p. 976, last line of footnote 3. For "0 degree" read "0°."

p. 976, line 10 from the bottom, p. 978, line 10, and p. 979, line 19. for "$[a]$" read "$[a]^-$."

p. 978, line 16. For "equal to 0" read "identically 0."

p. 978, line 23. For "f^n" read "$\bar{f}^n$."

p. 978, line 30. For "and on" read "and of."

p. 979, line 24. For "congruent to 0" read "identically 0."

p. 980, line 2. For "equal to" read "different from."

p. 981, lines 19 and 21. For "(γ_1)" read "$(\gamma 1)$."

p. 982, lines 11 and 17 from the bottom. For "a_1" read "a_2."

T. Motzkin, *The pentagon in the projective plane, with a comment on Napier's rule.*

p. 986, line 22. For "30" read "90."

p. 987, line 24. For "$i, -i$" read "$-i, i$."

p. 988, line 16. Delete the sentence "λ and the cross ratio quintuple determine each other."

p. 989, line 22 (twice). For "$=$" read "$\neq$."

ERRATA, VOLUME 52

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p. 42, line 2. For "51-1-5" read "52-1-5."

A. Spitzbart, *Approximation in the sense of least pth powers with a single auxiliary condition of interpolation.*

p. 341, fourth display. For "$(1+\alpha w)$" read "$(1+\bar{\alpha} w)$."

H. S. M. Coxeter, *A simple proof of the eight square theorem*, abstract 52-7-222.

p. 612, line 4 of the abstract. For "$b\bar{a}$" read "$\bar{b}\bar{a}$."

Howard Levene, *A test of randomness in two dimensions*, abstract 52-7-257.

p. 621, line 7 of the abstract. The second equation in the line should read "$\sigma^2(V) \sim N^2pq^2(4-20p+45p^2-27p^3)/4$."

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